# Lecture22:Game Theory II CS486/686 Intro to Artificial Intelligence

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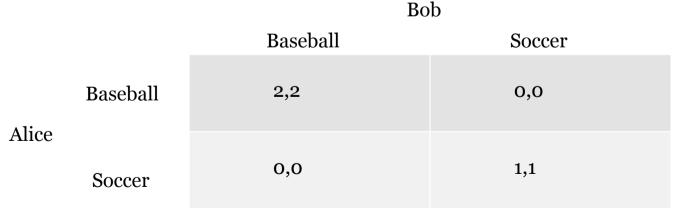
# Outline

- Game Theory
- Normal form game
  - Pareto Optimality
  - Mixed strategy Nash equilibrium



# **Consider the previous game**

- Both (B,B) and (S,S) and Nash equilibria
- However, (B,B) is better than (S,S)
- Nash equilibria does not capture this (which is better?)
- Formalize that (B,B) is better



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#### Pareto dominance

Pareto dominance:

An outcome o Pareto dominates another outcome o' if and only if every player is weakly better off in o and at least one player is strictly better off in o

 $u_i(o) \ge u_i(o'), \forall i$  $u_i(o) > u_i(o'), \exists i$ 



# **Pareto Optimality**

- An outcome *o* is Pareto optimal if and only if no other outcome *o'* Pareto dominates *o*
- Two statements:

wrong!

- Outcome *o* Pareto dominates all other outcomes
- An outcome *o* is NOT Pareto dominated by any other outcome

## Which are Pareto Optimal?



# Prisoner's dilemma

- Alice and Bob caught by the police
- If one testifies, the other gets caught
- If both testifies, both will be convicted of major charge
- If neither testifies, both will be convicted of a minor charge

			Cooperate	Defect
other	. 1.	Cooperate	-1,-1	-3,0
th will	Alice	Defect	0,-3	-2,-2



Bob

Defect

## Which is the dominant strategy equilibrium?

		Cooperate	Defect
Alice	Cooperate	-1,-1	-3,0
	Defect	0,-3	-2,-2

Bob

#### Which are pure-strategy Nash equilibria



## Which are Pareto optimal?



## **Another example - Matching quarters**

 Alice and Bob show one Bob side of a quarter Heads Tails • Alice wants the two Heads 1,0 0.1 sides to match Alice Tails 0,1 1,0 • Bob wants the sides to NOT match



#### How many pure strategy Nash equilibria

		Bob	
		Heads	Tails
Alice	Heads	1,0	0,1
	Tails	0,1	1,0

WATERLOO

# Mixed-strategy NE

Best Response in mixed strategies

 $\mathbb{E}[u_i(\sigma_i, \sigma_{-i})] \ge \mathbb{E}[u_i(\sigma_i', \sigma_{-i})], \forall \sigma_i' \neq \sigma_i$ 

Mixed-strategy NE (same definition): A (mixed) strategy profile *σ* is a Nash equilibrium (NE) if and only if each agent *i*'s (mixed) strategy *σ<sub>i</sub>* is a best response to the other agents' (mixed) strategies *σ<sub>-i</sub>*



#### **Nash Theorem**

#### Every finite game has at least one (mixed) strategy Nash Equilibrium



# Mixed strategy Eq. Of Matching Quarters

- Let Alice plays head with probability **p** and Bob plays head with probability **q** 

- Alice chooses a value for p such that Bob is indifferent between their actions

$$\begin{split} u_{Bob}(heads) &= p \times 0 + (1-p) = 1-p \\ u_{Bob}(tails) &= p \times 1 + (1-p) \times 0 = p \implies 1-p = p \implies p = 0.5 \end{split}$$

