Reasoning Over Time
[RN2] Sec 15.1-15.3, 15.5
[RN3] Sec 15.1-15.3, 15.5

CS 486/686

University of Waterloo

Lecture 11: June 5, 2017

Outline

- Reasoning under uncertainty over time
- Hidden Markov Models
- Dynamic Bayesian Networks

Static Inference

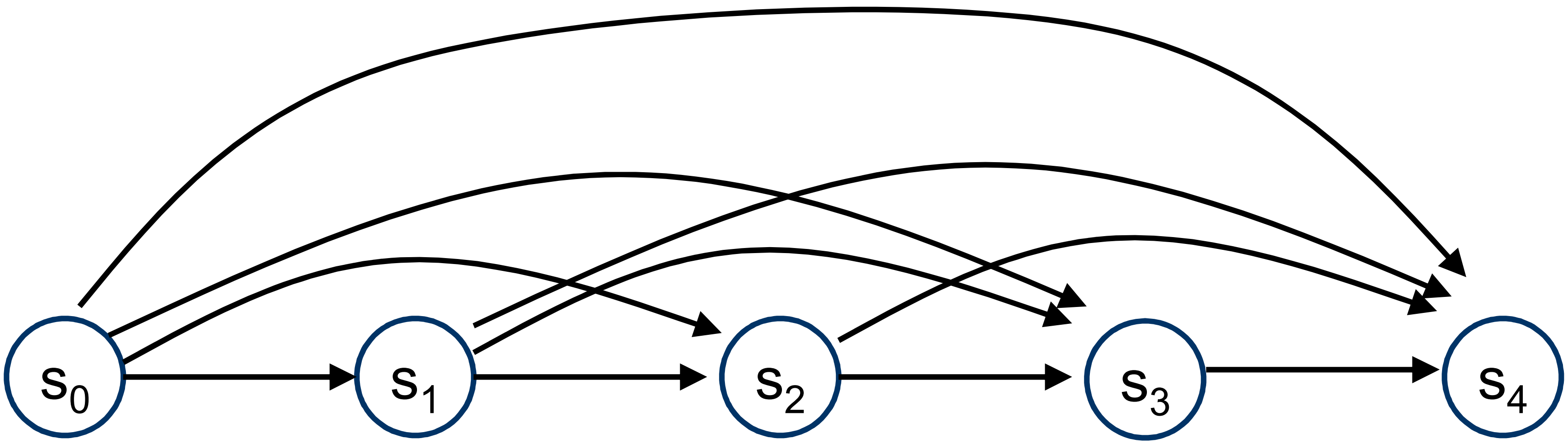
- So far…
  - Assume the world doesn't change
  - Static probability distribution
  - Ex: when repairing a car, whatever is broken remains broken during the diagnosis
- But the world evolves over time…
  - How can we use probabilistic inference for weather predictions, stock market predictions, patient monitoring, etc?

Dynamic Inference

- Need to reason over time
  - Allow the world to evolve
  - Set of states (encoding all possible worlds)
  - Set of time-slices (snapshots of the world)
  - Different probability distribution over states at each time slice
  - Dynamics encoding how distributions change over time

Stochastic Process

- Definition
  - Set of States: S
  - Stochastic dynamics: $Pr(s_t | s_{t-1}, \ldots, s_0)$



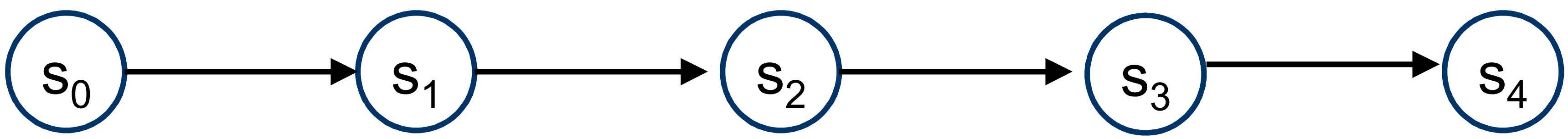
  - Can be viewed as a Bayes net with one random variable per time slice

