Lecture 15: Decision Networks
CS486/686 Intro to Artificial Intelligence

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Outline

- Utility theory
- Decision Networks
  - Aka Influence diagrams
- Value of information

Decision Making Under Uncertainty

- Options for your next smartphone: screen size, battery capacity, weight, brandname, etc.
- Many configurations:
  - Large screen, big battery, heavy, brandname
  - Small screen, small battery, light, brandname
  - Small screen, small battery, light, noname
  - etc.
- Which configuration do you prefer?
- How much would you pay for each configuration?

Preferences

- A *preference ordering* $\succcurlyeq$ is a ranking of all possible states of affairs (worlds) S
  - these could be outcomes of actions, truth assignments, states in a search problem, etc.
  - s $\succcurlyeq$ t: means that state s is *at least as good as* t
  - s $\succ$ t: means that state s is *strictly preferred to* t
  - s ~ t: means that the agent is *indifferent* between states s and t

Preferences

- If an agent's actions are deterministic then we know what states will occur

- If an agent's actions are not deterministic then we represent this by lotteries
  - Probability distribution over outcomes
  - Lottery $L=[p_1,s_1;\ p_2,s_2;\ \ldots;\ p_n,s_n]$
  - s_1 occurs with probability p_1, s_2 occurs with probability p_2,…

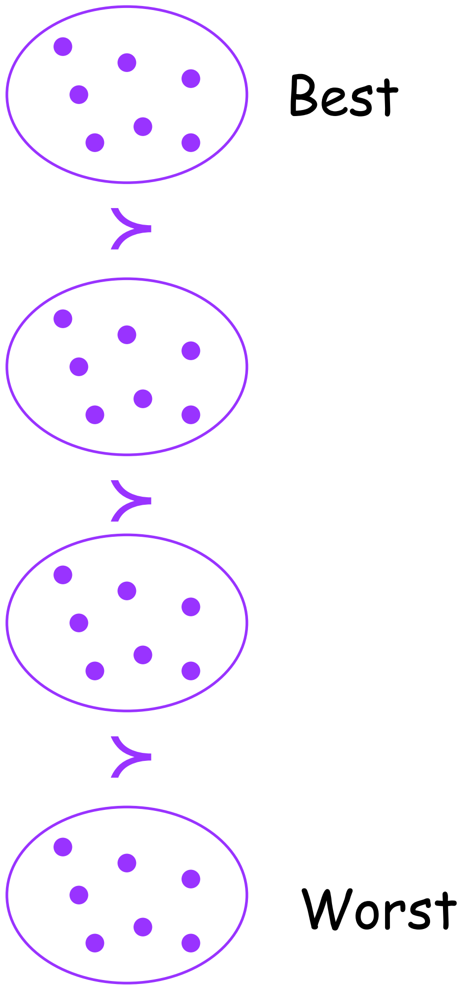
Axioms

- **Orderability:** Given 2 states A and B
  - $(A \succ B) \vee (B \succ A) \vee (A \sim B)$
- **Transitivity:** Given 3 states, A, B, and C
  - $(A \succ B) \wedge (B \succ C) \Rightarrow (A \succ C)$
- **Continuity:**
  - $A \succ B \succ C \Rightarrow \exists p\ [p,A;1\text{-}p,C] \sim B$
- **Substitutability:**
  - $A \sim B \rightarrow [p,A;1\text{-}p,C] \sim [p,B;1\text{-}p,C]$
- **Monotonicity:**
  - $A \succ B \Rightarrow (p \geq q \Leftrightarrow [p,A;1\text{-}p,B] \succ [q,A;1\text{-}q,B])$
- **Decomposibility:**
  - $[p,A;1\text{-}p,[q,B;1\text{-}q,C]] \sim [p,A;(1\text{-}p)q,B;\ (1\text{-}p)(1\text{-}q),C]$

Why Impose These Conditions?

- Structure of preference ordering imposes certain “rationality requirements” (it is a weak ordering)

- E.g., why transitivity?
  - Suppose you (strictly) prefer coffee to tea, tea to OJ, OJ to coffee
  - If you prefer X to Y, you’ll trade me Y plus $1 for X
  - I can construct a “money pump” and extract arbitrary amounts of money from you



Utilities

- Instead of ranking outcomes, we quantify our preferences
  - e.g., how much more valuable is coffee than tea
- A *utility function* $U:S \rightarrow \mathbb{R}$ associates a real-valued *utility* with each outcome.
  - $U(s)$ measures your *degree* of preference for s
- Note: U induces a preference ordering $\succcurlyeq_U$ over S defined as: $s \succcurlyeq_U t$ iff $U(s) \geq U(t)$
  - obviously $\succcurlyeq_U$ will be reflexive and transitive

Expected Utility

- Under conditions of uncertainty, each decision d induces a distribution $\Pr_d$ over possible outcomes
  - $\Pr_d(s)$ is probability of outcome s under decision d

