Lecture 4: Constraint Satisfaction
CS486/686 Intro to Artificial Intelligence

2023-5-18

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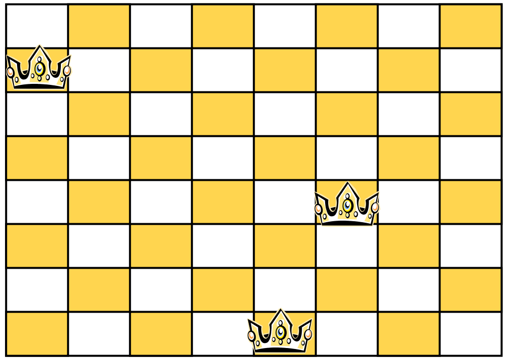
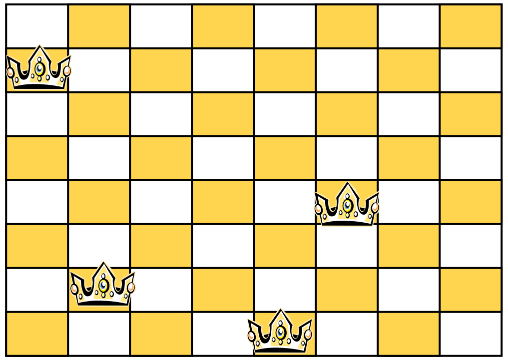
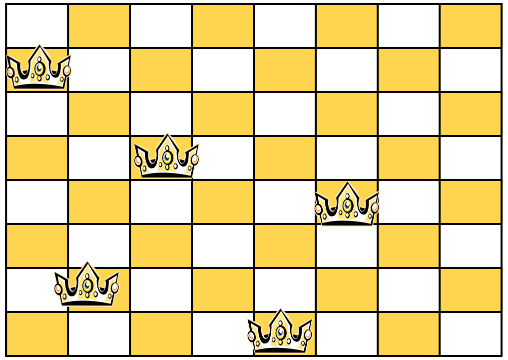
Outline

- What are Constraint Satisfaction Problems (CSPs)?
- Standard search and CSPs
- Improvements
  - Backtracking
  - Backtracking + heuristics
  - Forward checking

Introduction

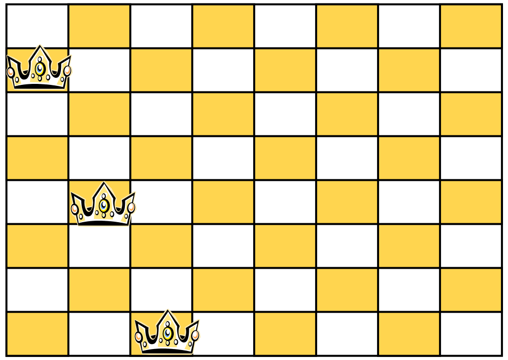
- In the last couple of lectures we have been solving problems by searching in a space of states
  - Treating states as black boxes, ignoring any structure inside them
  - Using problem-specific routines

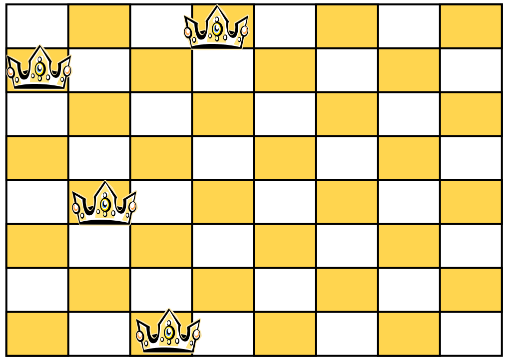
- Today we study problems where the state structure is important

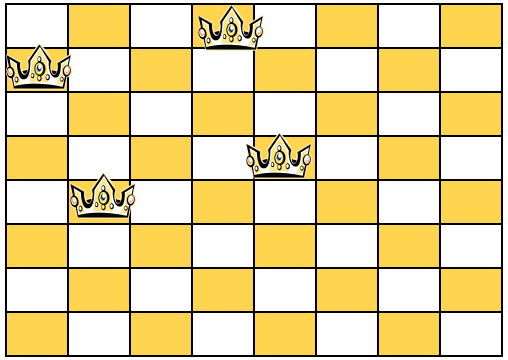




- **States:** all arrangements of 0,1,…, or 8 queens on the board
- **Initial state:** 0 queens on the board
- **Successor function:** Add a queen to the board
- **Goal test:** 8 queens on the board with no two of them attacking each other

$$64 \times 63 \times \ldots 57 \approx 3 \times 10^{14} \text{ states}$$







- **States:** all arrangements k queens ($0 \le k \le 8$), one per column in the leftmost k columns, with no queen attacking another
- **Initial state:** 0 queens on the board
- **Successor function**: Add a queen to the leftmost empty column such that it is not attacked
- **Goal test:** 8 queens on the board

2057 States

Introduction

- Earlier search methods studied often make choices in an arbitrary order

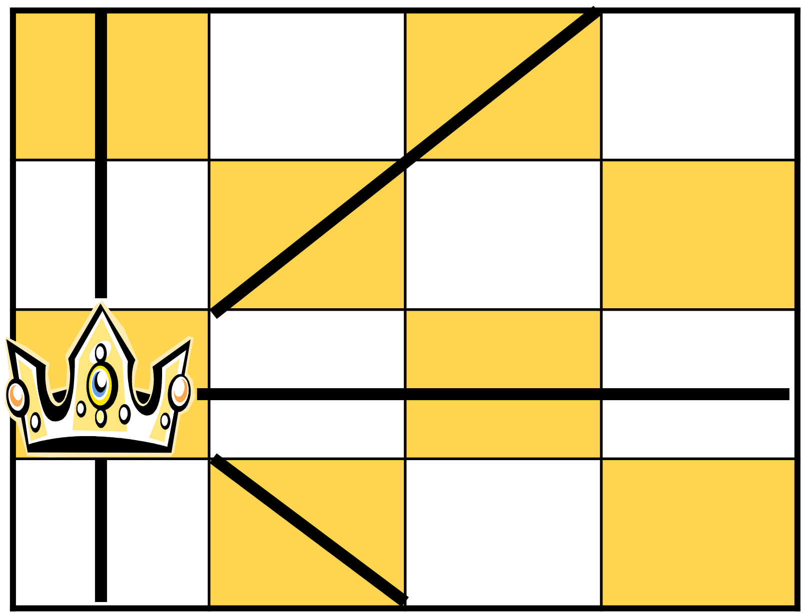
- In many problems the same state can be reached independent of the order in which the moves are chosen (commutative actions)

- Can we solve problems efficiently by being smart in the order in which we take actions?

4-Queens Constraint Propagation

Place a queen in a square

Remove conflicting squares from consideration



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4-Queens Constraint Propagation

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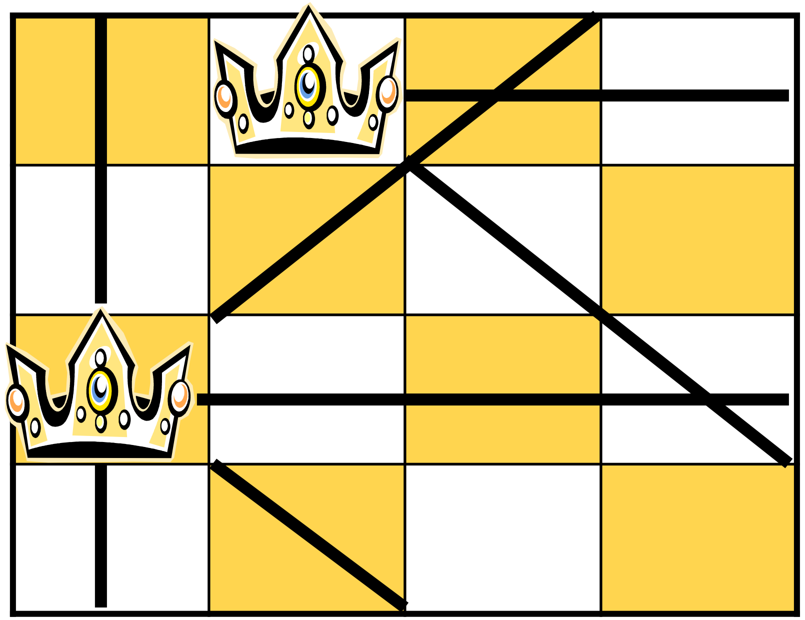

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4-Queens Constraint Propagation

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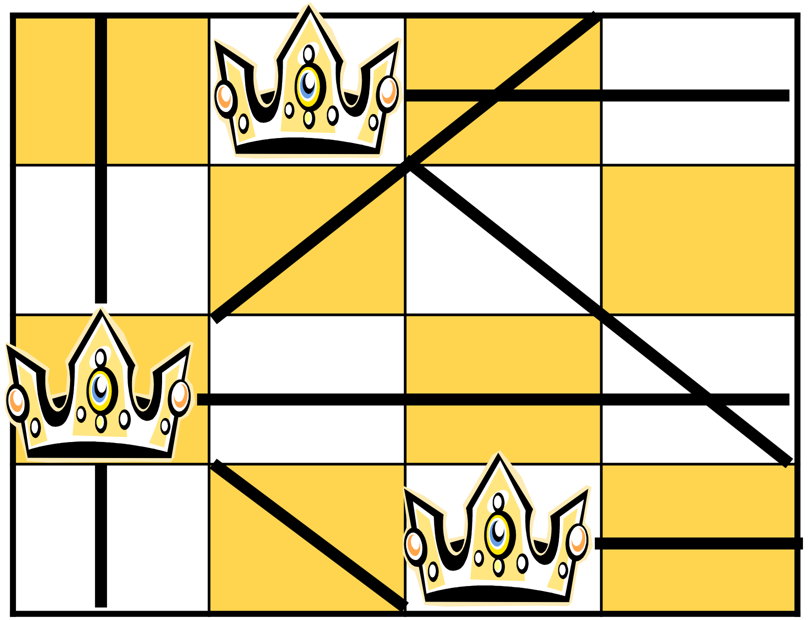
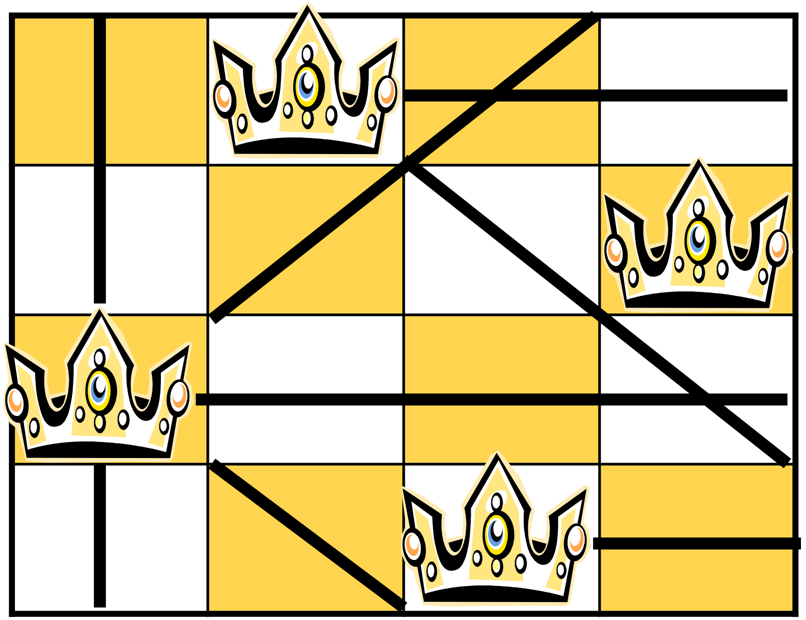
Remove conflicting squares from consideration



4-Queens Constraint Propagation

Place a queen in a square

Remove conflicting squares from consideration



CSP Definition

- A constraint satisfaction problem (CSP) is defined by $\{V, D, C\}$ where
  - $V = \{V_1, V_2, \dots, V_n\}$ is a set of variables
  - $D = \{D_1, \dots, D_n\}$ is the set of domains, D_i is the domain of possible values for variable V_i
  - $C = \{C_1, \dots, C_m\}$ is the set of constraints
    - Each constraint involves some subset of the variables and specifies the allowable combinations of values for that subset

CSP Definition

- A state is an assignment of values to some or all of the variables

$$\{V_i = x_i, V_j = x_j, \ldots\}$$

- An assignment is consistent if it does not violate any constraints

- A solution is a complete, consistent assignment ("hard constraints")
  - Some CSPs also require an objective function to be optimized ("soft constraints")