Stochastic Process

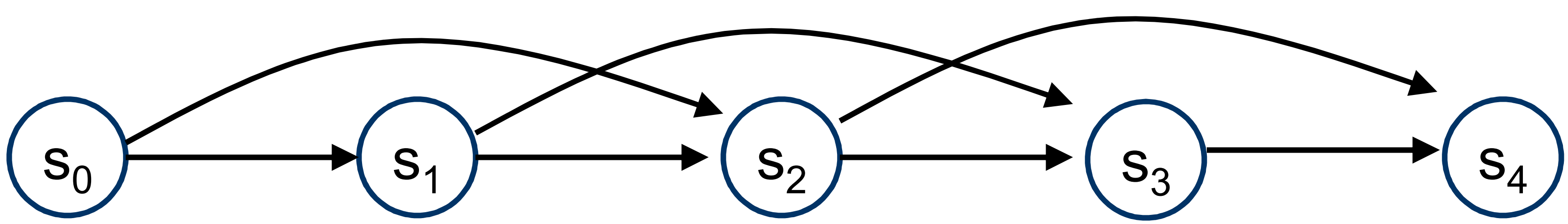
- Problems:
  - Infinitely many variables
  - Infinitely large conditional probability tables
- Solutions:
  - Stationary process: dynamics do not change over time
  - Markov assumption: current state depends only on a finite history of past states

K-order Markov Process

- Assumption: last k states sufficient
- First-order Markov Process
  - $\Pr(s_t|s_{t-1}, \ldots, s_0) = \Pr(s_t|s_{t-1})$



- Second-order Markov Process
  - $\Pr(s_t|s_{t-1}, \ldots, s_0) = \Pr(s_t|s_{t-1}, s_{t-2})$



