Lecture 17: Reinforcement Learning
CS486/686 Intro to Artificial Intelligence

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Outline

- Reinforcement Learning
  - Model-based RL, model-free RL
  - Value-based RL, policy-based RL, actor-critic

- Algorithms:
  - Monte-Carlo evaluation
  - Temporal Difference (TD) evaluation
  - Control: Q-learning

Recap: Markov Decision Process

- Formal Definition
  - States: $s \in S$
  - Actions: $a \in A$
  - Rewards: $r \in \Re$
  - Transition model: $Pr(s_t | s_{t-1}, a_{t-1})$
  - Reward model: $R(s_t, a_t) = Pr(r_t | s_t, a_t)$
  - Discount factor: $0 \leq \gamma \leq 1$
    - discounted: $\gamma < 1$ undiscounted: $\gamma = 1$
  - Horizon (i.e., # of time steps): h
    - Finite horizon: $h \in \mathbb{N}$ infinite horizon: $h = \infty$

- Goal: find optimal policy such that $\pi^* = \arg\max_{\pi} \sum_{t=0}^{h} \gamma^t \mathbb{E}_\pi[r_t]$

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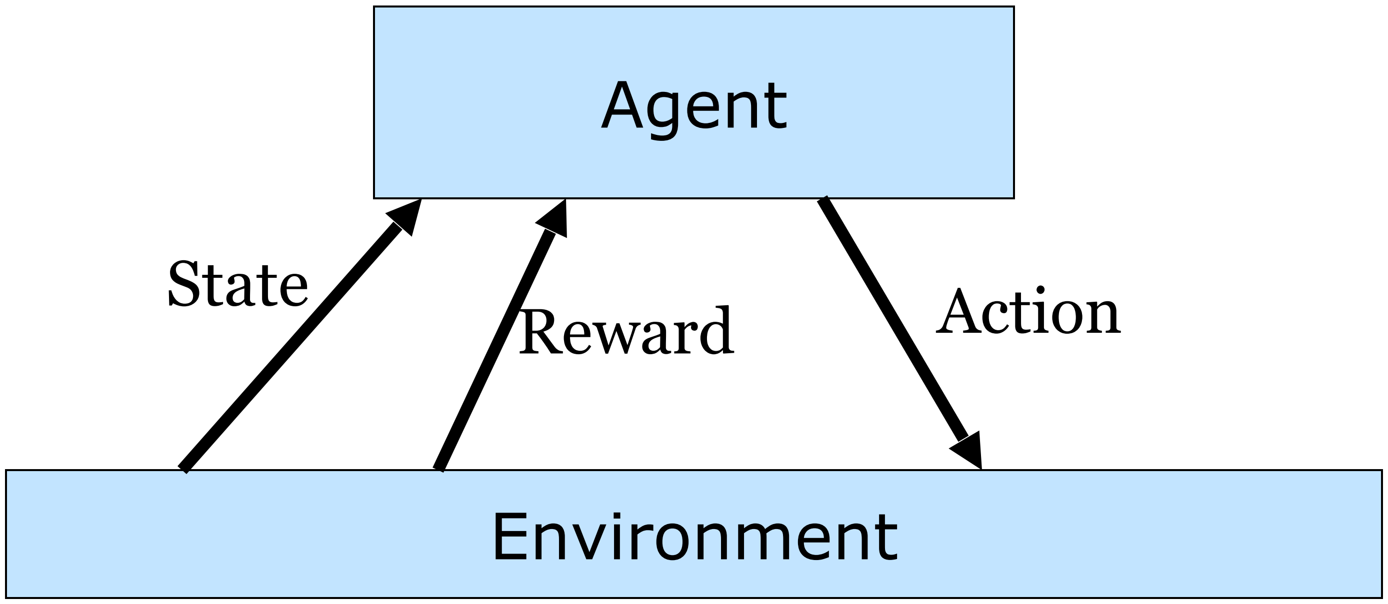
Reinforcement Learning Problem



Goal: Learn to choose actions that maximize rewards

Reinforcement Learning

- Formal Definition
  - States: $s \in S$
  - Actions: $a \in A$
  - Rewards: $r \in \Re$
  - ~~Transition model: $Pr(s_t | s_{t-1}, a_{t-1})$~~
  - ~~Reward model: $R(s_t, a_t)$~~

  Unknown Models

  - Discount factor: $0 \leq \gamma \leq 1$
    - discounted: $\gamma < 1$ undiscounted: $\gamma = 1$
  - Horizon (i.e., # of time steps): h
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Policy Optimization

- Markov Decision Process:
  - Known transition and reward model
    - Value and Policy Iteration
    - Find optimal policy using planning/dynamic programming
    - Execute the policy found
  - Unknown transition and reward model
    - Reinforcement learning
    - Learn optimal policy while interacting with environment