Example 1: 8-Queens

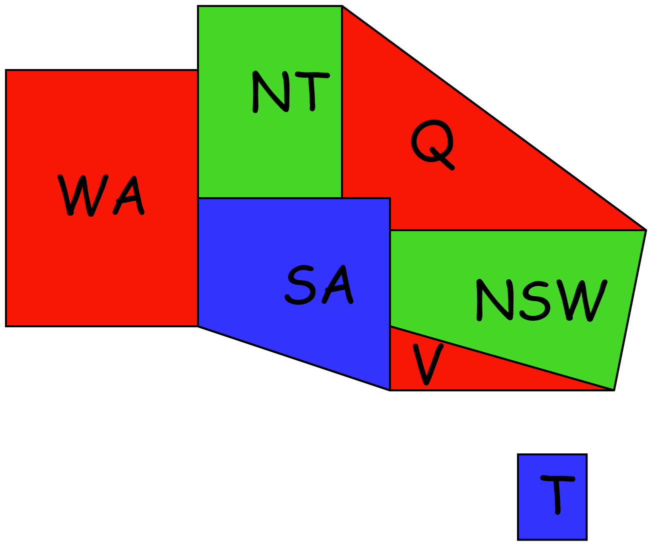
- 64 variables V_{ij}, $i = 1$ to 8, $j = 1$ to 8
- Domain of each variable is {0,1}
- Constraints
  - $V_{ij} = 1 \rightarrow V_{ik} = 0$ for all $k \neq j$
  - $V_{ij} = 1 \rightarrow V_{kj} = 0$ for all $k \neq i$
  - Similar constraint for diagonals
  - $\sum_{ij} V_{ij} = 8$

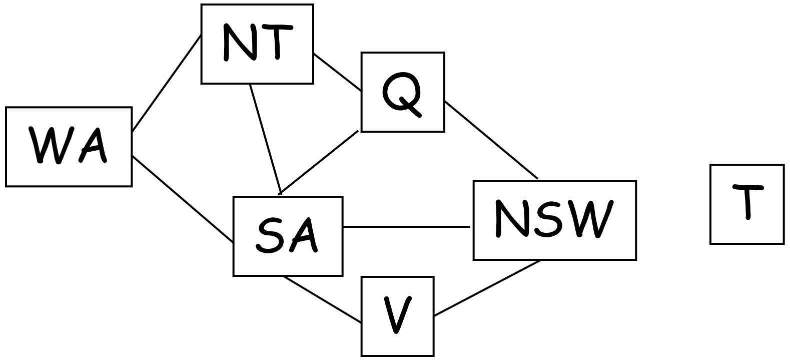
Binary constraints relate two variables

Example 2 – 8 queens

- 8 variables V_i, $i = 1$ to 8
- Domain of each variable is $\{1,2,\ldots,8\}$
- Constraints
  - $V_i = k \;\rightarrow\; V_j \neq k$ for all $j \neq i$
  - Similar constraints for diagonals

Example 3 - Map Coloring





Constraint graph

- 7 variables $\{WA, NT, SA, Q, NSW, V, T\}$
- Each variable has the same domain: $\{red, green, blue\}$
- No two adjacent variables have the same value:

$$WA{\neq}NT, WA{\neq}SA, NT{\neq}SA, NT{\neq}Q, SA{\neq}Q, SA{\neq}NSW, SA{\neq}V, Q{\neq}NSW, NSW{\neq}V$$

Example 4 - Street Puzzle

1	2	3	4	5

N_i = {English, Spaniard, Japanese, Italian, Norwegian}
C_i = {Red, Green, White, Yellow, Blue}
D_i = {Tea, Coffee, Milk, Fruit-juice, Water}
J_i = {Painter, Sculptor, Diplomat, Violinist, Doctor}
A_i = {Dog, Snails, Fox, Horse, Zebra}

The Englishman lives in the Red house
The Spaniard has a Dog
The Japanese is a Painter
The Italian drinks Tea
The Norwegian lives in the first house on the left
The owner of the Green house drinks Coffee
The Green house is on the right of the White house
The Sculptor breeds Snails
The Diplomat lives in the Yellow house
The owner of the middle house drinks Milk
The Norwegian lives next door to the Blue house
The Violinist drinks Fruit juice
The Fox is in the house next to the Doctor's
The Horse is next to the Diplomat's

Who owns the Zebra?
Who drinks Water?

Example 4 - Street Puzzle

1	2	3	4	5

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The Englishman lives in the Red house → $(N_i = \text{English}) \Leftrightarrow (C_i = \text{Red})$
The Spaniard has a Dog
The Japanese is a Painter → $(N_i = \text{Japanese}) \Leftrightarrow (J_i = \text{Painter})$
The Italian drinks Tea
The Norwegian lives in the first house on the left → $(N_1 = \text{Norwegian})$
The owner of the Green house drinks Coffee
The Green house is on the right of the White house →
$$\begin{cases} (C_i = \text{White}) \Leftrightarrow (C_{i+1} = \text{Green}) \\ (C_5 \neq \text{White}) \\ (C_1 \neq \text{Green}) \end{cases}$$
The Sculptor breeds Snails
The Diplomat lives in the Yellow house
The owner of the middle house drinks Milk
The Norwegian lives next door to the Blue house → left as an exercise
The Violinist drinks Fruit juice
The Fox is in the house next to the Doctor's
The Horse is next to the Diplomat's

Example 4 - Street Puzzle

1 2 3 4 5

N_i = {English, Spaniard, Japanese, Italian, Norwegian}
C_i = {Red, Green, White, Yellow, Blue}
D_i = {Tea, Coffee, Milk, Fruit-juice, Water}
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A_i = {Dog, Snails, Fox, Horse, Zebra}

$(N_i = \text{English}) \Leftrightarrow (C_i = \text{Red})$

The Englishman lives in the Red house
The Spaniard has a Dog
The Japanese is a Painter → $(N_i = \text{Japanese}) \Leftrightarrow (J_i = \text{Painter})$
The Italian drinks Tea
The Norwegian lives in the first house on the left → $(N_1 = \text{Norwegian})$
The owner of the Green house drinks Coffee
The Green house is on the right of the White house → $\left\{ \begin{array}{l} (C_i = \text{White}) \Leftrightarrow (C_{i+1} = \text{Green}) \\ (C_5 \neq \text{White}) \\ (C_1 \neq \text{Green}) \end{array} \right.$
The Sculptor breeds Snails
The Diplomat lives in the Yellow house
The owner of the middle house drinks Milk
The Norwegian lives next door to the Blue house
The Violinist drinks Fruit juice
The Fox is in the house next to the Doctor's
The Horse is next to the Diplomat's

unary constraints

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Example 4 - Street Puzzle

1	2	3	4	5

N_i = {English, Spaniard, Japanese, Italian, Norwegian}
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Implicit constraints:

$\forall i,j \in [1,5],\ i \neq j,\ N_i \neq N_j$

$\forall i,j \in [1,5],\ i \neq j,\ C_i \neq C_j$

...

Example 4 - Street Puzzle

1	2	3	4	5

N_i = {English, Spaniard, Japanese, Italian, Norwegian}
C_i = {Red, Green, White, Yellow, Blue}
D_i = {Tea, Coffee, Milk, Fruit-juice, Water}
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A_i = {Dog, Snails, Fox, Horse, Zebra}

The Englishman lives in the Red house
The Spaniard has a Dog
The Japanese is a Painter
The Italian drinks Tea
The Norwegian lives in the first house on the left → **N_1 = Norwegian**
The owner of the Green house drinks Coffee
The Green house is on the right of the White house
The Sculptor breeds Snails
The Diplomat lives in the Yellow house
The owner of the middle house drinks Milk → **D_3 = Milk**
The Norwegian lives next door to the Blue house
The Violinist drinks Fruit juice
The Fox is in the house next to the Doctor's
The Horse is next to the Diplomat's

Use inference to derive new facts

Example 4 - Street Puzzle

1	2	3	4	5

N_i = {English, Spaniard, Japanese, Italian, Norwegian}
C_i = {Red, Green, White, Yellow, Blue}
D_i = {Tea, Coffee, Milk, Fruit-juice, Water}
J_i = {Painter, Sculptor, Diplomat, Violinist, Doctor}
A_i = {Dog, Snails, Fox, Horse, Zebra}

The Englishman lives in the Red house → **$C_1 \neq$ Red**
The Spaniard has a Dog → **$A_1 \neq$ Dog**
The Japanese is a Painter
The Italian drinks Tea
The Norwegian lives in the first house on the left → **N_1 = Norwegian**
The owner of the Green house drinks Coffee
The Green house is on the right of the White house
The Sculptor breeds Snails
The Diplomat lives in the Yellow house
The owner of the middle house drinks Milk → **D_3 = Milk**
The Norwegian lives next door to the Blue house
The Violinist drinks Fruit juice → **$J_3 \neq$ Violinist**
The Fox is in the house next to the Doctor's
The Horse is next to the Diplomat's

Use inference to derive new facts

Example 5 - Scheduling

Four tasks T_1, T_2, T_3, and T_4 are related by time constraints:

- T_1 must be done during T_3
- T_2 must be achieved before T_1 starts
- T_2 must overlap with T_3
- T_4 must start after T_1 is complete

- Are the constraints compatible?
- What are the possible time relations between two tasks?
- What if the tasks use resources in limited supply?

Example 6 - 3-Sat

- n Boolean variables, $V_1, \ldots, V_n$

- K constraints of the form $V_i \lor V_j \lor V_k$ where V_i is either true or false

- NP-complete

Properties of CSPs

- Types of variables
  - Discrete and finite
    - Map colouring, 8-queens, Boolean CSPs
  - Discrete variables with infinite domains
    - Scheduling jobs in a calendar
    - Require a constraint language ($Job_1 + 3 \leq Job_2$)
  - Continuous domains
    - Scheduling on the Hubble telescope
    - Linear programming

Properties of CSPs

- Types of constraints
  - Unary constraint relates a single variable to a value
    - $Queensland = Blue, SA \neq Green$
  - Binary constraint relates two variables
    - $SA \neq NSW$
    - Can use a constraint graph to represent CSPs with only binary constraints
  - Higher order constraints involve three of more variables
    - $Alldiff(V_1, \ldots, V_n)$
    - Can use a constraint hypergraph to represent the problem

CSPs and search

- N variables $V_1, \dots, V_n$
- Valid assignment: $\{V_1 = x_1, \dots, V_k = x_k\}$ for $0 \leq k \leq n$ such that values satisfy constraints on the variables
- States: valid assignments
- Initial state: empty assignment
- Successor: $\{V_1 = x_1, \dots, V_k = x_k\} \rightarrow \{V_1 = x_1, \dots, V_k = x_k, V_{k+1} = x_{k+1}\}$
- Goal test: complete assignment
- If all domains have size d, then there are $O(d^n)$ complete assignments

CSPs and commutativity

- CSPs are commutative!
  - The order of application of any given set of actions has no effect on the outcome
  - When assigning values to variables we reach the same partial assignment, no matter the order
  - All CSP search algorithms generate successors by considering possible assignments for only a single variable at each node in the search tree

CSPs and commutativity

- 3 variables V_1, V_2, V_3
- Let the current assignment be $A = \{V_1 = x_1\}$
- Pick variable V_3
- Let domain of V_3 be $\{a, b, c\}$
- The successors of A are

$$\{V_1 = x_1, V_3 = a\}$$
$$\{V_1 = x_1, V_3 = b\}$$
$$\{V_1 = x_1, V_3 = c\}$$

Backtracking Search

```
function BACKTRACKING-SEARCH( csp) returns a solution, or failure
   return RECURSIVE-BACKTRACKING({}, csp)

function RECURSIVE-BACKTRACKING( assignment,csp) returns a solution, or
failure
   if assignment is complete then return assignment
   var ← SELECT-UNASSIGNED-VARIABLE( Variables[csp], assignment, csp)
   for each value in ORDER-DOMAIN-VALUES(var, assignment, csp) do
      if value is consistent with assignment according to Constraints[csp] then
         add { var = value } to assignment
         result ← RECURSIVE-BACKTRACKING(assignment, csp)
         if result ≠ failue then return result
         remove { var = value } from assignment
   return failure
```

