Lecture21:Game Theory
CS486/686 Intro to Artificial Intelligence

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Outline

- Game Theory
- Normal form game
  - Dominant strategy equilibria
  - Pure strategy Nash equilibria

Multi-agent Decision Making

- Sequential Decision Making
  - Markov Decision Processes
  - Reinforcement Learning
  - Multi-Armed Bandits

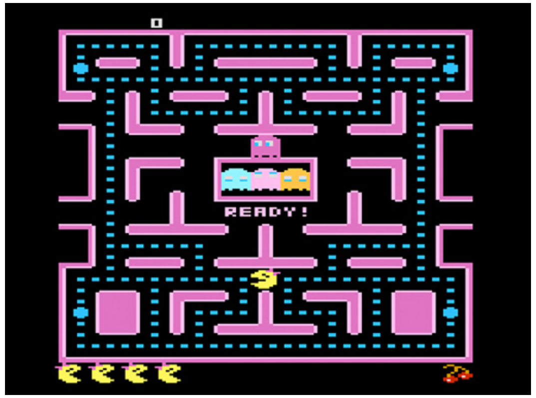
- All in single agent environments

- Real world environments: Mostly more than one agent?
  - Each agent needs to account for other agents' actions/behaviours

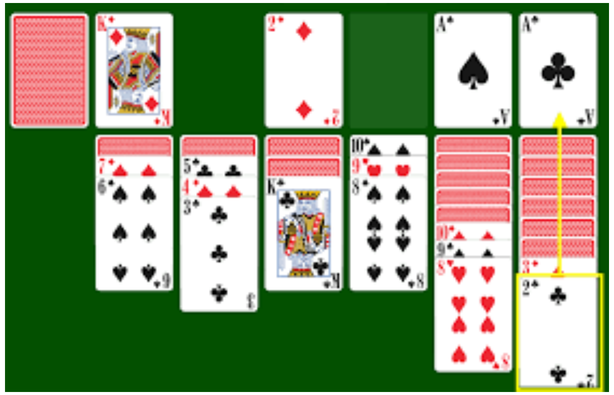
Game

- Game: Any set of circumstances, where outcomes depend on actions of two or more rational and self-interested players
  - Players (Decision Makers)
    - Agent within the game (observe state and take actions)
  - Rational
    - Agents choose their best actions (unless exploring)
  - Self-interested
    - Only care about their own benefits
    - May/May not harm others
    - Own description of states (actions based on this description)

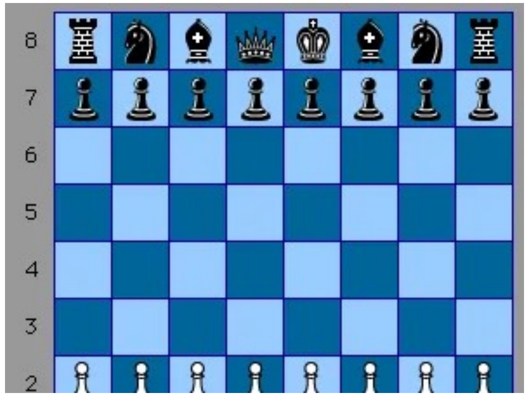
Which of these are games?



Atari



Solitaire



Chess

Go

Game Theory

- Game Theory: Mathematical model of strategic interactions among more than one rational agents in a game
  - Interaction:
    - One agent directly affects other agent(s)
    - Utility for one agent depends on other agent(s)
  - Strategic:
    - Agents maximize their utility by taking into account their influence (through actions) on the game
  - Multiple:
    - At-least two agents
    - If only one player, then it is a decision problem (not game!)

Game Theory Applications

- Auctions
- Diplomacy
- Sports analytics
- Autonomous Driving
- Sustainability (Disaster response, Wildfire fighting)

Learning in multi-agent frameworks

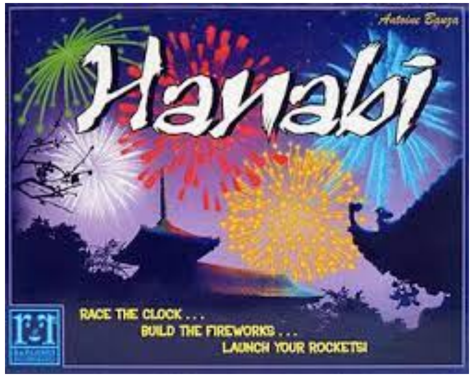
- Each agent decides to act based on
  - Information about the world
  - Information about other agents
  - Utility function