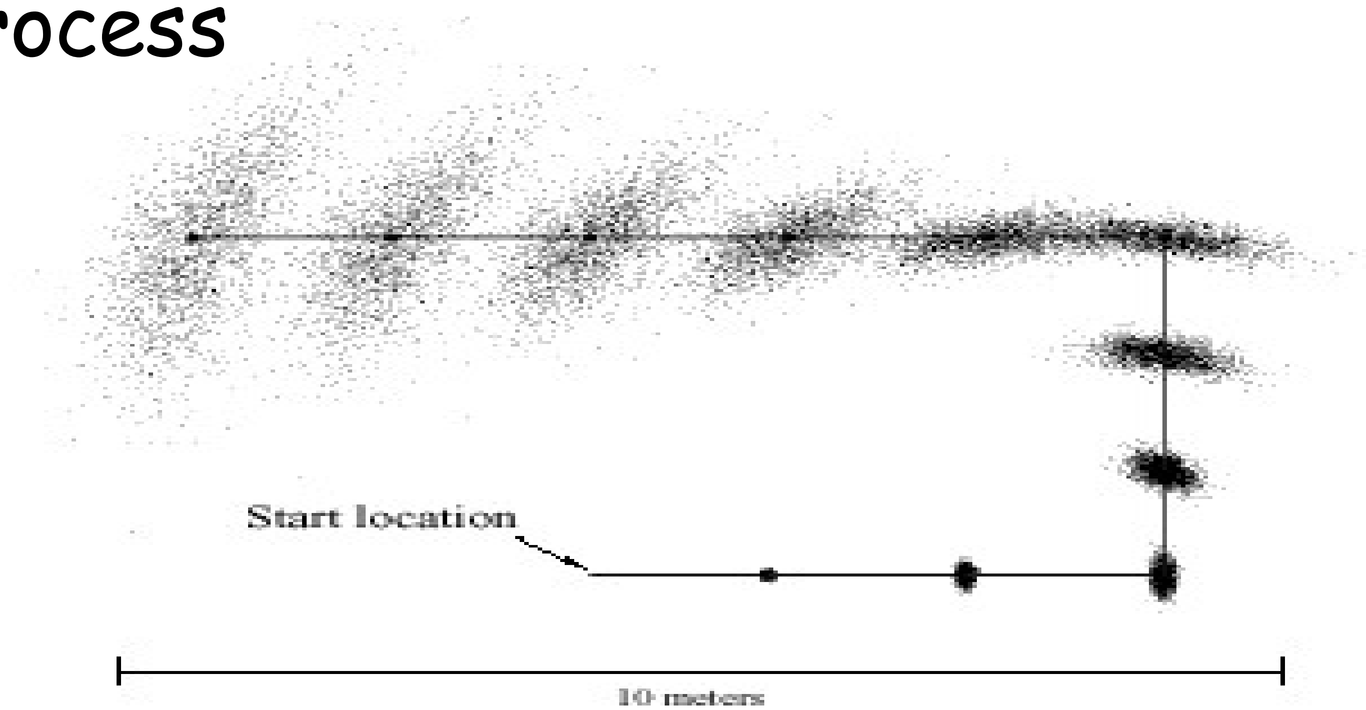
K-order Markov Process

- Advantage:
  - Can specify entire process with finitely many time slices
- Two slices sufficient for a first-order Markov process…
  - Graph: 

  - Dynamics: $Pr(s_t|s_{t-1})$
  - Prior: $Pr(s_0)$

Mobile Robot Localisation

- Example of a first-order Markov process



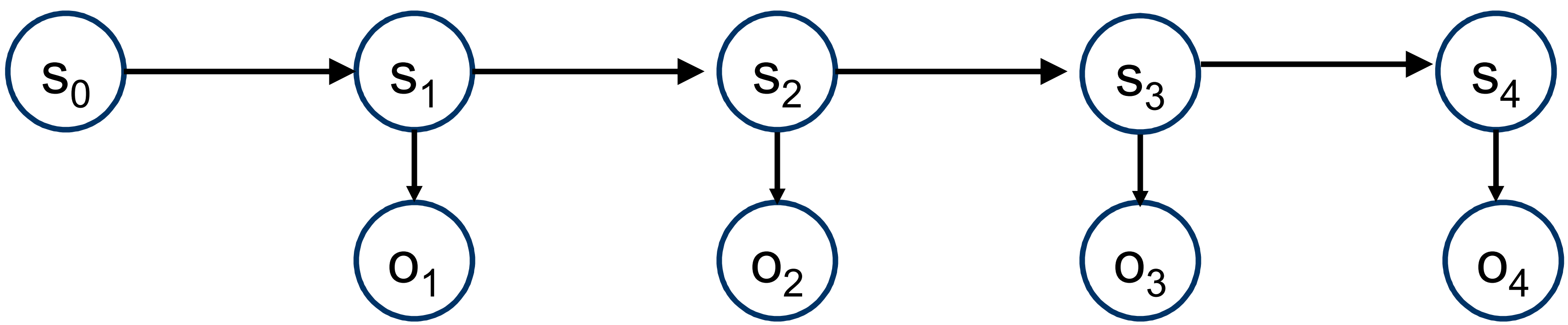
- Problem: uncertainty grows over time…

Hidden Markov Models

- Robot could use sensors to reduce location uncertainty…
- In general:
  - States not directly observable, hence uncertainty captured by a distribution
  - Uncertain dynamics increase state uncertainty
  - Observations made via sensors reduce state uncertainty
- Solution: Hidden Markov Model

First-order Hidden Markov Model

- Definition:
  - Set of states: S
  - Set of observations: O
  - Transition model: $Pr(s_t|s_{t-1})$
  - Observation model: $Pr(o_t|s_t)$
  - Prior: $Pr(s_0)$



Mobile Robot Localisation

- (First-order) Hidden Markov Model:
  - S: (x,y) coordinates of the robot on a map
  - O: distances to surrounding obstacles (measured by laser range finders or sonars)
  - $Pr(s_t|s_{t-1})$: movement of the robot with uncertainty
  - $Pr(o_t|s_t)$: uncertainty in the measurements provided by laser range finders and sonars
- Localisation corresponds to the query: $Pr(s_t|o_t, ..., o_1)$?