Depth first search that chooses values for one variable at a time

Backtracks when a variable has no legal values to assign

Backtracking

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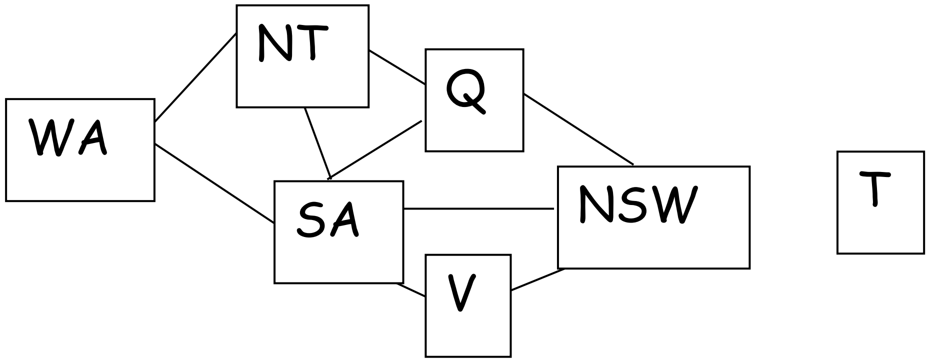
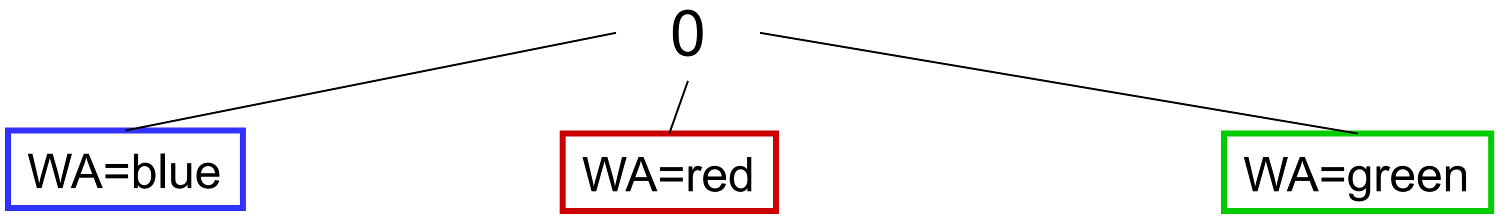
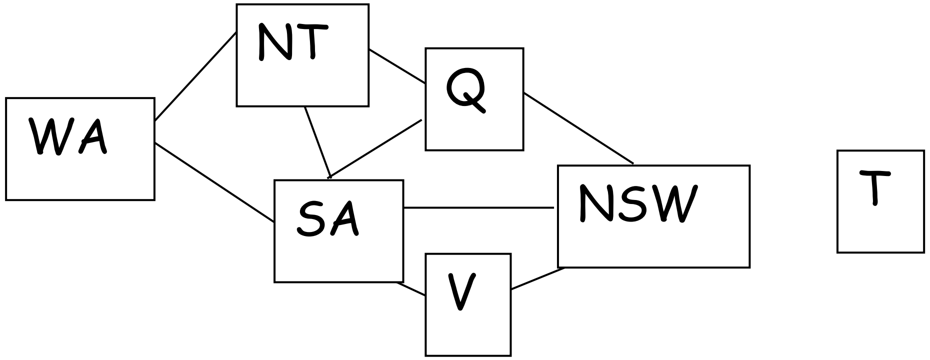


Backtracking





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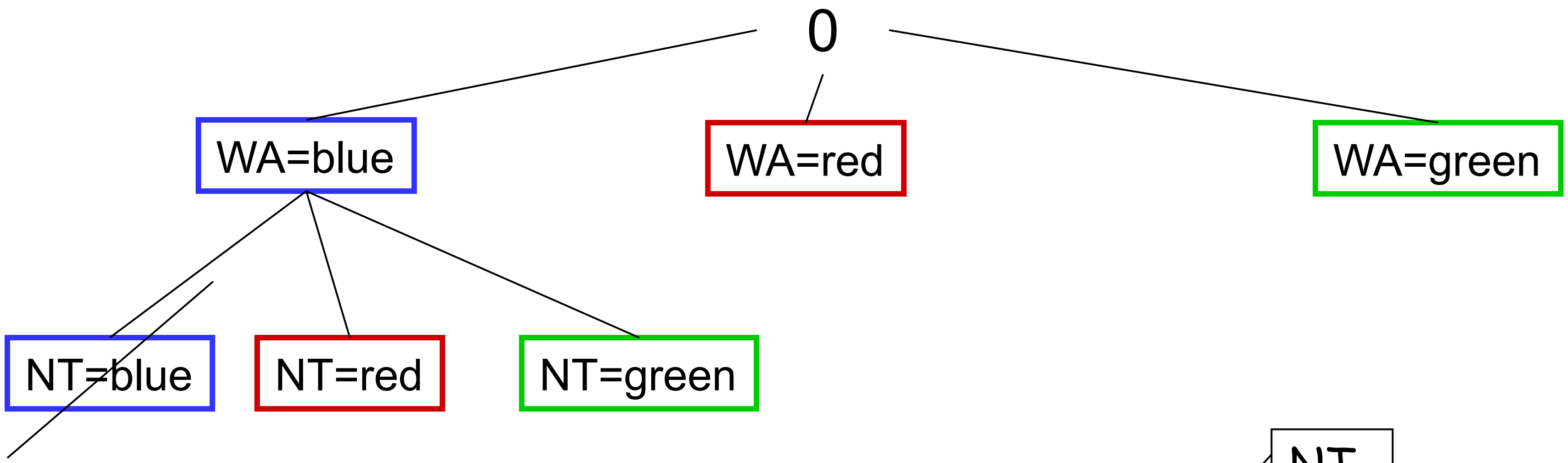

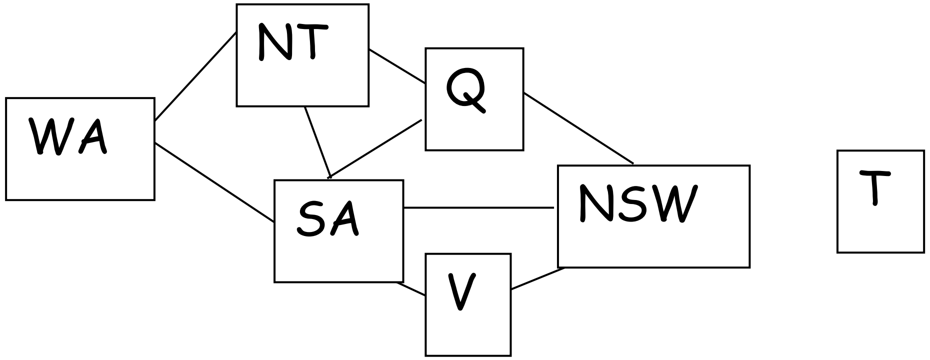

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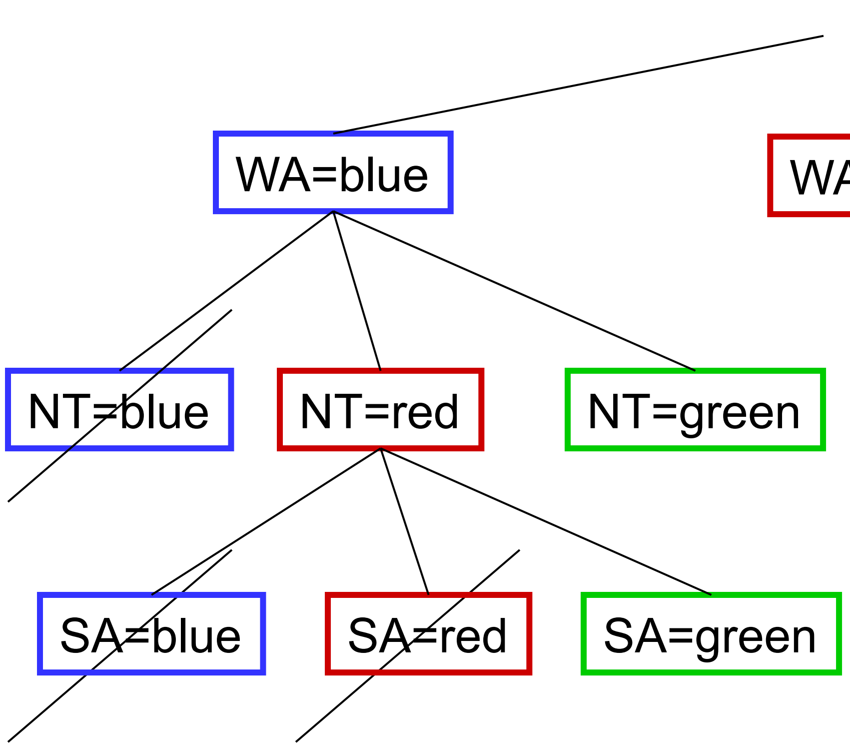
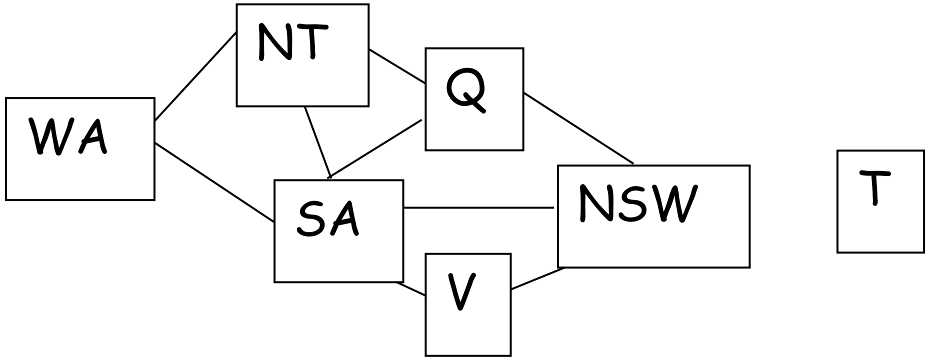




Backtracking and efficiency

- Backtracking search is an uninformed search method
  - Not very efficient

- We can do better by thinking about the following questions
  - Which variable should be assigned next?
  - In which order should its values be tried?
  - Can we detect inevitable failure early (and avoid same failure in other paths)?

Most constrained variable

- Choose the variable which has the fewest "legal" moves
  - AKA minimum remaining values (MRV) heuristic

$D_{NT} = \{green, blue\}$

$D_{SA} = \{green, blue\}$

$D_{others} = \{red, green, blue\}$

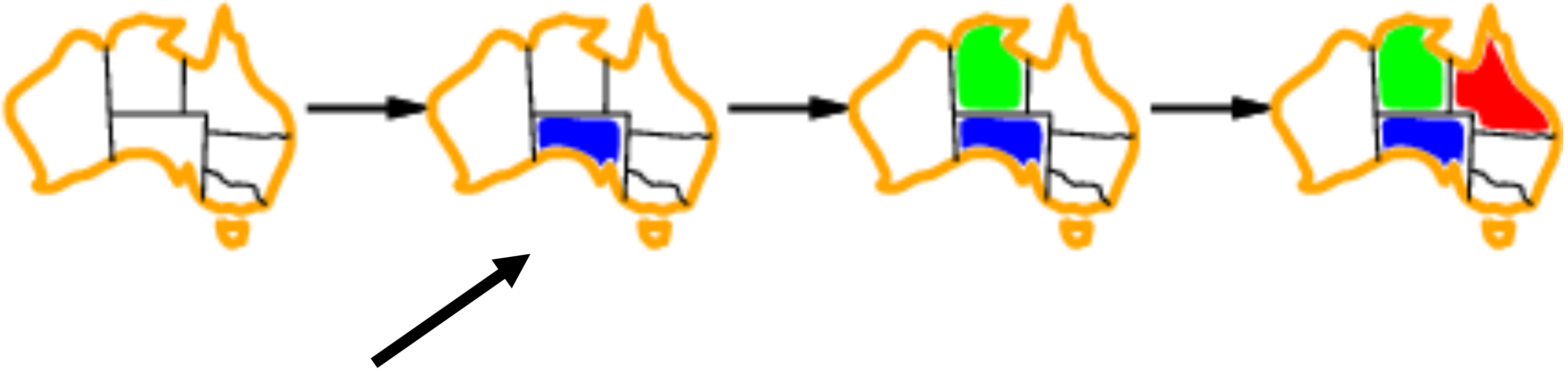
$D_{SA} = \{blue\}$

$D_{Q} = \{blue, red\}$

$D_{others} = \{red, green, blue\}$

Most constraining variable

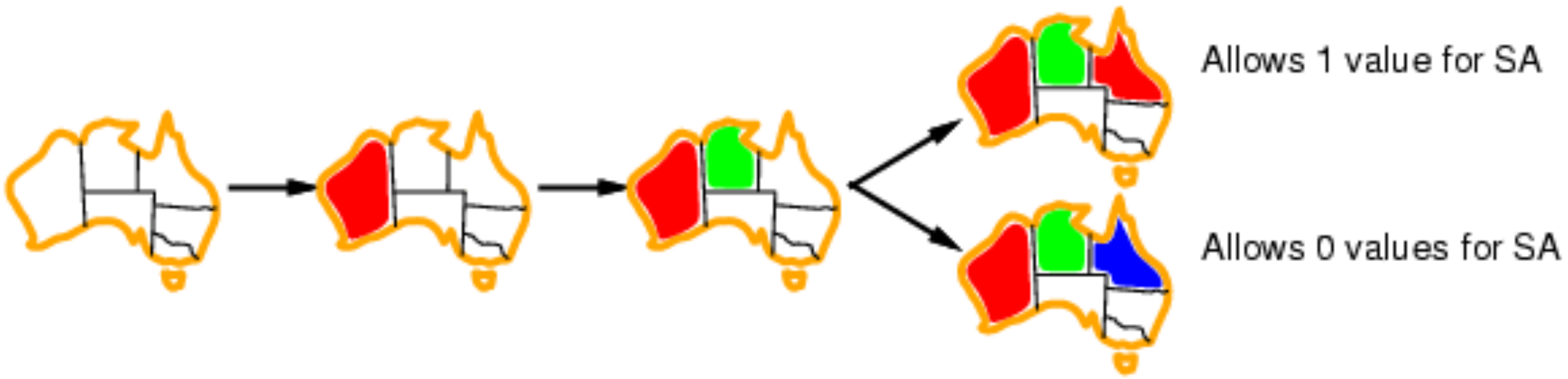
- Most constraining variable:
  - choose the variable with the most constraints on remaining variables
- Tie-breaker among most constrained variables



SA is involved in 5 constraints

Least-constraining value

- Given a variable, choose the least constraining value:
  - the one that rules out the fewest values in the remaining variables

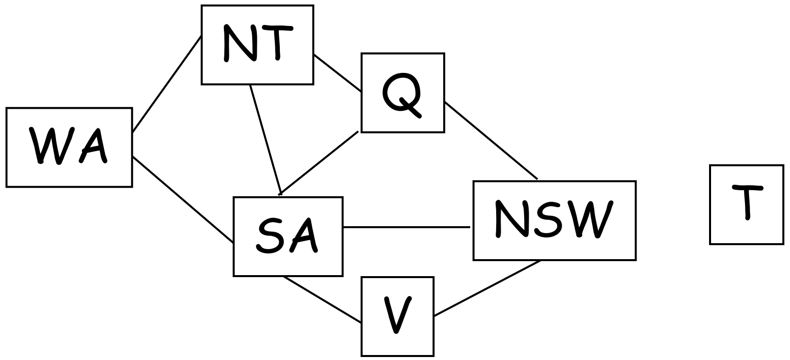


Forward checking

- The third question was
  - Is there a way to detect failure early?