- The *expected utility* of decision d is defined

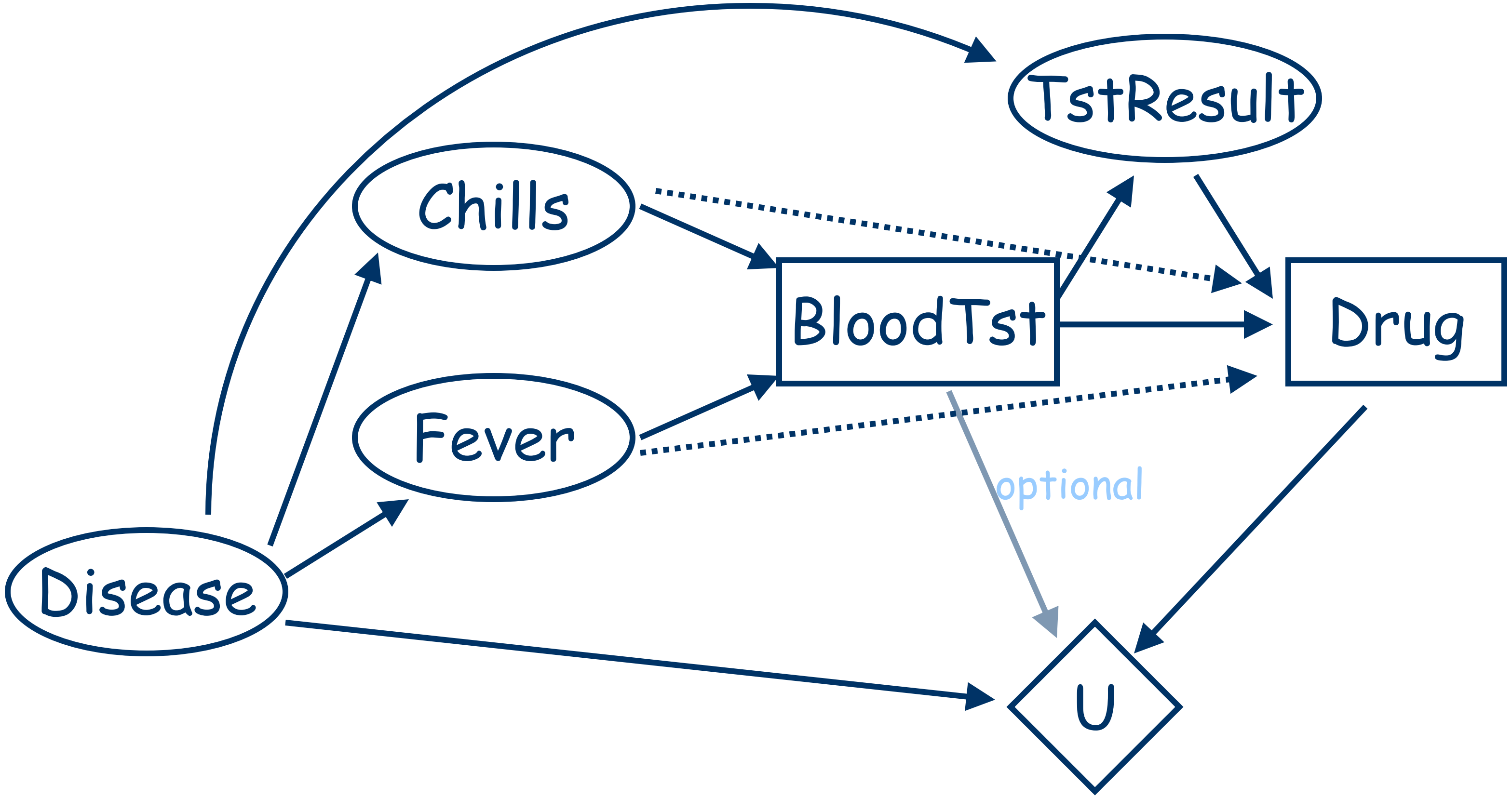
$$EU(d) = \sum_{s \in S} \Pr_d(s) U(s)$$

- The *principle of maximum expected utility (MEU)* states that the optimal decision under conditions of uncertainty is that with the greatest expected utility.