- Outcome of each agent depends on action of all agents

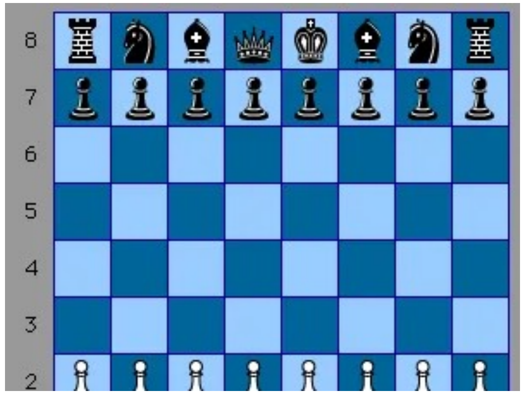
Based on utility function

- Games can be
  - Cooperative where agents have a common goal
  - Competitive where agents have conflicting goals
  - Mixed — somewhere in between

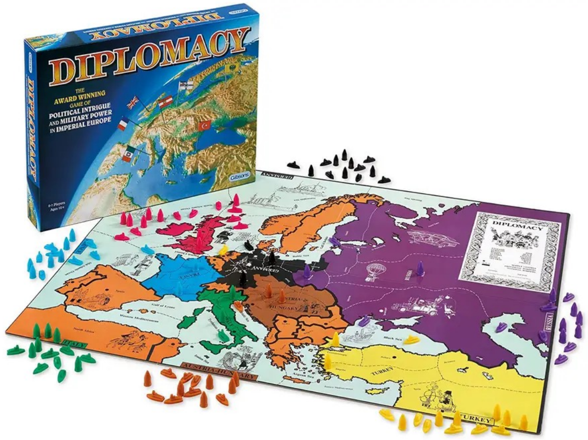
Cooperative



Competitive



Mixed



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Normal Form Games

- Set of agents

  $I = 1,2,\ldots,N$, where $N \geq 2$
- Set of actions for each agent

  $A_i = \{a_i^1, \ldots, a_i^m\}$
- Outcome of game is defined by a profile (joint action)

  $\boldsymbol{a} = (a_1, \ldots, a_n)$
- Total space of joint actions
  - $\boldsymbol{a} \in \{A_1 \times A_2 \times \cdots \times A_N\}$
- Agents have preferences over outcomes
  - Utility functions $u_i : \boldsymbol{A} \rightarrow \mathfrak{R}$, where $\boldsymbol{A} = \{A_1 \times A_2 \times \cdots \times A_N\}$

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Payoff Matrices

- Normal form games are represented by payoff matrices
- Elements
  - $I = \{1,2\}$
  - $A_i = \{One, Two\}$
  - Outcomes: $(One, One); (Two, Two); (One, Two); (Two, One)$
  - $u_1(One, Two) = -3$
  - $u_2(One, Two) = 3$

Agent 2

Agent 1		One	Two
	One	2,-2	-3,3
	Two	-3,3	4,-4

Zero - sum game:

$$\sum_{i=1}^{n} u_i(a_1, \ldots, a_n) = 0$$

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Another example

- Alice and Bob like playing together
- Decide to play or stay at home
- Cannot communicate with each other
- Both prefer playing to staying at home

		Bob	
		Home	Play
Alice	Home	0,0	0,1
	Play	1,0	2,2

Playing a Normal-Form Game

- Players choose their actions at the same time
  - No communication with other agents
  - No observation of other player's actions

- Each player chooses a strategy σ_i which can be either
  - Mixed strategy: Distribution over actions
    - Eg: Stay at home with prob 80 % and go to play with prob 20 %
  - Pure strategy: One action with prob 100 %

- Solution for normal form game is a strategy profile, a set of strategies with one strategy for each player
  - Strategy profile: $\{\sigma_{\text{Alice}}, \sigma_{\text{Bob}}\}$

Solution for example?