- Forward checking
  - Keep track of remaining legal values for unassigned variables
  - Terminate search when any variable has no legal values

Forward Checking in Map Coloring



WA	NT	Q	NSW	V	SA	T
RGB	RGB	RGB	RGB	RGB	RGB	RGB

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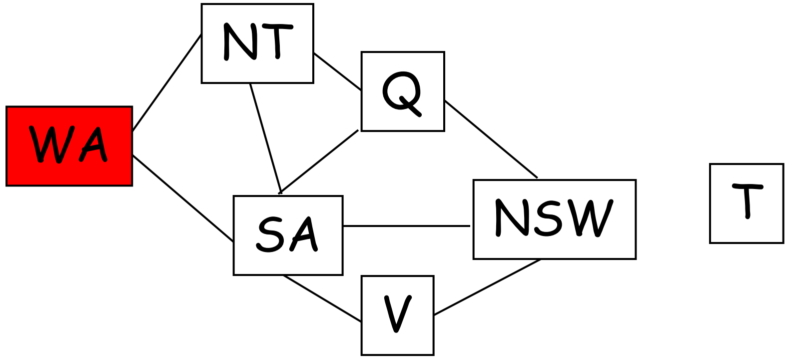
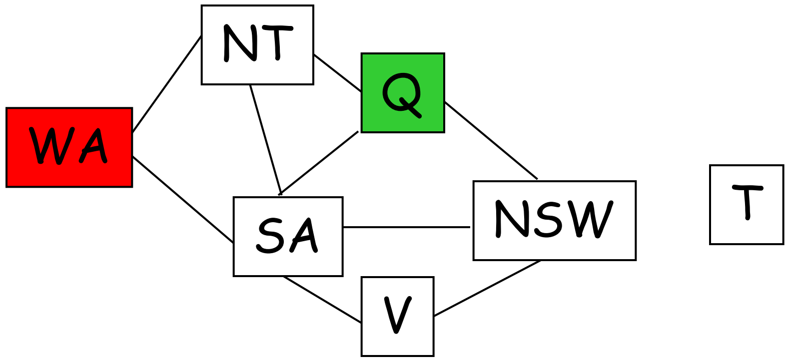
Forward Checking in Map Coloring



WA	NT	Q	NSW	V	SA	T
RGB	RGB	RGB	RGB	RGB	RGB	RGB
R	~~R~~GB	RGB	RGB	RGB	~~R~~GB	RGB

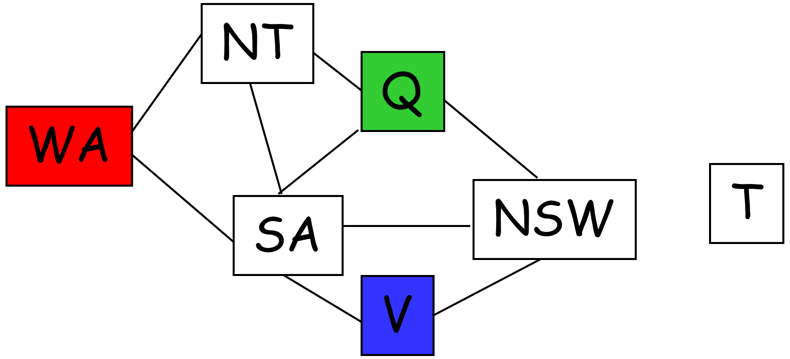
Forward checking removes the value Red of NT and of SA

Forward Checking in Map Coloring



WA	NT	Q	NSW	V	SA	T
RGB	RGB	RGB	RGB	RGB	RGB	RGB
R	GB	RGB	RGB	RGB	GB	RGB
R	~~G~~B	G	R~~G~~B	RGB	~~G~~B	RGB

Forward Checking in Map Coloring



WA	NT	Q	NSW	V	SA	T
RGB	RGB	RGB	RGB	RGB	RGB	RGB
R	GB	RGB	RGB	RGB	GB	RGB
R	B	G	RB	RGB	B	RGB
R	B	G	R~~B~~	B	~~B~~	RGB

Forward Checking in Map Coloring

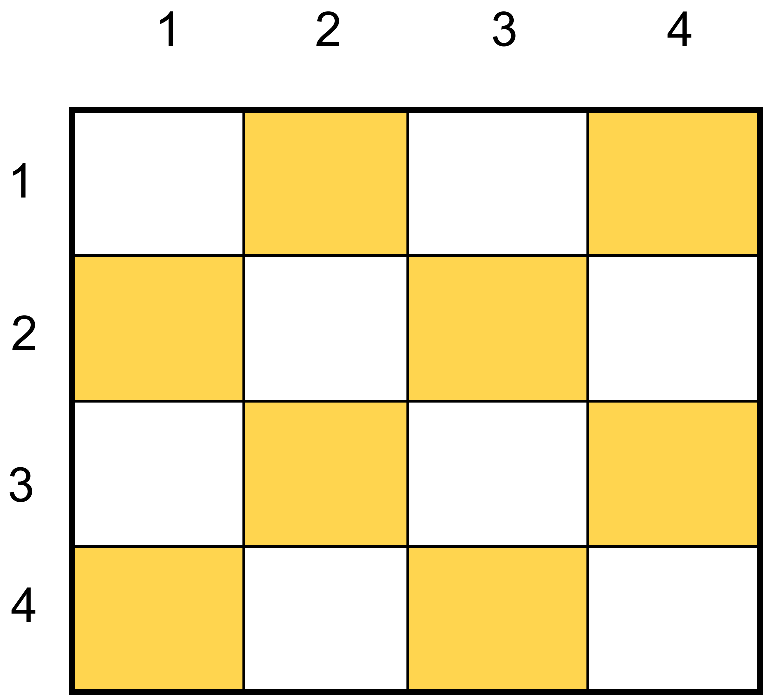
Empty set: the current assignment

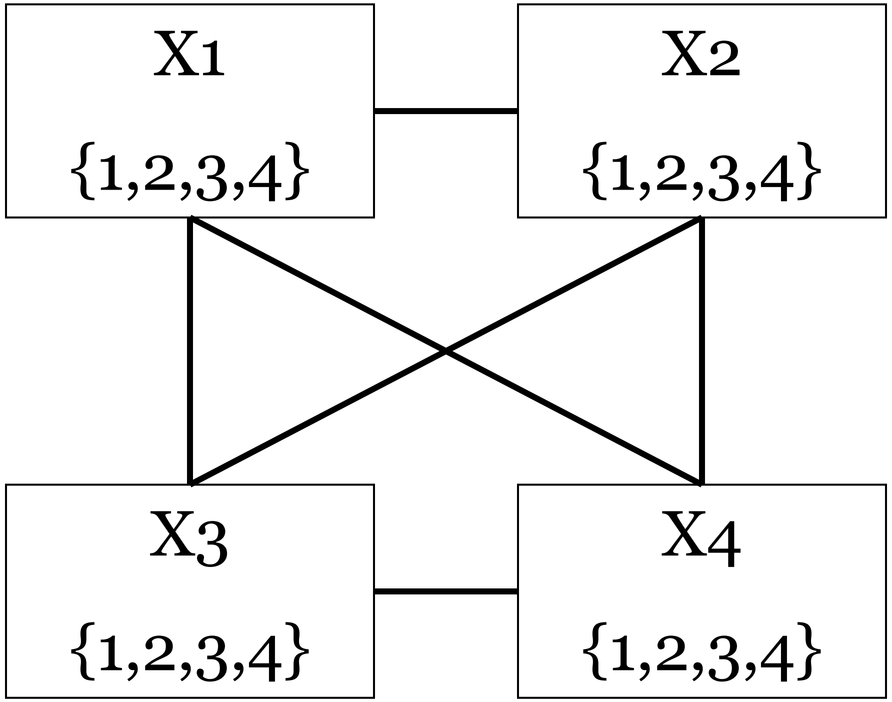
$$\{(WA \leftarrow R), (Q \leftarrow G), (V \leftarrow B)\}$$

does not lead to a solution

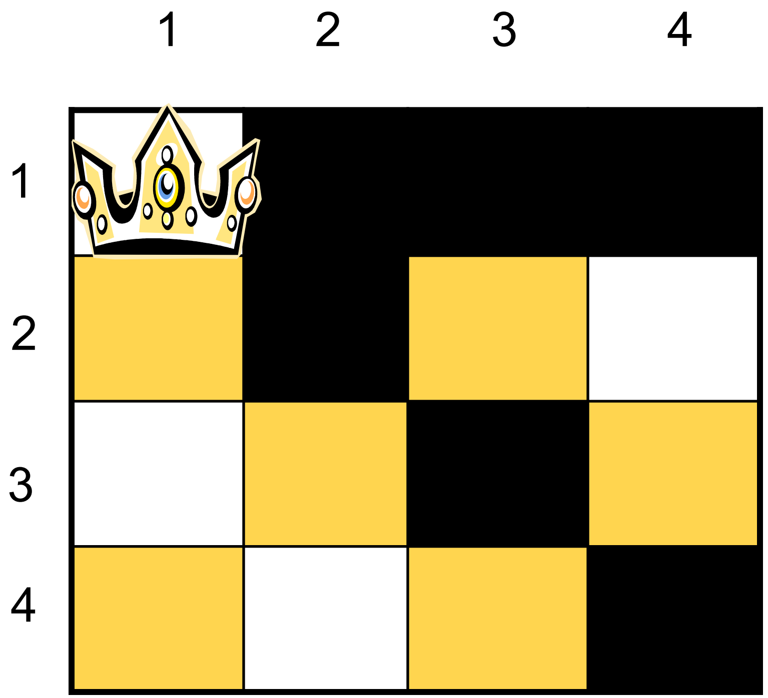
WA	NT	Q	NSW	V	SA	T
RGB	RGB	RGB	RGB	RGB	RGB	RGB
R	GB	RGB	RGB	RGB	GB	RGB
R	B	G	RB	RGB	B	RGB
R	B	G	R~~B~~	B	~~B~~	RGB