Decision Networks

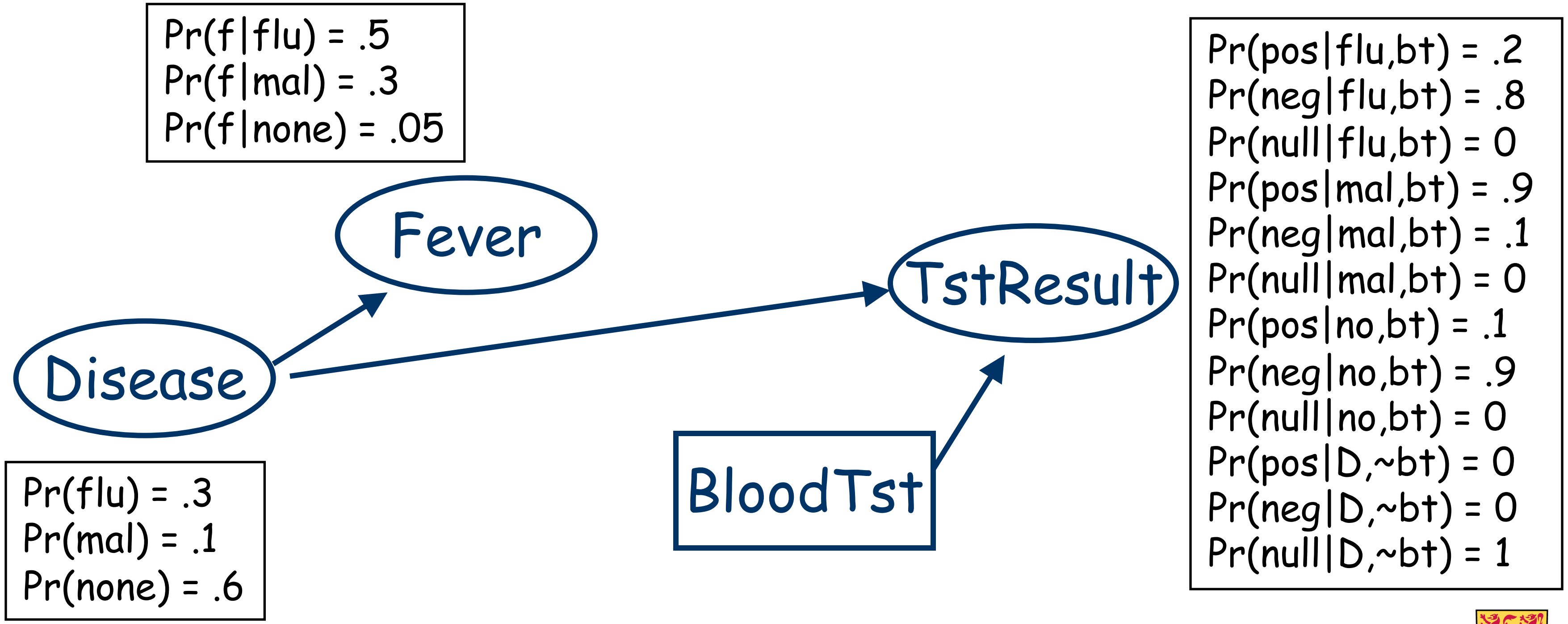
- *Decision networks* (also known as *influence diagrams*) provide a way of representing sequential decision problems
  - basic idea: represent the variables in the problem as you would in a Bayesian network
  - add decision variables – variables that you "control"
  - add utility variables – how good different states are

Sample Decision Network



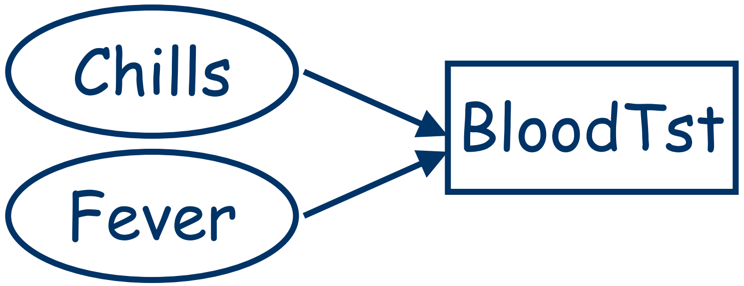
Decision Networks: Chance Nodes

- **Chance nodes**
  - random variables, denoted by circles
  - as in a BN, probabilistic dependence on parents



Decision Networks: Decision Nodes

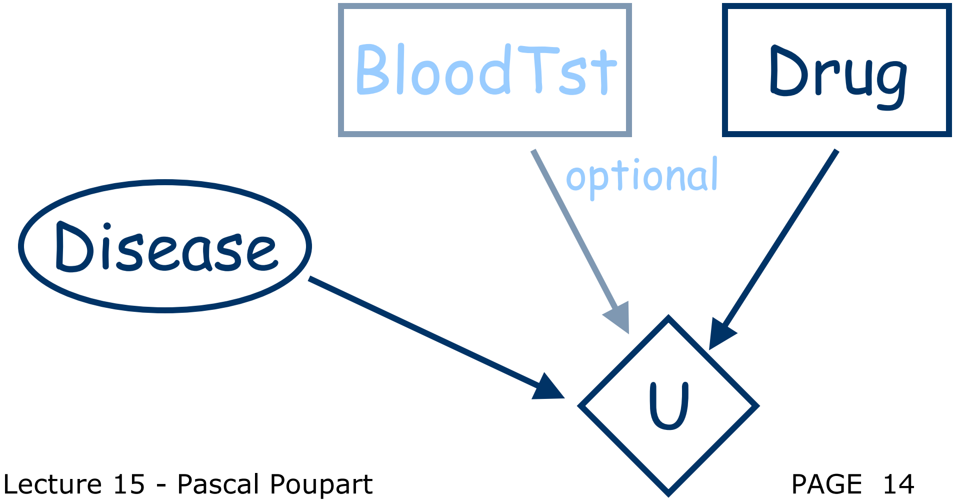
- **Decision nodes**
  - variables set by decision maker, denoted by squares
  - parents reflect *information available* at time decision is to be made
- Example: the actual values of Ch and Fev will be observed before the decision to take test must be made
  - agent can make *different decisions* for each instantiation of parents (i.e., policies)