		Bob	
		Home	Play
Alice	Home	0,0	0,1
	Play	1,0	2,2

Terminologies

- Terminologies for strategies:
  - σ_i: Strategy of player i
  - σ_{-i}: Strategy of all players except i

- Terminologies for utilities:
  - $u_i(\sigma) = u_i(\sigma_i, \sigma_{-i})$ denotes the utility of agent i under strategy profile σ

Dominant Strategy Equilibrium

- For player i, a strategy σ_i dominates strategy σ_i' if
  - $u_i(\sigma_i, \sigma_{-i}) \geq u_i(\sigma_i', \sigma_{-i}), \forall \sigma_{-i}$, and
  - $u_i(\sigma_i, \sigma_{-i}) > u_i(\sigma_i', \sigma_{-i}), \exists \sigma_{-i}$
- A dominant strategy dominates all other strategies
- A rational agent will never play a dominated strategy
- Each player may or may not have dominant strategy
- When each player has a dominant strategy, the set of those strategies (strategy profile) is a dominant strategy equilibrium (DSE)
  - DSE: $\{\sigma_1, \ldots, \sigma_N\}$ if σ_i, for all i, is a dominant strategy
- A game that has at least one DSE is dominance solvable

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Does this have a DSE?

		Bob: Home	Bob: Play
Alice	Home	0,0	0,1
	Play	1,0	2,2

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Does this have a DSE?

- Let's solve for Alice:
  - If Bob stays at home:

    $$u_{Alice}(Play, Home) > u_{Alice}(Home, Home)$$

  - If Bob goes to play:

    $$u_{Alice}(Play, Play) > u_{Alice}(Home, Play)$$

- In both cases Alice prefers to Play, therefore Play is a dominant strategy for Alice
- As the game is symmetric, Play is a dominant strategy for Bob too
- Thus, {Play, Play} is a DSE

Example

		Bob: Home	Bob: Play
Alice	Home	0,0	0,1
	Play	1,0	2,2

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Adding another action

		Bob: Home	Bob: Play	Bob: Dance
Alice	Home	0,0	0,1	5,-10
	Play	1,0	2,2	2,-10
	Dance	-10,5	-10, 2	-10,-10

- Is there a dominant strategy equilibrium for this game?

Iterative elimination of dominated strategies

- If Alice knows that Bob is rational, then Alice will eliminate the Bob's 'Dance' strategy
- Likewise for Bob
- After two rounds of elimination of strictly dominated strategies we are back to the previous game
- The previous game had one DSE {Play, Play}
- That is the DSE for this game as well

Another example

		Bob	
		Baseball	Soccer
Alice	Baseball	2,2	0,0
	Soccer	0,0	1,1

- Is there a dominant strategy equilibrium for this game?
- What are the players likely to do?

Best Response

- Given a strategy profile $\{\sigma_i, \sigma_{-i}\}$, agent i's strategy σ_i is a best response to the other agents' strategies σ_{-i} if and only if

$$u_i(\sigma_i, \sigma_{-i}) \geq u_i(\sigma_i', \sigma_{-i}), \forall \sigma_i' \neq \sigma_i$$

- A rational agent will always play a best response

Nash Equilibrium

- A strategy profile σ is a Nash equilibrium (NE) if and only if each agent i's strategy σ_i is a best response to the other agents' strategies σ_{-i}
  - Mixed strategy NE: At-least one σ_i is a distribution over actions
  - Pure strategy: Every σ_i chooses one action with 100% probability
- (Alternative Definition): No agent has any incentive to deviate from their current strategy σ_i if the strategy profile σ is a Nash equilibrium

Nash Equilibrium for the example

Is there a pure strategy Nash equilibrium for this example?

		Bob: Baseball	Bob: Soccer
Alice	Baseball	2,2	0,0
	Soccer	0,0	1,1

Solving for Nash equilibrium - I

- Follow the chain of best responses until we reach a stable point
- If some player is not playing a best response then we can switch the strategy to another strategy that is best response
- Keep repeating it until all players are playing the best response
- Pick an arbitrary strategy profile {Baseball, Soccer}
  - Now Alice changes strategy to soccer
  - Bob has no incentive to change strategy, so {Soccer, Soccer} is NE
- Similarly we can show that {Baseball, Baseball} is a NE

Solving for Nash equilibrium - II

- Another Idea: Fix strategy for one player and find the best response for other
- If Bob goes for Soccer, the best response for Alice is to go for Soccer
- If Bob goes for Baseball, the best response for Alice is to go for Baseball
- So we have two pure strategy NE: {Soccer, Soccer} and {Baseball, Baseball}

Summary

- Introduction to Multi-agent decision making (Game Theory)

- Normal form game (2 players)
  - Dominant strategy equilibria
  - Pure strategy Nash equilibria