Current Assumptions

- Uncertainty: stochastic process
- Time: sequential process
- Observability: fully observable states
- ~~No learning: complete model~~ Unknown Model
- Variable type: discrete (e.g., discrete states and actions)

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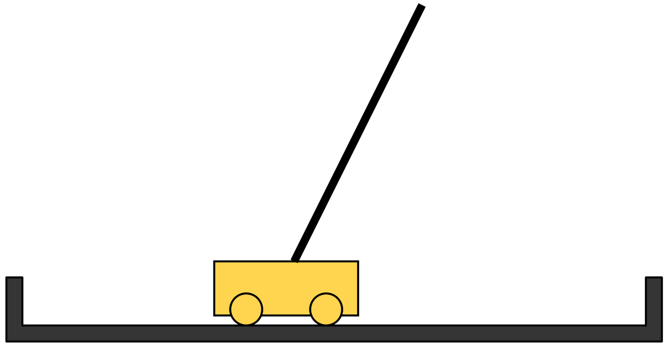
Example: Inverted Pendulum

- State: $x(t), x'(t), \theta(t), \theta'(t)$
- Action: Force F
- Reward: 1 for any step where pole balanced



Problem: Find $\pi : S \rightarrow A$ that maximizes rewards

Important Components in RL

RL agents may or may not include the following components:

- Model: $Pr(s' \mid s, a), R(s, a)$
  - Transition dynamics and rewards
- Policy: $\pi(s)$
  - Agent action choices
- Value function: $V(s)$
  - Expected total rewards of the agent policy

Categorizing RL agents

Value based
- No policy (implicit)
- Value function

Policy based
- Policy
- No value function

Actor critic
- Policy
- Value function

Model based
- Transition and reward model

Model free
- No transition and no reward model (implicit)

Online RL
- Learn by interacting with environment

Offline RL
- No environment
- Learn only from saved data

Toy Maze Example

3	r	r	r	+1
2	u		u	-1
1	u	l	l	l
	1	2	3	4

Start state: (1,1)
Terminal states: (4,2), (4,3)
No discount: $\gamma = 1$

Reward is -0.04 for non-terminal states

Four actions: up (u), left (l), right (r), down (d)
Do not know the transition probabilities

What is the value $V(s)$ of being in state s?

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Unfair Dice

- Consider an unfair die with the following distribution:

X	1	2	3	4	5	6
P(X)	1/6	2/6	0	2/6	0	1/6

- Objective: Determine the expected value of the dice
  - If $P(X)$ is given: $\mathbb{E}(X) = \sum_{x_i} X_i P(X_i) = 3.17$
  - If $P(X)$ is not given?
    - Roll the dice several times (N)

$$\mathbb{E}(X) \approx \frac{X_1 + X_2 + \ldots + X_N}{N}$$

    - Just an estimate

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Model Free Evaluation

- Given policy π, estimate $V^{\pi}(s)$ without any transition or reward model

- **Monte Carlo** evaluation

$$V_{\pi}(s) = \mathbb{E}_{\pi}[\sum_{t} \gamma^t r_t \,|\, s, \pi]$$

$$\approx \frac{1}{n(s)} \sum_{k=1}^{n(s)} \left[\sum_{t} \gamma^t r_t^{(k)} \,|\, s, \pi\right] \quad \text{(several sample approximation)}$$

- **Temporal difference (TD)** evaluation

$$V^{\pi}(s) = \mathbb{E}[r \,|\, s, \pi(s)] + \gamma \sum_{s'} Pr(s' \,|\, s, \pi(s)) V^{\pi}(s')$$

$$\approx r + \gamma V^{\pi}(s') \quad \text{(one sample approximation)}$$

Monte Carlo Evaluation

	1	2	3	4
3	r	r	r	+1
2	u		u	-1
1	u	l	l	l

- Let G_k be a one-trajectory Monte Carlo target

$$G_k = \sum_t \gamma^t r_t^{(k)}$$

- Examples:
  - $(1,1) \rightarrow (1,2) \rightarrow (1,3) \rightarrow (1,2) \rightarrow (1,3) \rightarrow (2,3) \rightarrow (3,3) \rightarrow (4,3)$

$$G_1 = 1 - (0.04 \times 7) = 0.72$$

  - $(1,1) \rightarrow (1,2) \rightarrow (1,3) \rightarrow (2,3) \rightarrow (3,3) \rightarrow (3,2) \rightarrow (3,3) \rightarrow (4,3)$

$$G_2 = 1 - (0.04 \times 7) = 0.72$$

  - $(1,1) \rightarrow (2,1) \rightarrow (3,1) \rightarrow (3,2) \rightarrow (4,2)$