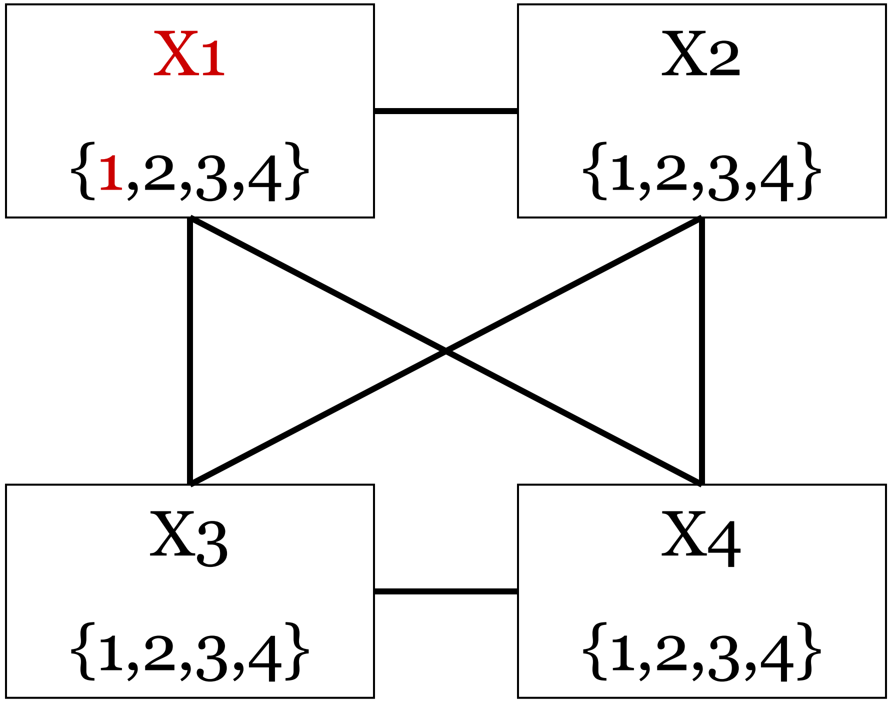
Example: 4 Queens



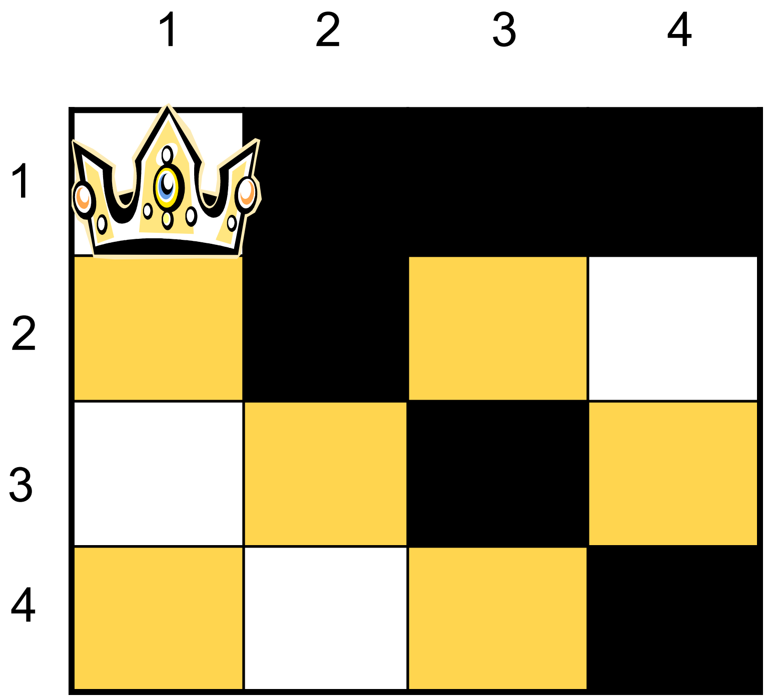


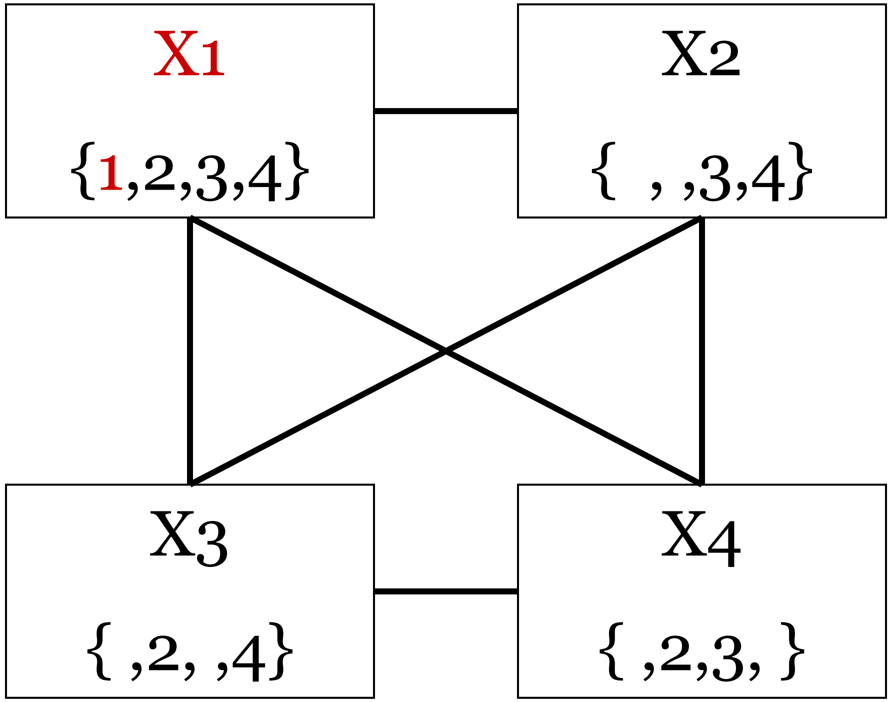
Example: 4 Queens



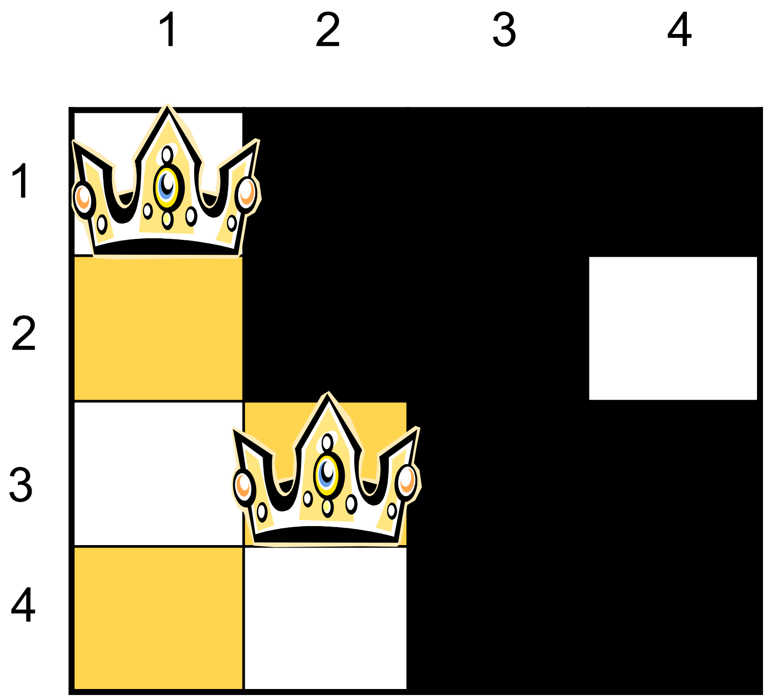


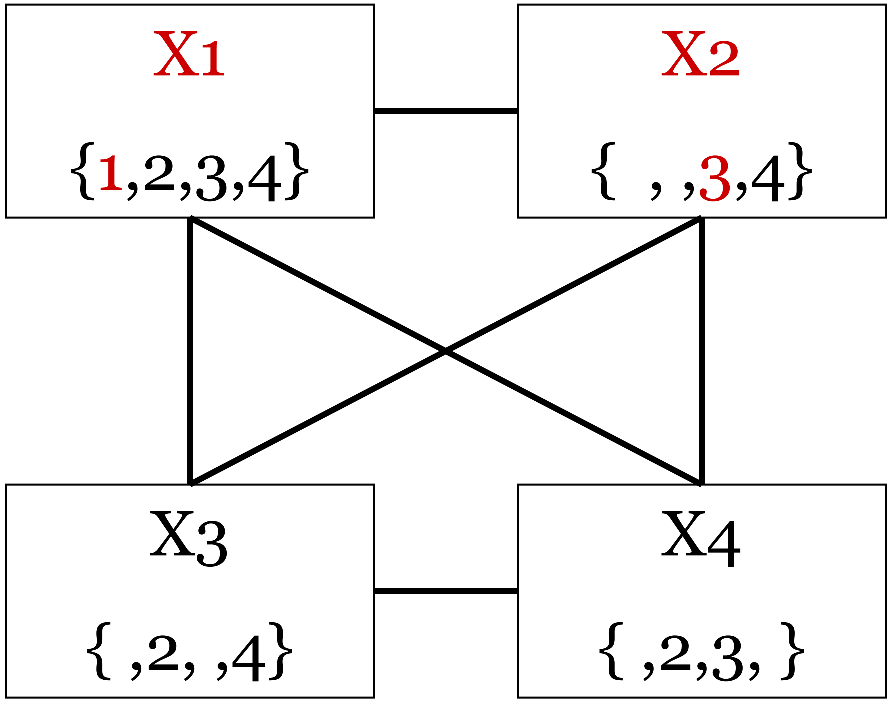
Example: 4 Queens





Example: 4 Queens





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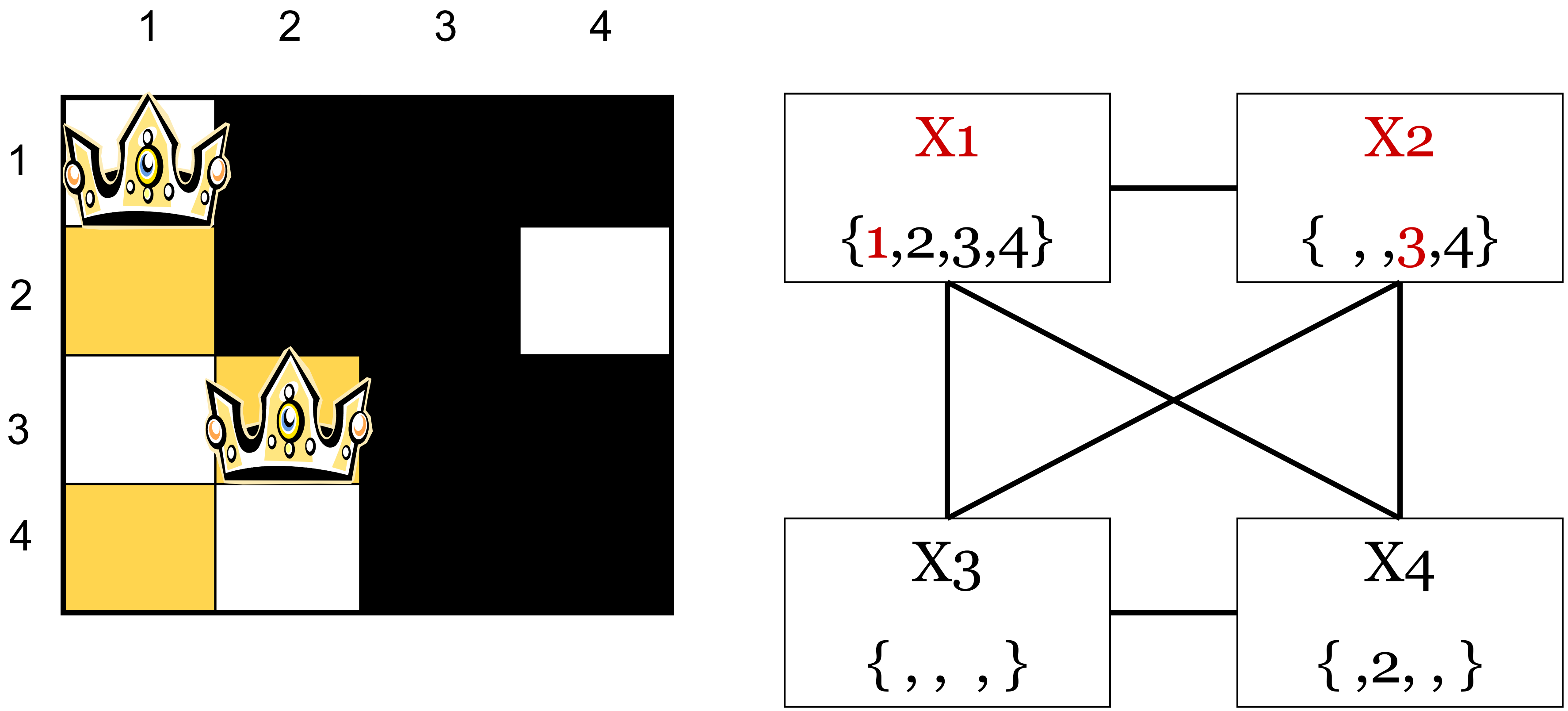
Example: 4 Queens



