Lecture22:Game Theory II
CS486/686 Intro to Artificial Intelligence

2023-7-25

Acknowledgements: Kate Larson, Alice Gao

Sriram Ganapathi Subramanian,
Vector Institute

UNIVERSITY OF WATERLOO

Outline

- Game Theory
- Normal form game
  - Pareto Optimality
  - Mixed strategy Nash equilibrium

Consider the previous game

- Both (B,B) and (S,S) and Nash equilibria
- However, (B,B) is better than (S,S)
- Nash equilibria does not capture this (which is better?)
- Formalize that (B,B) is better

		Bob: Baseball	Bob: Soccer
Alice	Baseball	2,2	0,0
	Soccer	0,0	1,1

UNIVERSITY OF WATERLOO

Pareto dominance

Pareto dominance:

An outcome o Pareto dominates another outcome o' if and only if every player is weakly better off in o and at least one player is strictly better off in o

$$u_i(o) \geq u_i(o'), \forall i$$

$$u_i(o) > u_i(o'), \exists i$$

Pareto Optimality

- An outcome o is Pareto optimal if and only if no other outcome o' Pareto dominates o

- Two statements:
  - ~~Outcome o Pareto dominates all other outcomes~~ wrong !
  - An outcome o is NOT Pareto dominated by any other outcome

UNIVERSITY OF WATERLOO

Which are Pareto Optimal?

Bob

Alice		One	Two
	One	2,2	0,0
	Two	0,0	1,1

UNIVERSITY OF WATERLOO

Prisoner’s dilemma

- Alice and Bob caught by the police
- If one testifies, the other gets caught
- If both testifies, both will be convicted of major charge
- If neither testifies, both will be convicted of a minor charge

		Bob	
		Cooperate	Defect
Alice	Cooperate	-1,-1	-3,0
	Defect	0,-3	-2,-2

Which is the dominant strategy equilibrium?

Bob

Alice		Cooperate	Defect
	Cooperate	-1,-1	-3,0
	Defect	0,-3	-2,-2

UNIVERSITY OF WATERLOO

Which are pure-strategy Nash equilibria

		Bob	
		Cooperate	Defect
Alice	Cooperate	-1,-1	-3,0
	Defect	0,-3	-2,-2

UNIVERSITY OF WATERLOO

Which are Pareto optimal?

		Bob: Cooperate	Bob: Defect
Alice	Cooperate	-1,-1	-3,0
	Defect	0,-3	-2,-2

Another example - Matching quarters

- Alice and Bob show one side of a quarter
- Alice wants the two sides to match
- Bob wants the sides to NOT match

Alice \ Bob	Heads	Tails
Heads	1,0	0,1
Tails	0,1	1,0

How many pure strategy Nash equilibria

Bob

		Heads	Tails
Alice	Heads	1,0	0,1
	Tails	0,1	1,0

Mixed-strategy NE

- Best Response in mixed strategies

$$\mathbb{E}[u_i(\sigma_i, \sigma_{-i})] \geq \mathbb{E}[u_i(\sigma_i', \sigma_{-i})], \forall \sigma_i' \neq \sigma_i$$

- Mixed-strategy NE (same definition): A (mixed) strategy profile σ is a Nash equilibrium (NE) if and only if each agent i's (mixed) strategy σ_i is a best response to the other agents' (mixed) strategies σ_{-i}

Nash Theorem

Every finite game has at least one (mixed) strategy Nash Equilibrium

Mixed strategy Eq. Of Matching Quarters

- Let Alice plays head with probability p and Bob plays head with probability q

- Alice chooses a value for p such that Bob is indifferent between their actions

$$u_{Bob}(heads) = p \times 0 + (1-p) = 1-p$$
$$u_{Bob}(tails) = p \times 1 + (1-p) \times 0 = p \implies 1-p = p \implies p = 0.5$$

- Bob chooses a value for q such that Alice is indifferent between their actions

$$u_{Alice}(heads) = q \times 1 + (1-q) \times 0 = q$$
$$u_{Alice}(tails) = q \times 0 + (1-q) \times 1 = 1-q \implies 1-q = q \implies q = 0.5$$

- Strategy profile under mixed strategy NE:
    - Alice plays heads with prob 0.5 and tails with prob 0.5
    - Bob plays heads with prob 0.5 and tails with prob 0.5
    - Strategy profile: {(0.5,0.5), (0.5,0.5)}
- Expected utilities under mixed strategy NE:
    - Bob: 0.5
    - Alice: 0.5

Mixed Strategy NE for the example

		Bob: Heads (q)	Bob: Tails ($1-q$)
Alice	Heads (p)	1,0	0,1
	Tails ($1-p$)	0,1	1,0

$u_A(H) = q + 0(1-q) = q$

$u_A(T) = 0 + 1-q = 1-q$

$u_A(H) = u_A(T)$

$q = 1-q$

$\Rightarrow q = 1/2$

$u_B(H) = 0 + 1-p = 1-p$

$u_B(T) = p + 0 = p$

$u_B(H) = u_B(T) \Rightarrow p = 1-p$

$\Rightarrow p = 1/2$

When Mixed strategy NE

Three possibilities:

- Player's expected utility of playing head is greater than that of tails
- Player's expected utility of playing head is less than that of tails
- Player's expected utility of playing heads is same as that of playing tails

Mixed strategy NE

- Two things for each player
  - Each player chooses mixing probability
  - Make others indifferent between actions
- Once indifferent they can choose actions with any mixing probability
  - But the other should be indifferent as well

Another example - Dancing or concert?

- Alice and Bob sign up for an activity together
- They both prefer to play the same activity
- Alice prefers dancing over concert
- Bob prefers concert over dancing

		Bob: Dancing	Bob: Concert
Alice	Dancing	2,1	0,0
	Concert	0,0	1,2

Dancing or Concert, Mixed Strategy NE

$B = 2/3$
$A = 2/3$

		Bob: Dancing (q)	Bob: Concert ($1-q$)
Alice	Dancing (p)	2,1	0,0
	Concert ($1-p$)	0,0	1,2

$u_A(D) = 2q + 0(1-q) = 2q$

$u_A(C) = 0q + 1(1-q) = 1-q$

$u_A(D) = u_A(C)$

$\Rightarrow 2q = 1-q$

$\Rightarrow q = 1/3$

$u_B(D) = p + 0(1-p) = p$

$u_B(C) = 0p + 2(1-p) = 2-2p$

$2-2p = p$

$\Rightarrow p = 2/3$

Calculation

- Suppose that Alice goes dancing with probability p and Bob goes dancing with probability q

- Bob is indifferent between the two actions
  - $u_{Bob}(dancing) = p \times 1 + (1 - p) \times 0 = p$
  - $u_{Bob}(Concert) = p \times 0 + (1 - p) \times 2 = 2 - 2p \implies p = 2 - 2p \implies p = 2/3$

- Alice is indifferent between the two actions
  - $u_{Alice}(dancing) = q \times 2 + (1 - q) \times 0 = 2q$
  - $u_{Alice}(concert) = q \times 0 + (1 - q) \times 1 = 1 - q \implies q = 1/3$

Summary

- Normal Form Games
  - Pareto Optimality
  - Mixed strategy Nash Equilibrium