BT $\in$ {bt, ~bt}

Decision Networks: Value Node

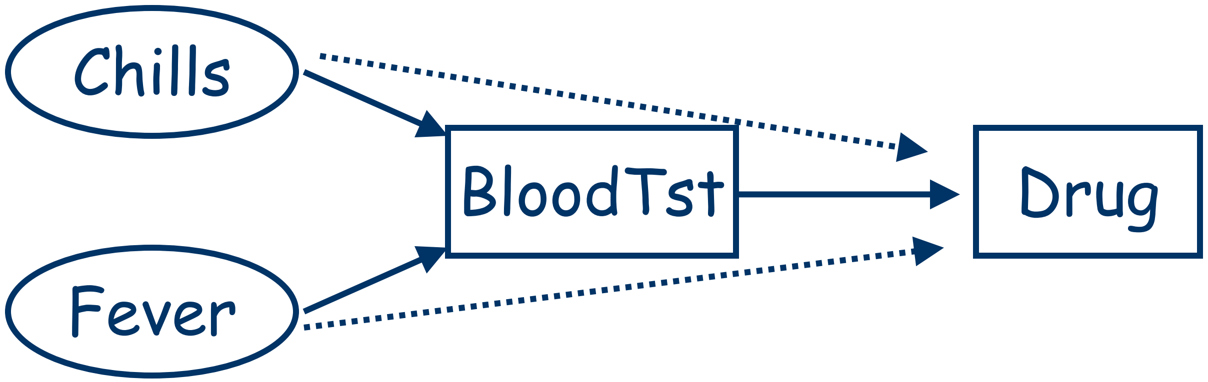
- **Value node**
  - specifies utility of a state, denoted by a diamond
  - utility depends *only on state of parents* of value node
  - generally: only one value node in a decision network
- Utility depends only on disease and drug



U(fludrug, flu) = 20
U(fludrug, mal) = -300
U(fludrug, none) = -5
U(maldrug, flu) = -30
U(maldrug, mal) = 10
U(maldrug, none) = -20
U(no drug, flu) = -10
U(no drug, mal) = -285
U(no drug, none) = 30

Decision Networks: Assumptions

- Decision nodes are totally ordered
  - decision variables $D_1, D_2, \ldots, D_n$
  - decisions are made in sequence
  - e.g., BloodTst (yes,no) decided before Drug (fd,md,no)
- *No-forgetting property*
  - any information available when decision D_i is made is available when decision D_j is made (for $i < j$)
  - thus all parents of D_i are parents of D_j



Dashed arcs ensure the no-forgetting property

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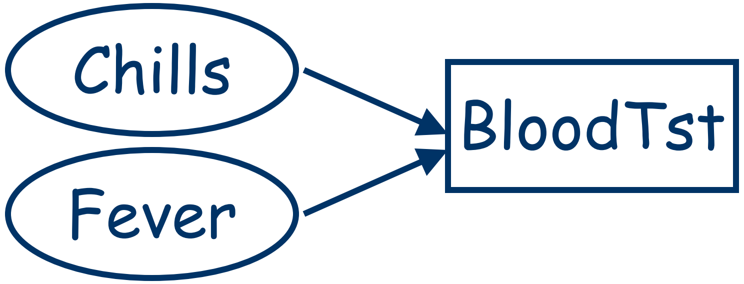
Policies

- Let $Par(D_i)$ be the parents of decision node D_i
  - $Dom(Par(D_i))$ is the set of assignments to parents
- A policy δ is a set of mappings δ_i, one for each decision node D_i
  - $\delta_i : Dom(Par(D_i)) \rightarrow Dom(D_i)$
  - δ_i associates a decision with each parent asst for D_i
- For example, a policy for BT might be:
  - $\delta_{BT}(c,f) = bt$
  - $\delta_{BT}(c,\sim f) = \sim bt$
  - $\delta_{BT}(\sim c,f) = bt$
  - $\delta_{BT}(\sim c,\sim f) = \sim bt$



Value of a Policy

- *Value of policy δ* is the expected utility given that decisions are executed according to δ

- Given asst **x** to the set **X** of all chance variables, let $\delta(\mathbf{x})$ denote the asst to decision variables dictated by δ
  - e.g., asst to D_1 determined by it's parents' asst in **x**
  - e.g., asst to D_2 determined by it's parents' asst in **x** along with whatever was assigned to D_1
  - etc.
- Value of δ: $EU(\delta) = \Sigma_{\mathbf{X}}\, P(\mathbf{X}, \delta(\mathbf{X}))\, U(\mathbf{X}, \delta(\mathbf{X}))$