$$G_3 = -1 - (0.04 \times 4) = -1.16$$

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Monte Carlo Evaluation

- Let G_k be a one-trajectory Monte Carlo target

$$G_k = \sum_t \gamma^t r_t^{(k)}$$

- Approximate value function

$$\begin{aligned} V_n^\pi(s) &\approx \frac{1}{n(s)} \sum_{k=1}^{n(s)} G_k \\ &= \frac{1}{n(s)} \left(G_{n(s)} + \sum_{k=1}^{n(s)-1} G_k \right) \\ &= \frac{1}{n(s)} \left(G_{n(s)} + \left(n(s) - 1 \right) V_{n-1}^\pi(s) \right) \\ &= V_{n-1}^\pi(s) + \frac{1}{n(s)} \left(G_{n(s)} - V_{n-1}^\pi(s) \right) \end{aligned}$$

- **Incremental update**

$$V_n^\pi(s) \leftarrow V_{n-1}^\pi(s) + \alpha_n \left(G_n - V_{n-1}^\pi(s) \right), \text{ where } \alpha_n = \frac{1}{n(s)}$$

Temporal Difference Evaluation

- Approximate value function: $V^\pi(s) \approx r + \gamma V^\pi(s')$

- **Incremental update**

  $V_n^\pi(s) \leftarrow V_{n-1}^\pi(s) + \alpha_n(r + \gamma V_{n-1}^\pi(s') - V_{n-1}^\pi(s))$

- **Theorem:** If α_n is appropriately decreased with # of times a state is visited then $V_n^\pi(s)$ converges to correct value.
  - **Sufficient conditions** for α_n: (1) $\sum_n \alpha_n \to \infty$ (2) $\sum_n (\alpha_n)^2 < \infty$
  - Often $\alpha_n(s) = \frac{1}{n(s)}$ where $n(s) =$ # of times s is visited

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Temporal Difference (TD) Evaluation

TDevaluation(π, V^π)

Repeat

Execute $\pi(s)$

Observe s' and r

Update counts: $n(s) \leftarrow n(s) + 1$

Learning rate: $\alpha \leftarrow \frac{1}{n(s)}$

Update value: $V^\pi(s) \leftarrow V^\pi(s) + \alpha(r + \gamma V^\pi(s') - V^\pi(s))$

$s \leftarrow s'$

Until convergence of V^π

Return V^π

Comparison

- Monte Carlo evaluation:
  - Unbiased estimate
  - High variance
  - Needs many trajectories
- Temporal difference evaluation:
  - Biased estimate
  - Lower variance
  - Needs less trajectories

Model Free Control

- Instead of evaluating the state value function, $V^\pi(s)$, evaluate the state-action value function, $Q^\pi(s, a)$

$Q^\pi(s, a)$: value of executing a followed by π

$$Q^\pi(s, a) = \mathbb{E}[r \mid s, a] + \gamma \sum_{s'} Pr(s' \mid s, a) V^\pi(s')$$

- Greedy policy π':

$$\pi'(s) = \arg\max_a Q^\pi(s, a)$$

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Bellman's Equation

- Optimal state value function $V^*(s)$

$$V^*(s) = \max_a \mathbb{E}[r \,|\, s, a] + \gamma \sum_{s'} Pr(s' \,|\, s, a) V^*(s')$$

- Optimal state-action value function $Q^*(s, a)$

$$Q^*(s, a) = \mathbb{E}[r \,|\, s, a] + \gamma \sum_{s'} Pr(s' \,|\, s, a) \max_{a'} Q^*(s', a')$$

$$\text{where } V^*(s) = \max_a Q^*(s, a)$$

$$\pi^*(s) = \arg\max_a Q^*(s, a)$$

Monte Carlo Control

- Let G_k^a be a one-trajectory Monte Carlo target

$$G_k^a = r_0^{(k)} + \sum_{t=1} \gamma^t r_t^{(k)}$$

- Alternate between
  - **Policy evaluation**

$$Q_k^\pi(s,a) \leftarrow Q_{k-1}^\pi(s,a) + \alpha_n \left(G_k^a - Q_{k-1}^\pi(s,a) \right)$$

  - **Policy improvement**

$$\pi'(s) \leftarrow \arg\max_a Q^\pi(s,a)$$

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Temporal Difference Control

- Approximate Q-function:

$$Q^*(s,a) = \mathbb{E}[r \,|\, s,a] + \gamma \sum_{s'} Pr(s' \,|\, s,a) \max_{a'} Q^*(s',a')$$

$$\approx r + \gamma \max_{a'} Q^*(s',a')$$

- **Incremental update**

$$Q^*_n(s,a) \leftarrow Q^*_{n-1}(s,a) + \alpha_n \left(r + \gamma \max_{a'} Q^*_{n-1}(s',a') - Q^*_{n-1}(s,a) \right)$$