No possibilities for X3, backtrack trying different value for X2

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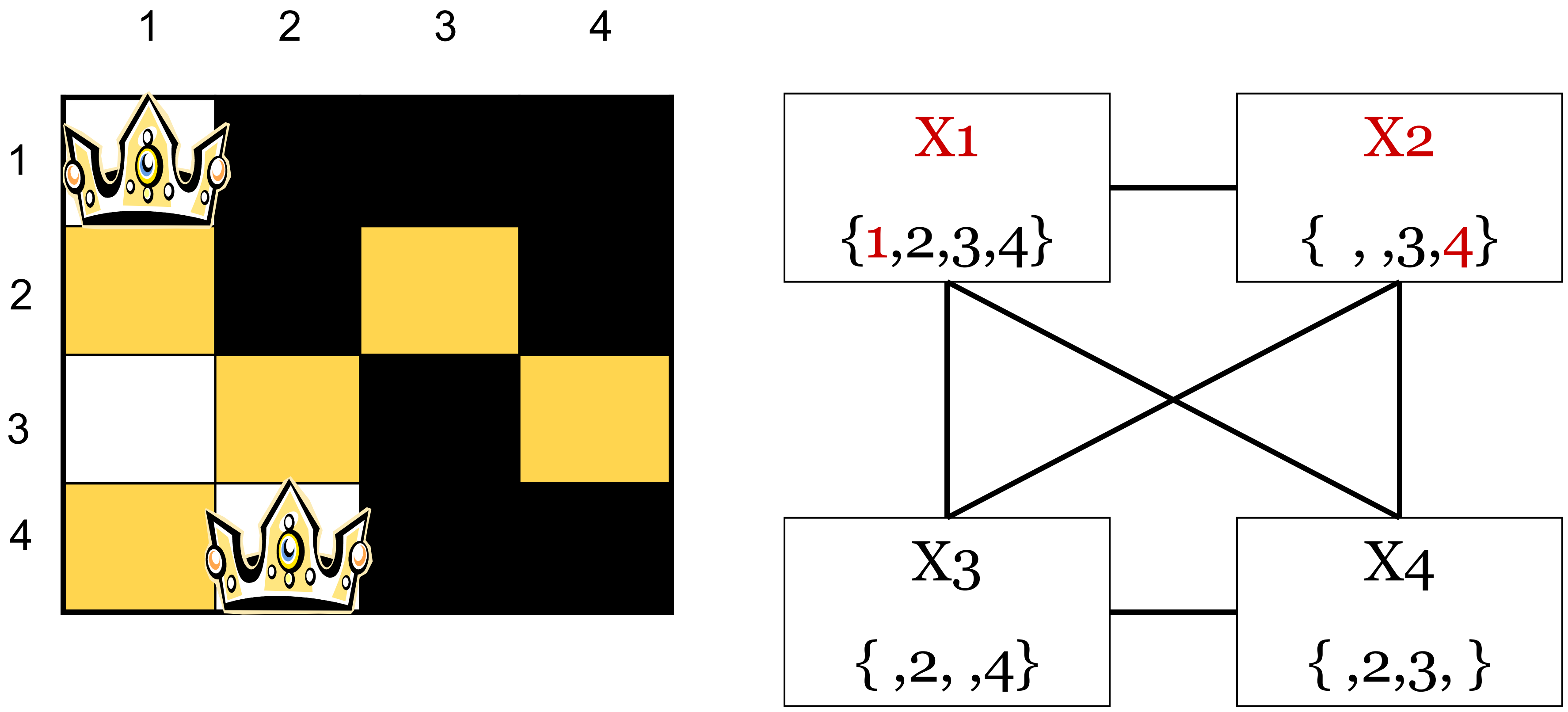
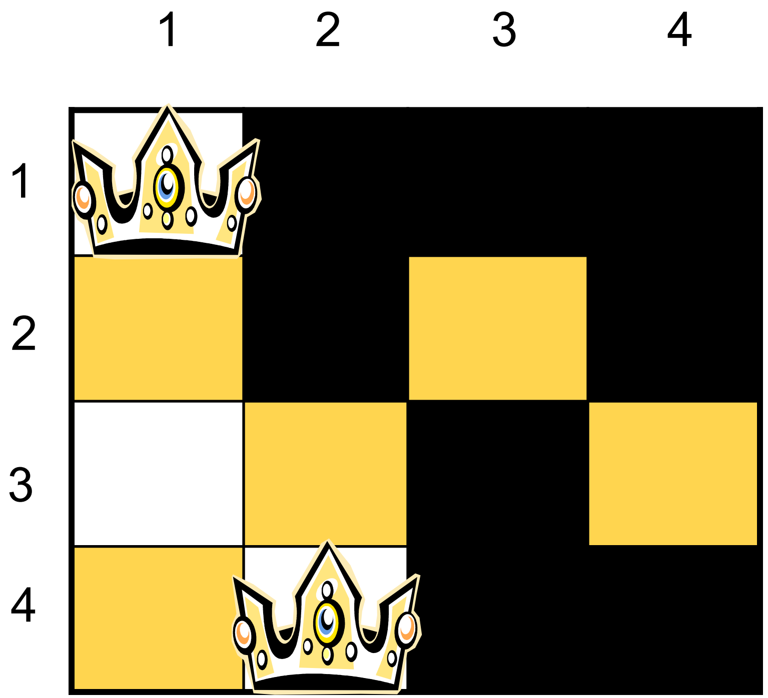
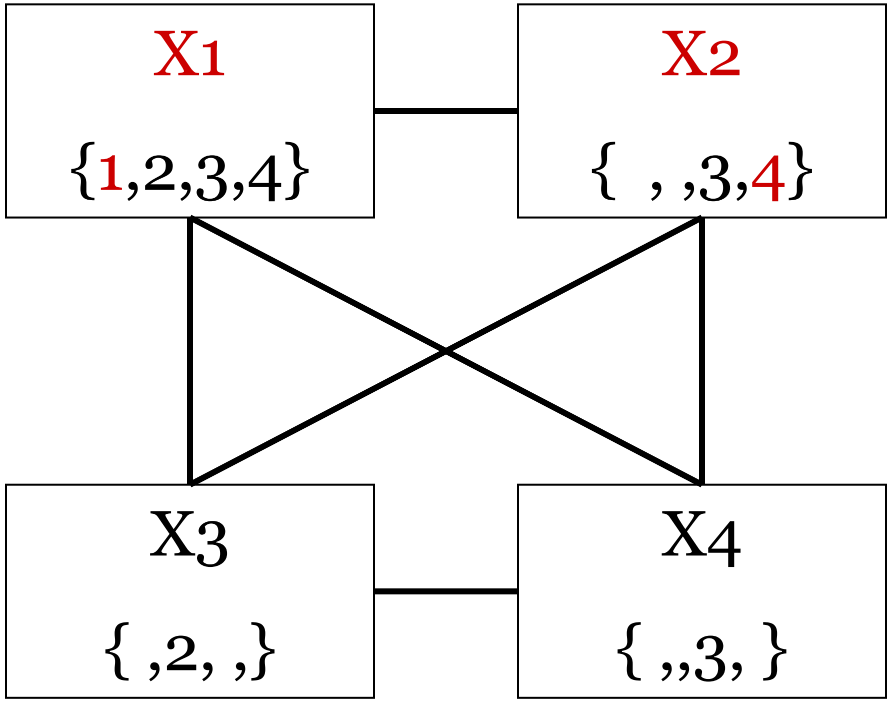
Example: 4 Queens



Example: 4 Queens





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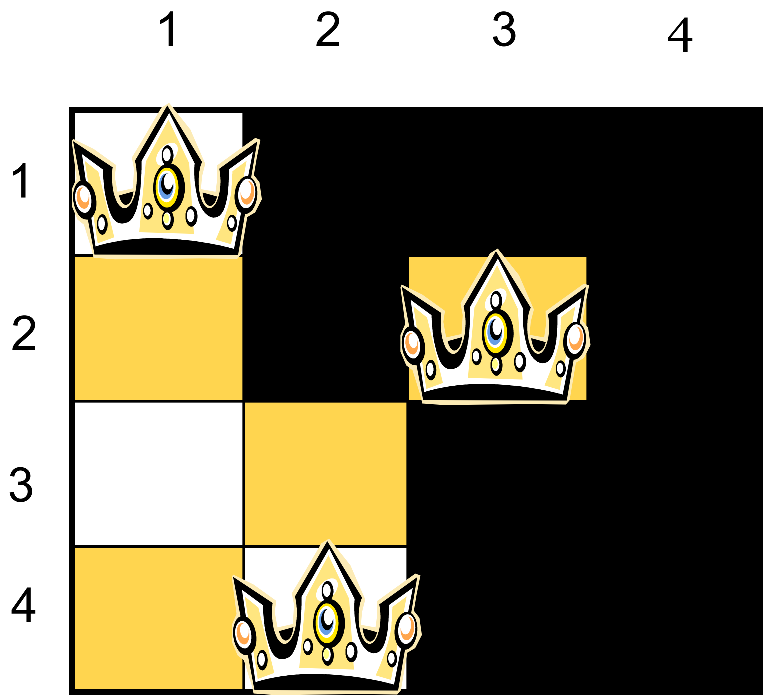
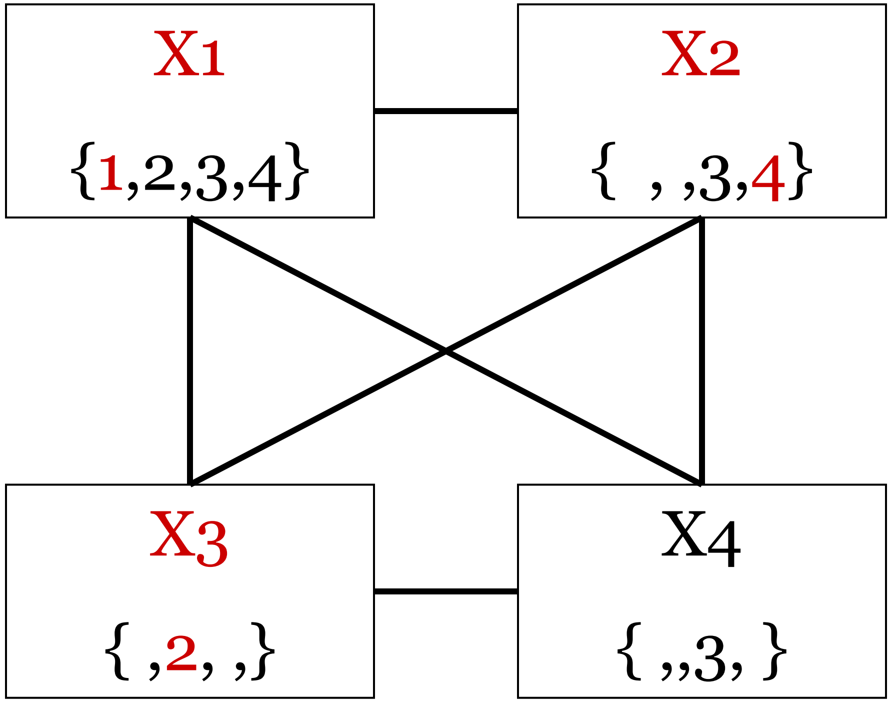
Example: 4 Queens





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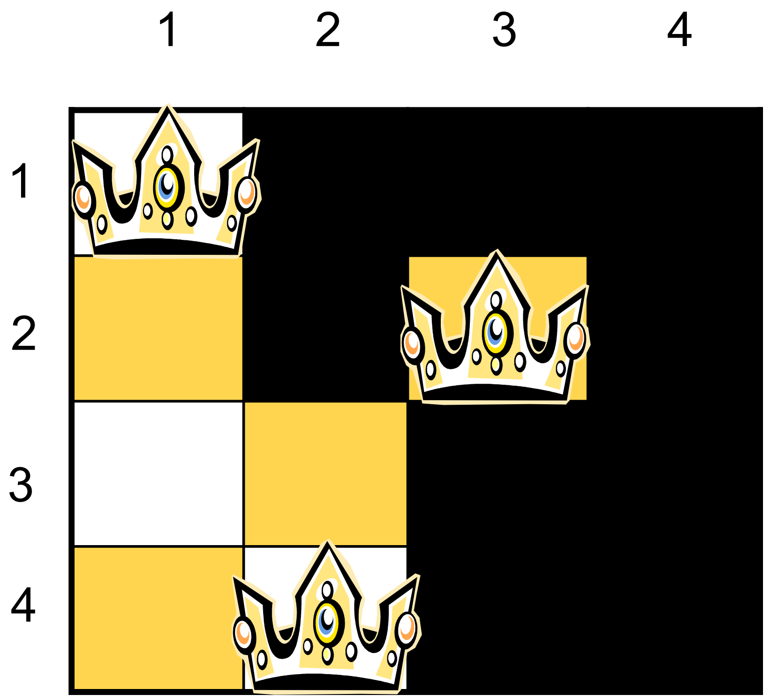
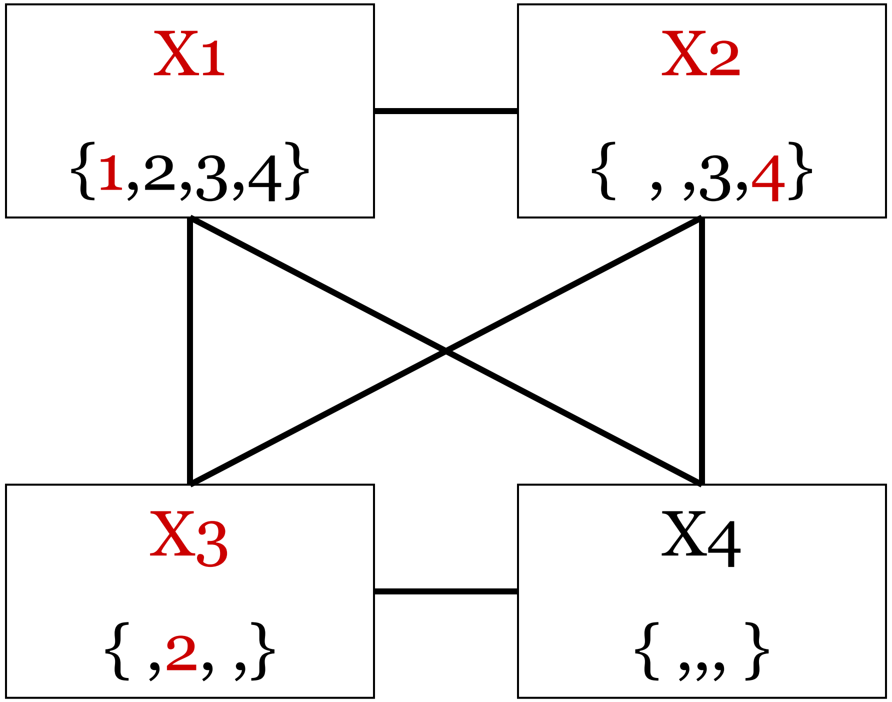
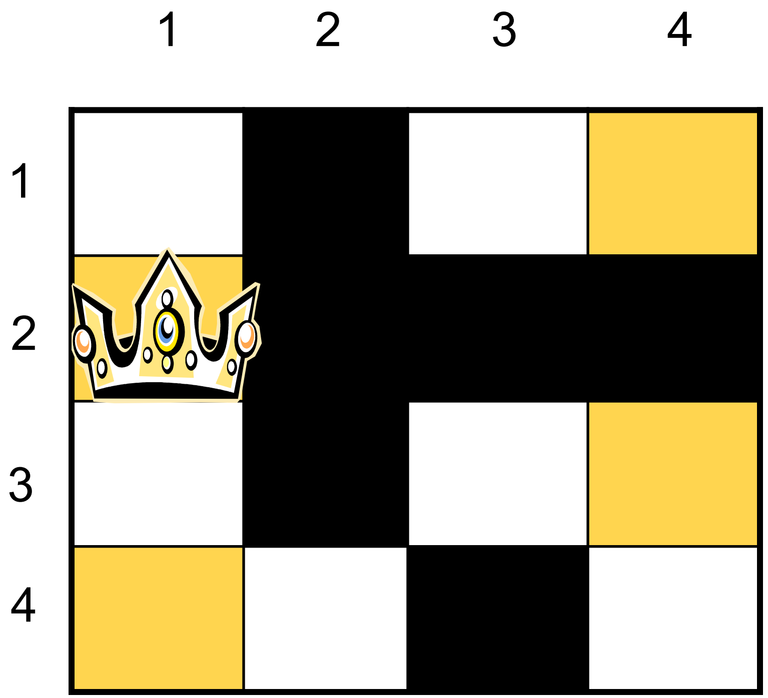
Example: 4 Queens