Optimal Policies

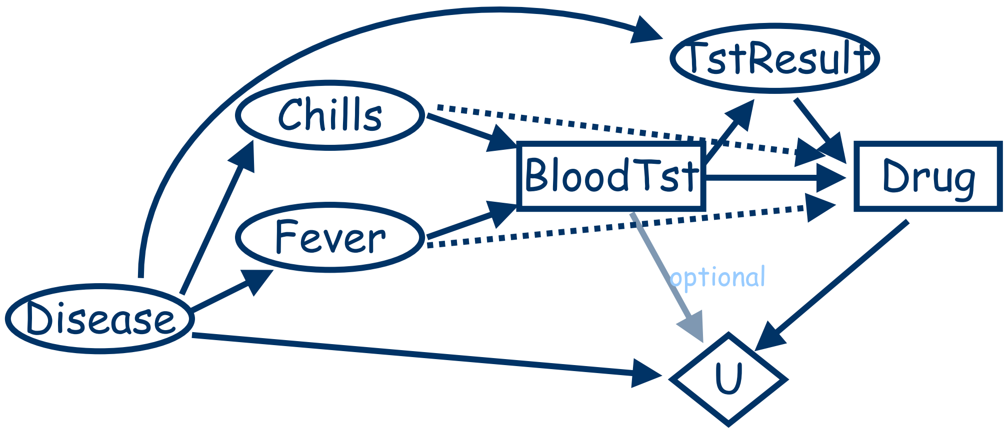
- An *optimal policy* is a policy δ^* such that $EU(\delta^*) \geq EU(\delta)$ for all policies δ

- We can use the dynamic programming principle yet again to avoid enumerating all policies

- We can also use the structure of the decision network to use variable elimination to aid in the computation

Computing the Best Policy

- We can work backwards as follows
- First compute optimal policy for Drug (last dec'n)
  - for each asst to parents (C,F,BT,TR) and for each decision value (D = md,fd,none), *compute the expected value* of choosing that value of D
  - set policy choice for each value of parents to be the value of D that has max value
  - eg: $\delta_D(c,f,bt,pos) = md$



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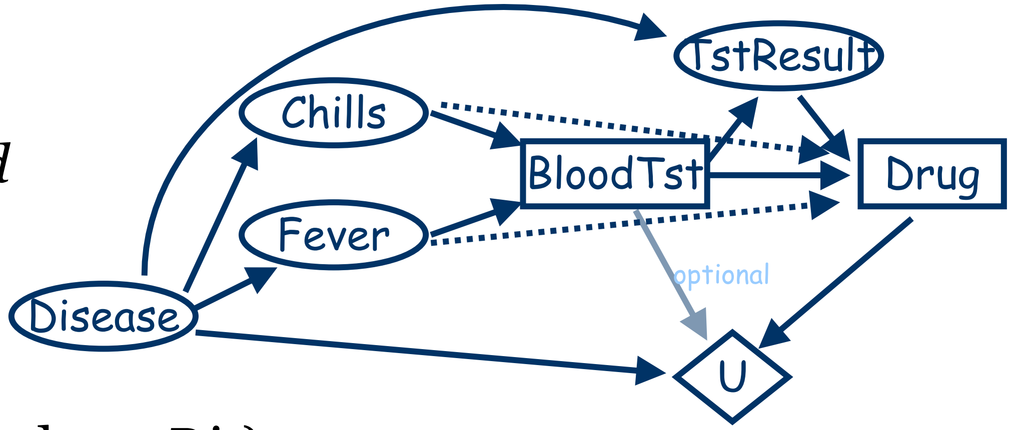
Computing the Best Policy

- Next compute policy for BT given policy $\delta_D(C,F,BT,TR)$ just determined for Drug
  - since $\delta_D(C,F,BT,TR)$ is fixed, we can treat Drug as a normal random variable with deterministic probabilities
  - i.e., for any instantiation of parents, value of Drug is fixed by policy δ_D
  - this means we can solve for optimal policy for BT just as before
  - only uninstantiated vars are random vars (once we fix *its* parents)

Computing the Best Policy

- How do we compute these expected values?
  - suppose we have asst <*c,f,bt,pos*> to parents of *Drug*
  - we want to compute EU of deciding to set *Drug* = *md*
  - we can run variable elimination!