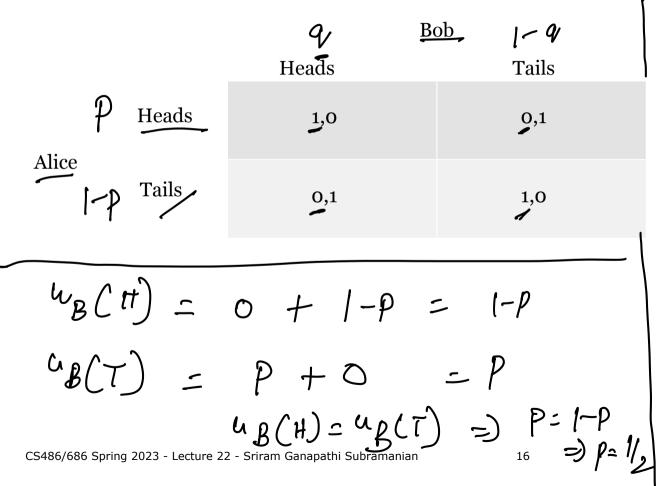
- Bob chooses a value for q such that Alice is indifferent between their actions

$$\begin{split} u_{Alice}(heads) &= q \times 1 + (1-q) \times 0 = q \\ u_{Alice}(tails) &= q \times 0 + (1-q) \times 1 = 1-q \implies 1-q = q \implies q = 0.5 \end{split}$$

- Strategy profile under mixed strategy NE:
  - Alice plays heads with prob 0.5 and tails with prob 0.5
  - Bob plays heads with prob 0.5 and tails with prob 0.5
  - Strategy profile: {(0.5,0.5), (0.5,0.5)}
- Expected utilities under mixed strategy NE:
  - Bob: 0.5
  - Alice: 0.5



#### Mixed Strategy NE for the example



 $W_{A}(H) = q + O((-2) = q)$  $w_{A}(T) = 0 + l - q = l - q$  $u_A(H) = u_A(T)$ 9 - 1-9, =) 9 = 1/2



# When Mixed strategy NE

Three possibilities:

- Player's expected utility of playing head is greater than that of tails
- Player's expected utility of playing head is less than that of tails
- Player's expected utility of playing heads is same as that of playing tails



# Mixed strategy NE

- Two things for each player
  - Each player chooses mixing probability
  - Make others indifferent between actions
- Once indifferent they can choose actions with any mixing probability
  - But the other should be indifferent as well



## **Another example - Dancing or concert?**

- Alice and Bob sign up for an activity together
- They both prefer to play the same activity
- Alice prefers dancing over concert
- Bob prefers concert over dancing

		Dancing	Concert
Alice	Dancing	2,1	0,0
	Concert	0,0	1,2

Bob



Dancing or	Concert, N	<b>/lixed Stra</b>	
$\begin{cases} B \in \frac{2}{3} \\ A = \frac{2}{3} \end{cases}$	<b>♀</b> Bo Dancing	ob <b>I9/</b> Concert	$u_{A}(D) = 2q + O(1-2)$ = 2q
P Dancing Alice	2,1	0,0	$u_{A}(c) = 0q + 1(1-q)$ = 1-q
l-p Concert	0,0	1,2	$u_A(D) = u_A(C)$
	0(1-P) =		=) 2g = 1-9
$(\mathcal{B}(\mathcal{C})) = \mathcal{O}\mathcal{P}^{-1}$ CS486/686 Spring 2023 - Lecture 22 - Sriran	2-2	2 - 2p 2 - 2 p 2 - P 20 = $P = \frac{2}{3}$	=) ? = 1/3 WATERLOO

## Calculation

- Suppose that Alice goes dancing with probability **p** and Bob goes dancing with probability **q**
- Bob is indifferent between the two actions
  - $u_{Bob}(dancing) = p \times 1 + (1-p) \times 0 = p$
  - $u_{Bob}(Concert) = p \times 0 + (1-p) \times 2 = 2 2p \implies p = 2 2p \implies p = 2/3$

- Alice is indifferent between the two actions
  - $u_{Alice}(dancing) = q \times 2 + (1 q) \times 0 = 2q$
  - $u_{Alice}(concert) = q \times 0 + (1 q) \times 1 = 1 q \implies q = 1/3$



# Summary

- Normal Form Games
  - Pareto Optimality
  - Mixed strategy Nash Equilibrium