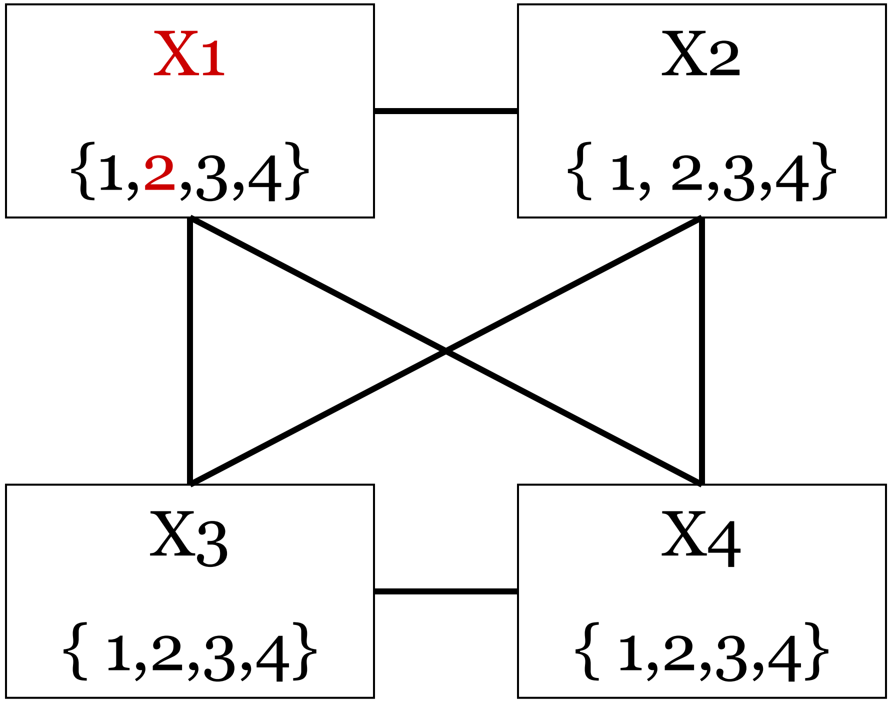




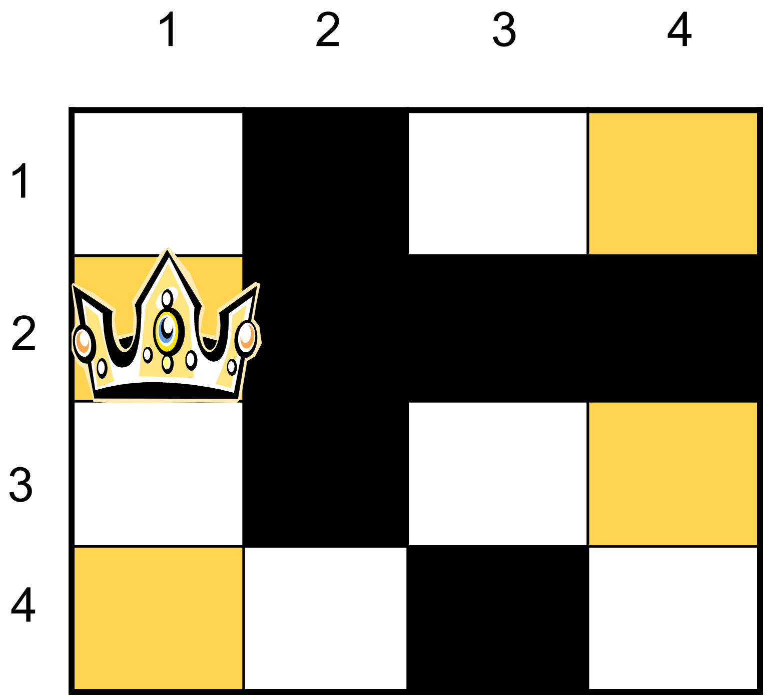
No possibilities for X4, backtrack trying different value for X1

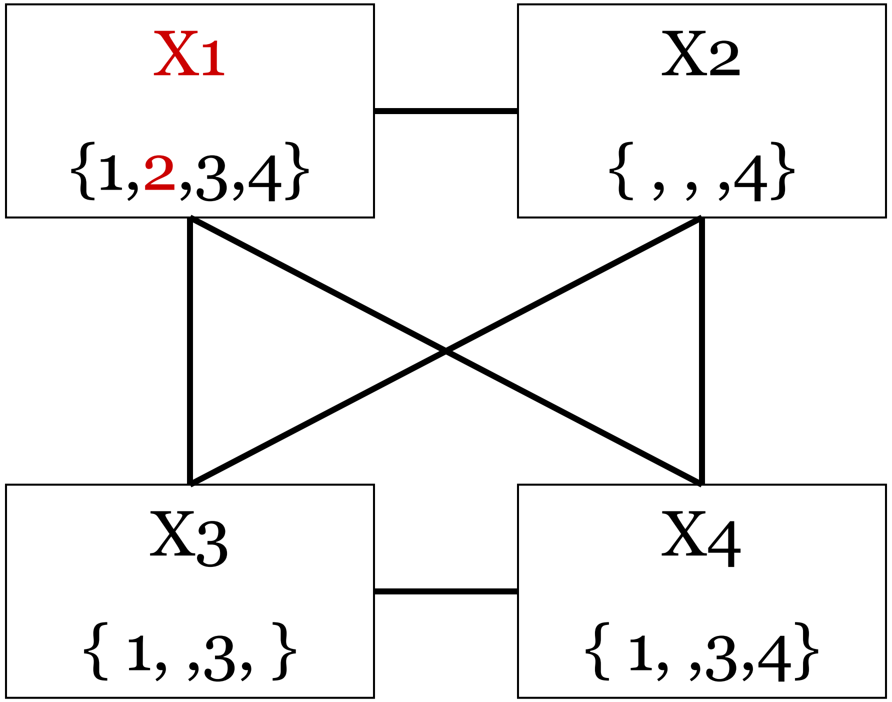
Example: 4 Queens





Example: 4 Queens





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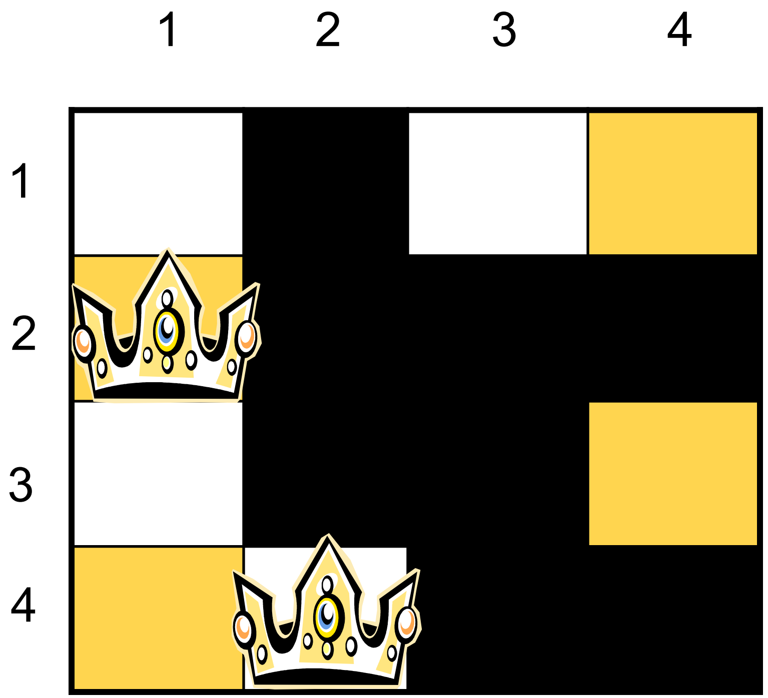
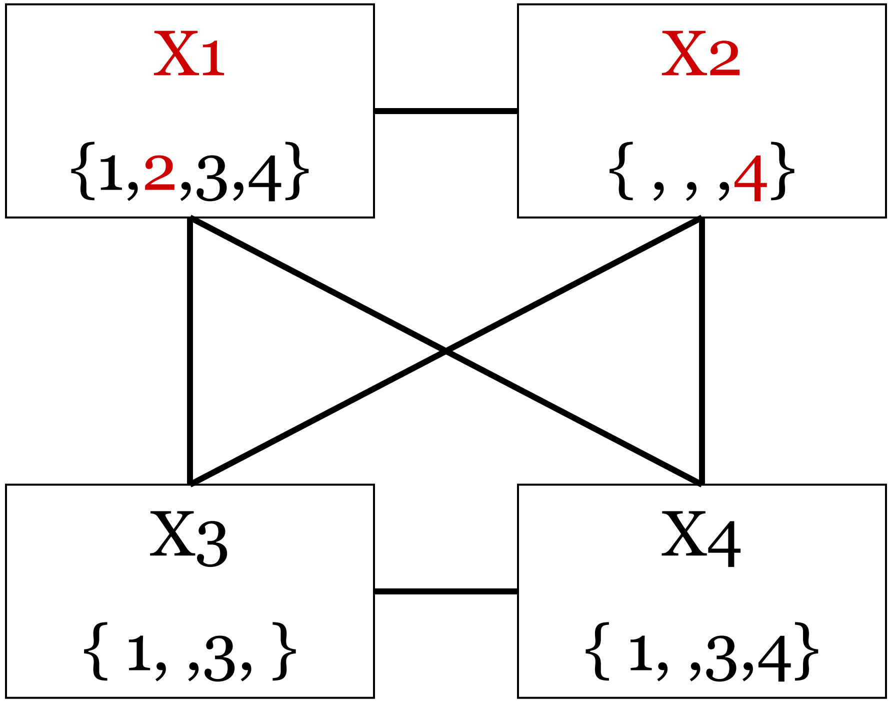
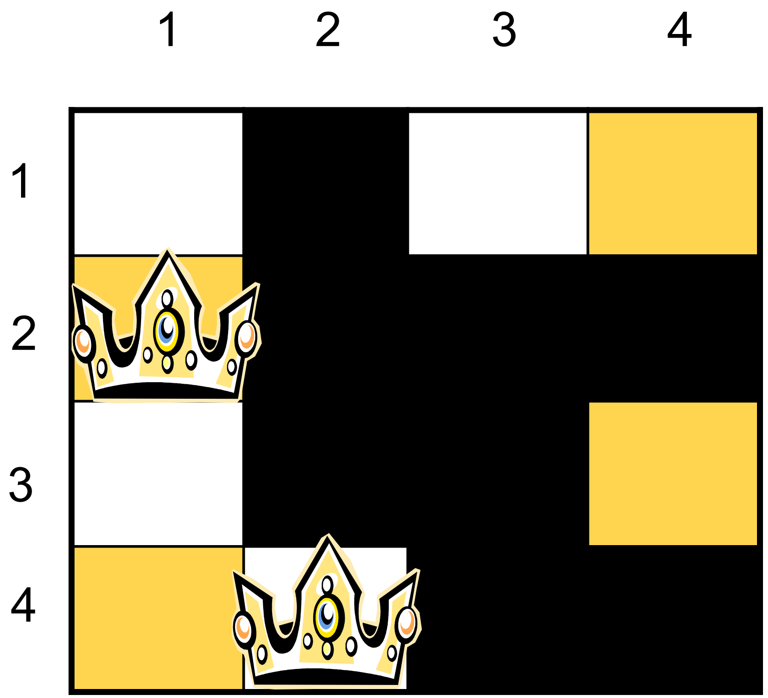
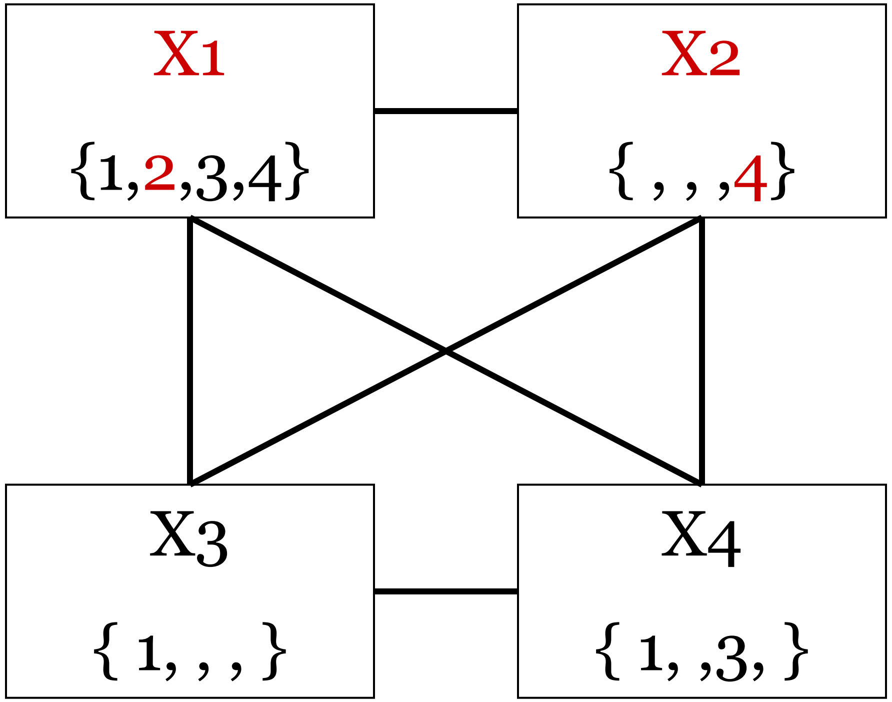
Example: 4 Queens