- Treat *C,F,BT,TR,Dr* as evidence
  - this reduces factors (e.g., *U* restricted to *bt,md*: depends on *Dis*)
  - eliminate remaining variables (e.g., only *Disease* left)
  - left with factor: **EU(md|c,f,bt,pos) = Σ_{Dis} P(Dis|c,f,bt,pos,md) U(Dis,bt,md)**

- We now know EU of doing *Dr=md* when *c,f,bt,pos* true
- Can do same for *fd,no* to decide which is best

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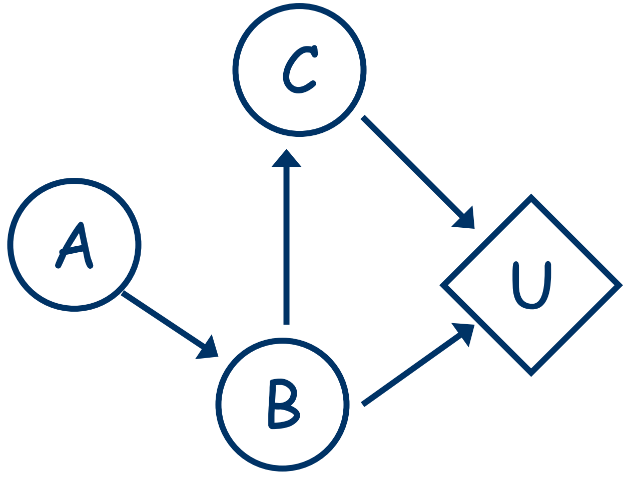
Computing Expected Utilities

- The preceding slide illustrates a general phenomenon
  - computing expected utilities with BNs is quite easy
  - utility nodes are just factors that can be dealt with using variable elimination

$$EU = \Sigma_{A,B,C}\, P(A,B,C)\, U(B,C)$$

$$= \Sigma_{A,B,C}\, P(C|B)\, P(B|A)\, P(A)\, U(B,C)$$



- Just eliminate variables in the usual way

Optimizing Policies: Key Points

- If a decision node D has no decisions that follow it, we can find its policy by instantiating each of its parents and computing the expected utility of each decision for each parent instantiation
  - no-forgetting means that all other decisions are instantiated (they must be parents)
  - its easy to compute the expected utility using VE
  - the number of computations is quite large: we run expected utility calculations (VE) for each parent instantiation together with each possible decision D might allow
  - policy: choose max decision for each parent instant’n

Optimizing Policies: Key Points

- When a decision D node is optimized, it can be treated as a random variable
  - for each instantiation of its parents we now know what value the decision should take
  - just treat policy as a new CPT: for a given parent instantiation $\mathbf{x}$, D gets $\delta(\mathbf{x})$ with probability 1 (all other decisions get probability zero)
- If we optimize from last decision to first, at each point we can optimize a specific decision by (a bunch of) simple VE calculations
  - it's successor decisions (optimized) are just normal nodes in the BNs (with CPTs)

A Decision Net Example

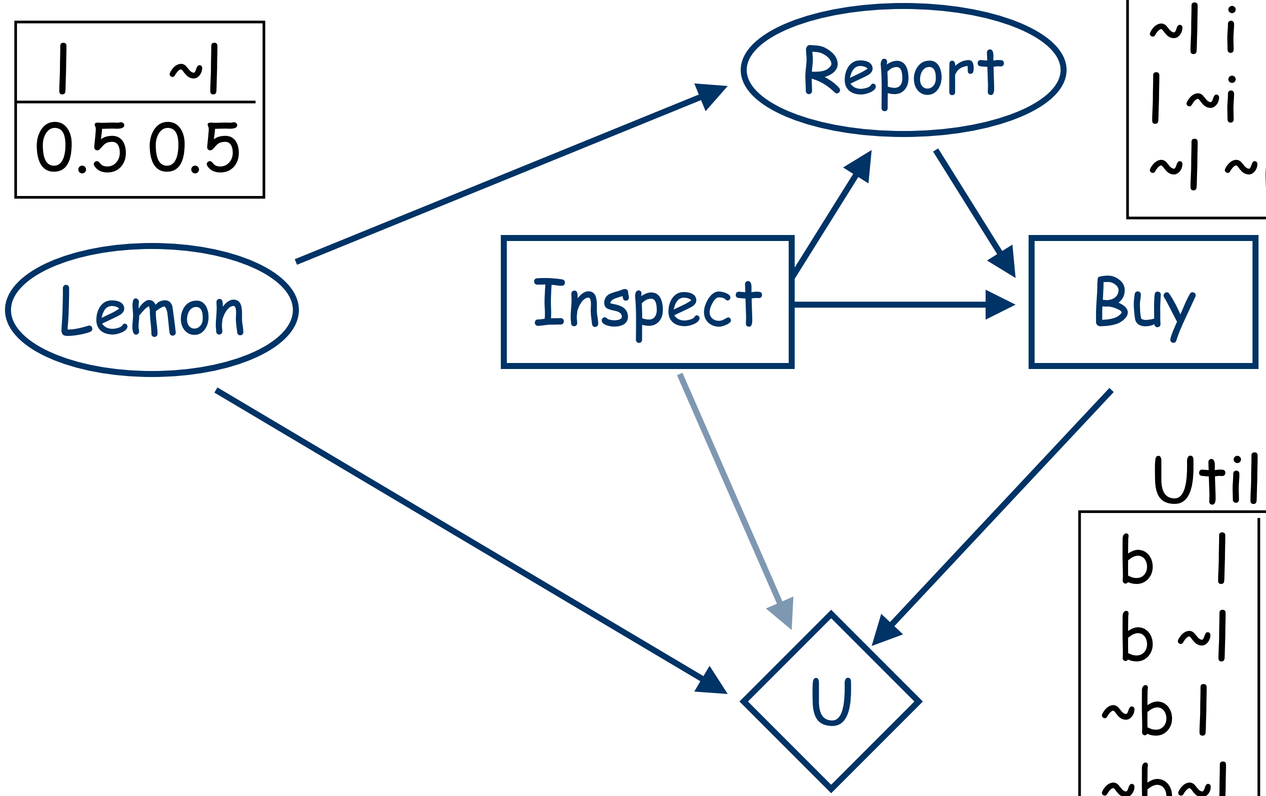
- Setting: you want to buy a used car, but there's a good chance it is a "lemon" (i.e., prone to breakdown). Before deciding to buy it, you can take it to a mechanic for inspection. S/he will give you a report on the car, labeling it either "good" or "bad". A good report is positively correlated with the car being sound, while a bad report is positively correlated with the car being a lemon.