Q-Learning

Qlearning(s, Q^*)

Repeat

Select and execute a

Observe s' and r

Update counts: $n(s,a) \leftarrow n(s,a) + 1$

Learning rate: $\alpha \leftarrow \dfrac{1}{n(s,a)}$

Update Q-value:

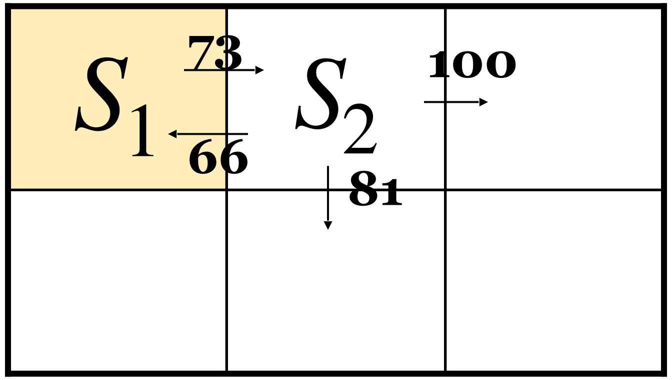
$Q^*(s,a) \leftarrow Q^*(s,a) + \alpha\left(r + \gamma \max_{a'} Q^*(s',a') - Q^*(s,a)\right)$

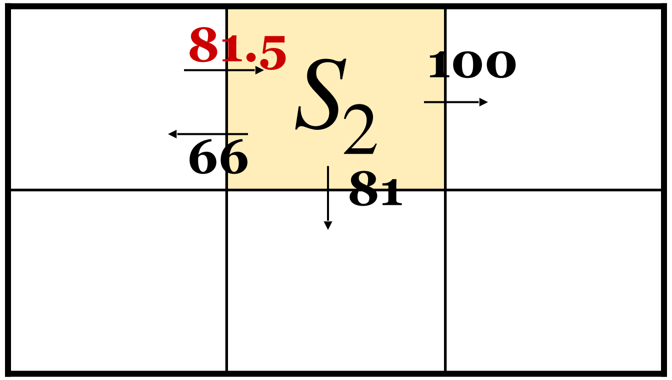
$s \leftarrow s'$

Until convergence of Q^*

Return Q^*

Q-learning Example





$$\gamma = 0.9, \quad \alpha = 0.5, \quad r = 0 \text{ for non-terminal states}$$

$$\begin{aligned} Q(s_1, \text{right}) &= Q(s_1, \text{right}) + \alpha\left(r + \gamma \max_{a'} Q(s_2, a') - Q(s_1, \text{right})\right) \\ &= 73 + 0.5\left(0 + 0.9 \max\{66, 81, 100\} - 73\right) \\ &= 73 + 0.5(17) = 81.5 \end{aligned}$$

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Q-Learning

Qlearning(s, Q^*)

Repeat

Select and execute a

Observe s' and r

Update counts: $n(s,a) \leftarrow n(s,a) + 1$

Learning rate: $\alpha \leftarrow \dfrac{1}{n(s,a)}$

Update Q-value:

$$Q^*(s,a) \leftarrow Q^*(s,a) + \alpha\left(r + \gamma \max_{a'} Q^*(s',a') - Q^*(s,a)\right)$$

$s \leftarrow s'$

Until convergence of Q^*

Return Q^*

Exploration vs Exploitation

- If agent always chooses action with highest value, then it is exploiting
  - The learned model is not accurate
  - Leads to suboptimal results

- By taking random actions (exploration), an agent may learn the model
  - But what is the use of learning a complete model if parts of it are never used?

- Need a balance between exploitation and exploration

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Common Exploration Methods

- ϵ-greedy:
  - With probability ϵ, execute random action
  - Otherwise execute best action $a^* = \arg\max_a Q(s, a)$

- Boltzmann exploration
  - Increasing temperature T increases stochasticity

$$Pr(a) = \frac{e^{\frac{Q(s,a)}{T}}}{\sum_a e^{\frac{Q(s,a)}{T}}}$$

Exploration and Q-learning

- Q-learning converges to optimal Q-values if
  - Every state is visited infinitely often (due to exploration)
  - The action selection becomes greedy as time approaches infinity
  - The learning rate α is decreased fast enough, but not too fast (sufficient conditions for α):

$$(1) \ \sum_n \alpha_n \to \infty \qquad (2) \ \sum_n (\alpha_n)^2 < \infty$$

Summary

- We can optimize a policy by RL when the transition and reward functions are unknown
- Model free, value based learning:
  - Monte Carlo learning (unbiased, but needs lots of data)
  - Temporal difference learning (low variance, less data)
- Active learning:
  - Exploration/exploitation dilemma