Example: 4 Queens





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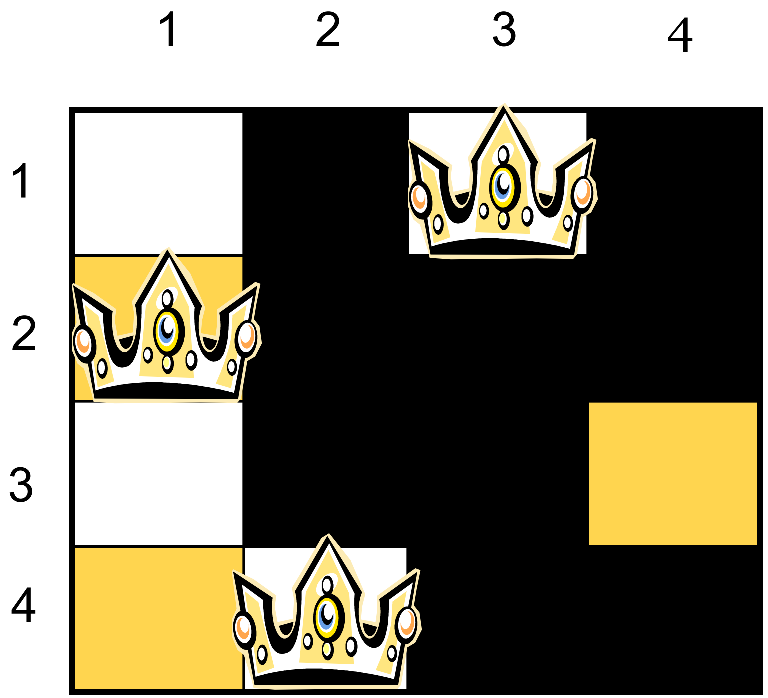
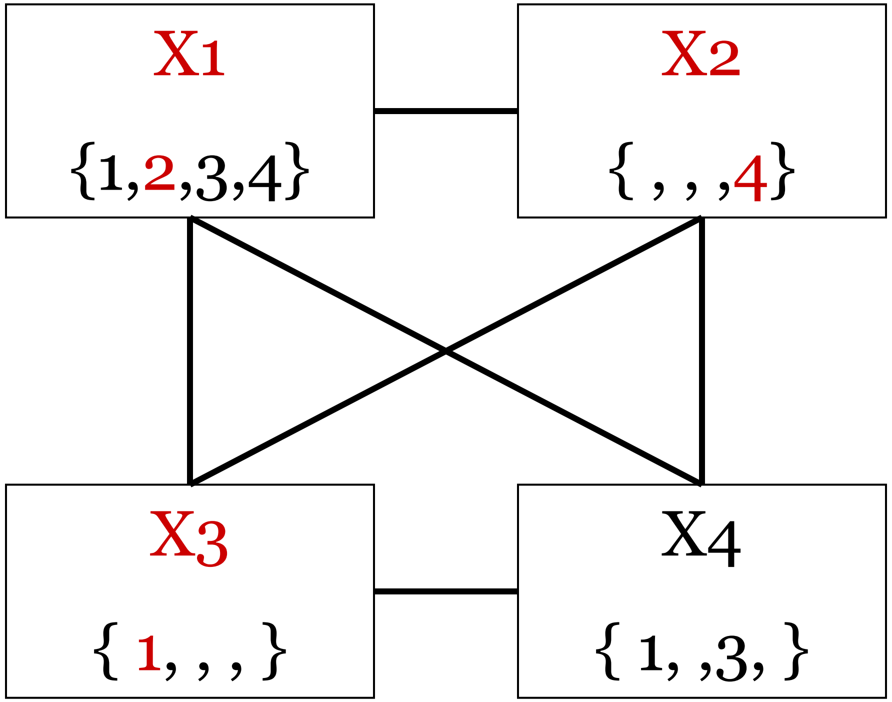
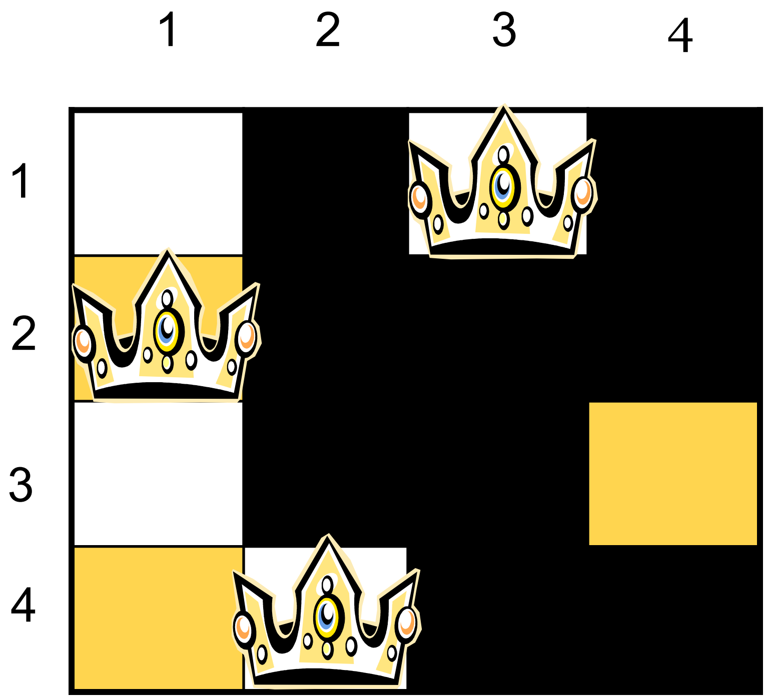
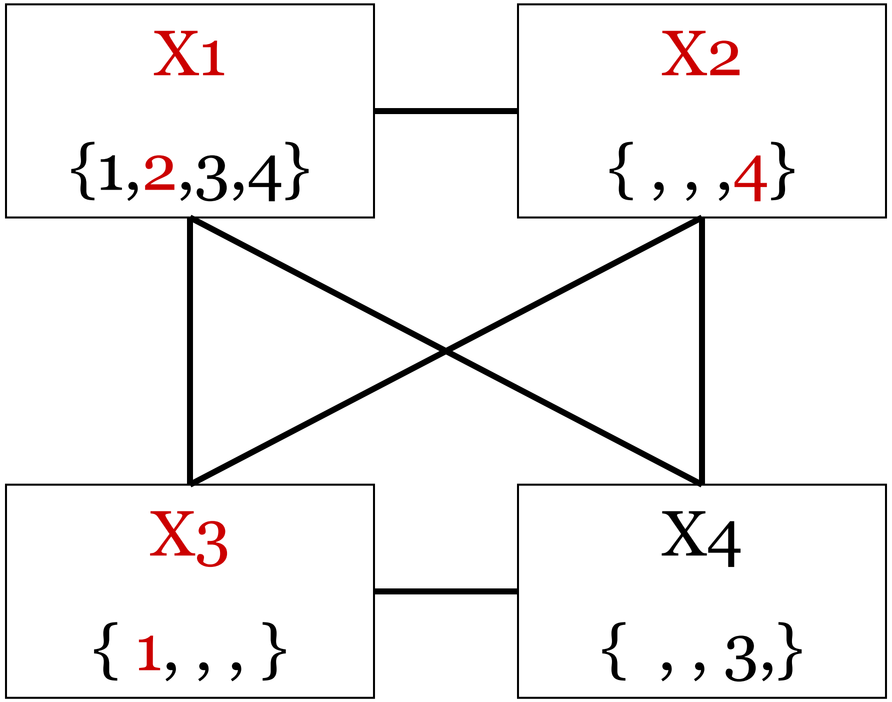
Example: 4 Queens



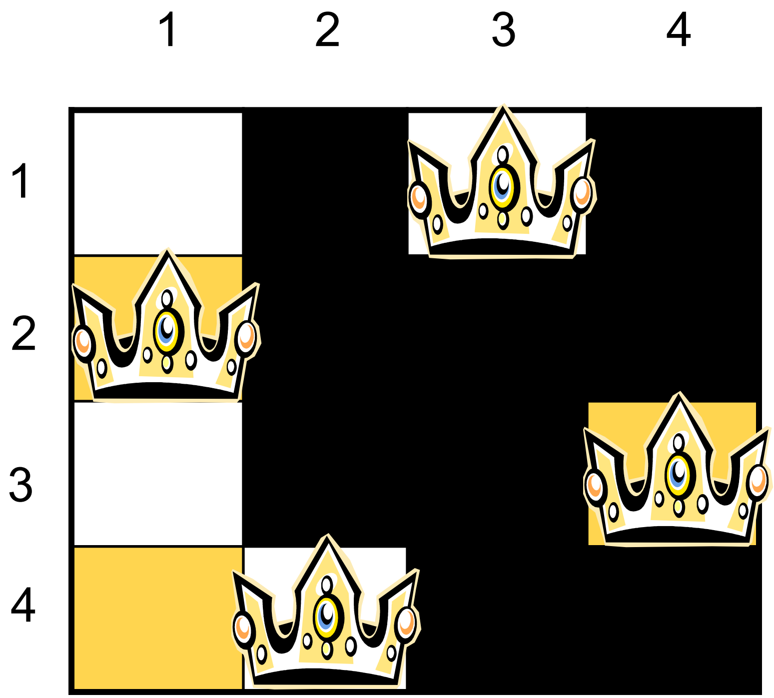


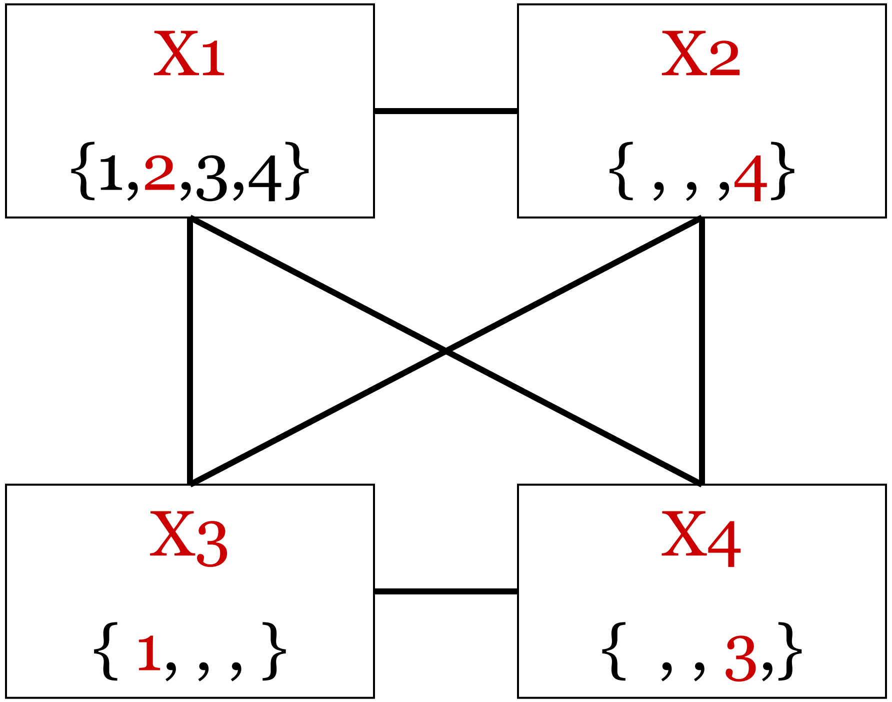
Example: 4 Queens





Example: 4 Queens





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Summary

- What you should know
  - How to formalize problems as CSPs
  - Backtracking search
  - Heuristics
    - Variable ordering
    - Value ordering
  - Forward checking

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