- The report costs $50 however. So you could risk it, and buy the car without the report.

- Owning a sound car is better than having no car, which is better than owning a lemon.

Car Buyer's Network

l	~l
0.5	0.5



Rep: good,bad,none

	g	b	n
l i	0.2	0.8	0
~l i	0.9	0.1	0
l ~i	0	0	1
~l ~i	0	0	1

Utility

b l	-600
b ~l	1000
~b l	-300
~b~l	-300

-50 if inspect

Evaluate Last Decision: Buy (1)

- EU(B|I,R) = Σ_L P(L|I,R,B) U(L,I,B)
- I = i, R = g:
  - EU(buy) = P(l|i,g,buy) U(l,i,buy) + P(~l|i,g,buy) U(~l,i,buy)

    = .18*-650 + .82*950 = 662
  - EU(~buy) = P(l|i,g,~buy) U(l,i,~buy) + P(~l|i,g,~buy) U(~l,i,~buy)

    = -300 - 50 = -350 (-300 indep. of lemon)
  - So optimal δ_{Buy} *(i,g) = buy*

Evaluate Last Decision: Buy (2)

- I = i, R = b:
  - EU(buy) = P(l|i,b,buy) U(l,i,buy) + P(~l|i,b,buy) U(~l,i,buy)

    = .89*-650 + .11*950 = -474
  - EU(~buy) = P(l|i,b,~buy) U(l,i,~buy) + P(~l|i, b,~buy) U(~l,i,~buy)

    = -300 - 50 = -350 (-300 indep. of lemon)
  - So optimal δ_{Buy} *(i,b) = ~buy*

Evaluate Last Decision: Buy (3)

- I = ~i, R = n
  - EU(buy) = P(l|~i,n,buy) U(l,~i,buy) + P(~l|~i,n,buy) U(~l,~i,buy)

    = .5*-600 + .5*1000 = 200
  - EU(~buy) = P(l|~i,n,~buy) U(l,~i,~buy) + P(~l|~i,n,~buy) U(~l,~i,~buy)

    = -300 (-300 indep. of lemon)
  - So optimal δ_{Buy} *(~i,n) = buy*
- So optimal policy for Buy is:
  - δ_{Buy} *(i,g) = buy* ; δ_{Buy} *(i,b) = ~buy* ; δ_{Buy} *(~i,n) = buy*
- Note: we don't bother computing policy for (i,~n), (~i, g), or (~i, b), since these occur with probability 0

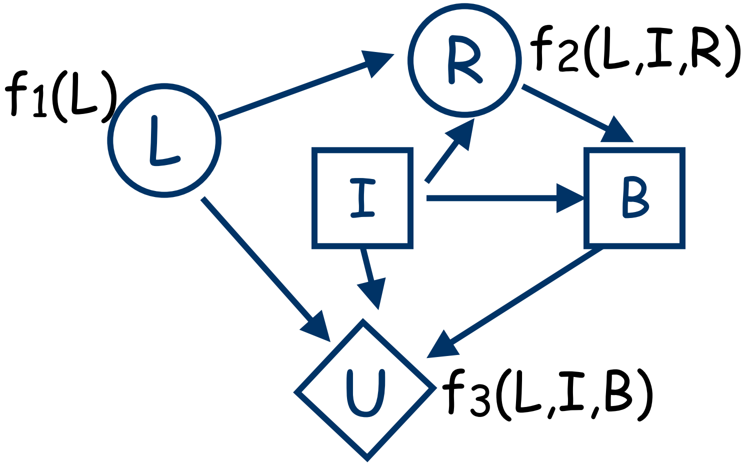
Using Variable Elimination

Factors: $f_1(L)\ f_2(L,I,R)\ f_3(L,I,B)$

Query: EU(B)?

Evidence: I = i, R = g

Elim. Order: L



Restriction: replace $f_2(L,I,R)$ by $f_4(L) = f_2(L,i,g)$

replace $f_3(L,I,B)$ by $f_5(L,B) = f_3(L,i,B)$

Step 1: Add $f_6(B) = \Sigma_L\, f_1(L)\ f_4(L)\ f_5(L,B)$

Remove: $f_1(L)$, $f_4(L)$, $f_5(L,B)$

Last factor: $f_6(B)$ is proportional to the expected utility of buy and ~buy. Select action with highest value.

Repeat for EU(B|i,b), EU(B|~i,n)

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Alternatively

- N.B.: variable elimination for decision networks computes expected utility that are not scaled…
- Can still pick best action, since utility scale is not important (relative magnitude is what matters)
- If we want exact expected utility:
  - Let $\mathbf{X}$ = parents(U)
  - EU(dec|evidence) = $\Sigma_{\mathbf{X}}$ Pr($\mathbf{X}$|dec,evidence) U($\mathbf{X}$)
  - Compute Pr($\mathbf{X}$|dec,evidence) by variable elimination
  - Multiply Pr($\mathbf{X}$|dec,evidence) by U($\mathbf{X}$)
  - Summout $\mathbf{X}$

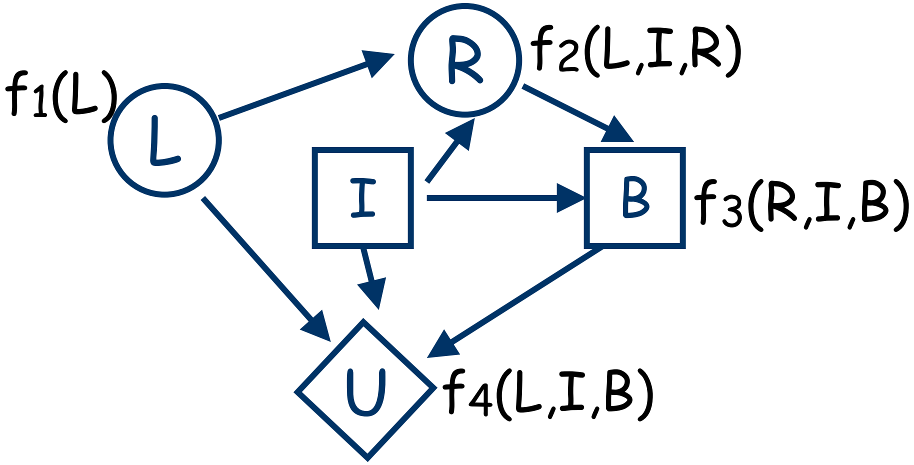
Evaluate First Decision: Inspect

- $EU(I) = \Sigma_{L,R}\ P(L,R|i)\ U(L,i,\boldsymbol{\delta_{Buy}(I,R)})$
  - where $P(R,L|i) = P(R|L,i)P(L|i)$
  - $EU(i) = (.1)(-650)+(.4)(-350)+(.45)(950)+(.05)(-350) = 205$
  - $EU(\sim i) = P(n,l|\sim i)\ U(l,\sim i,buy) + P(n,\sim l|\sim i)\ U(\sim l,\sim i,buy) = .5*-600 + .5*1000 = 200$
  - So optimal $\delta_{Inspect}\ () = inspect$

	$P(R,L \mid i)$	δ_{Buy}	$U(L, i, \delta_{Buy})$
g,l	0.1	buy	-600 - 50 = -650
b,l	0.4	~buy	-300 - 50 = -350
g,~l	0.45	buy	1000 - 50 = 950
b,~l	0.05	~buy	-300 - 50 = -350

Using Variable Elimination

Factors: $f_1(L)\ f_2(L,I,R)\ f_3(R,I,B)\ f_4(L,I,B)$
Query: EU(I)?
Evidence: none
Elim. Order: L, R, B



N.B. $f_3(R,I,B) = \delta_B(R,I)$

Step 1: Add $f_5(R,I,B)= \Sigma_L\ f_1(L)\ f_2(L,I,R)\ f_4(L,I,B)$

Remove: $f_1(L)\ f_2(L,I,R)\ f_4(L,I,B)$

Step 2: Add $f_6(I,B)= \Sigma_R\ f_3(R,I,B)\ f_5(R,I,B)$

Remove: $f_3(R,I,B)\ f_5(R,I,B)$

Step 3: Add $f_7(I)= \Sigma_B\ f_6(I,B)$

Remove: $f_6(I,B)$

Last factor: $f_7(I)$ is the expected utility of inspect and ~inspect. Select action with highest expected utility.

Value of Information

- So optimal policy is: inspect the car and if the report is good buy, otherwise don't buy
  - EU = 205
  - Notice that the EU of inspecting the car, then buying it iff you get a good report is 205 (i.e., 255 – 50 (cost of inspection)) which is greater than 200. So inspection improves EU.
  - Suppose inspection cost is $60: would it be worth it?
    - EU = 255 – 60 = 195 < EU(~i)
  - The *expected value of information* associated with inspection is 55 (it improves expected utility by this amount ignoring cost of inspection). How? Gives opportunity to change decision (~buy if bad).
  - You should be willing to pay up to $55 for the report