Inference in temporal models

- Four common tasks:
  - Monitoring: $\Pr(s_t | o_t, \ldots, o_1)$
  - Prediction: $\Pr(s_{t+k} | o_t, \ldots, o_1)$
  - Hindsight: $\Pr(s_k | o_t, \ldots, o_1)$ where $k < t$
  - Most likely explanation: $\text{argmax}_{st,\ldots,s1} \Pr(s_t, \ldots, s_1 | o_t, \ldots, o_1)$

- What algorithms should we use?
  - First 3 tasks can be done with variable elimination and 4^{th} task with a variant of variable elimination

Monitoring

- $Pr(s_t|o_t, \ldots, o_1)$: distribution over current state given observations
- Examples: robot localisation, patient monitoring
- Forward algorithm: corresponds to variable elimination
  - Factors: $Pr(s_0)$, $Pr(s_i|s_{i-1})$, $Pr(o_i|s_i)$, $1 \leq i \leq t$
  - Restrict $o_1, \ldots, o_t$ to the observations made
  - Summout $s_0, \ldots, s_{t-1}$
  - $\Sigma_{s0 \ldots st-1} Pr(s_0) \Pi_{1 \leq i \leq t} Pr(s_i|s_{i-1}) Pr(o_i|s_i)$

Prediction

- $Pr(s_{t+k}|o_t, \ldots, o_1)$: distribution over future state given observations
- Examples: weather prediction, stock market prediction
- Forward algorithm: corresponds to variable elimination
  - Factors: $Pr(s_0)$, $Pr(s_i|s_{i-1})$, $Pr(o_i|s_i)$, $1 \leq i \leq t+k$
  - Restrict $o_1, \ldots, o_t$ to the observations made
  - Summout $s_0, \ldots, s_{t+k-1}, o_{t+1}, \ldots, o_{t+k}$
  - $\Sigma_{s0 \ldots st+k-1, ot+1 \ldots ot+k} Pr(s_0) \Pi_{1 \leq i \leq t+k} Pr(s_i|s_{i-1}) Pr(o_i|s_i)$

Hindsight

- $Pr(s_k|o_t, \ldots, o_1)$ for $k<t$: distribution over a past state given observations
- Example: crime scene investigation
- **Forward-backward algorithm**: corresponds to variable elimination
  - Factors: $Pr(s_0)$, $Pr(s_i|s_{i-1})$, $Pr(o_i|s_i)$, $1 \le i \le t$
  - Restrict $o_1, \ldots, o_t$ to the observations made
  - Summout $s_0, \ldots, s_{k-1}, s_{k+1}, \ldots, s_t$
  - $\Sigma_{s0\ldots sk-1,sk+1,\ldots,st} Pr(s_0) \Pi_{1 \le i \le t} Pr(s_i|s_{i-1}) Pr(o_i|s_i)$

Most likely explanation

- $\text{Argmax}_{s0\ldots st} \Pr(s_0,\ldots,s_t|o_t, \ldots, o_1)$: most likely state sequence given observations
- Example: speech recognition
- Viterbi algorithm: corresponds to a variant of variable elimination
  - Factors: $\Pr(s_0)$, $\Pr(s_i|s_{i-1})$, $\Pr(o_i|s_i)$, $1 \leq i \leq t$
  - Restrict $o_1, \ldots, o_t$ to the observations made
  - Maxout $s_0, \ldots, s_t$
  - $\max_{s0\ldots st} \Pr(s_0) \Pi_{1 \leq i \leq t} \Pr(s_i|s_{i-1}) \Pr(o_i|s_i)$

Complexity of temporal inference

- Hidden Markov Models are Bayes nets with a polytree structure
- Hence, variable elimination is
  - Linear w.r.t. to # of time slices
  - Linear w.r.t. to largest conditional probability table ($Pr(s_t|s_{t-1})$ or $Pr(o_t|s_t)$)

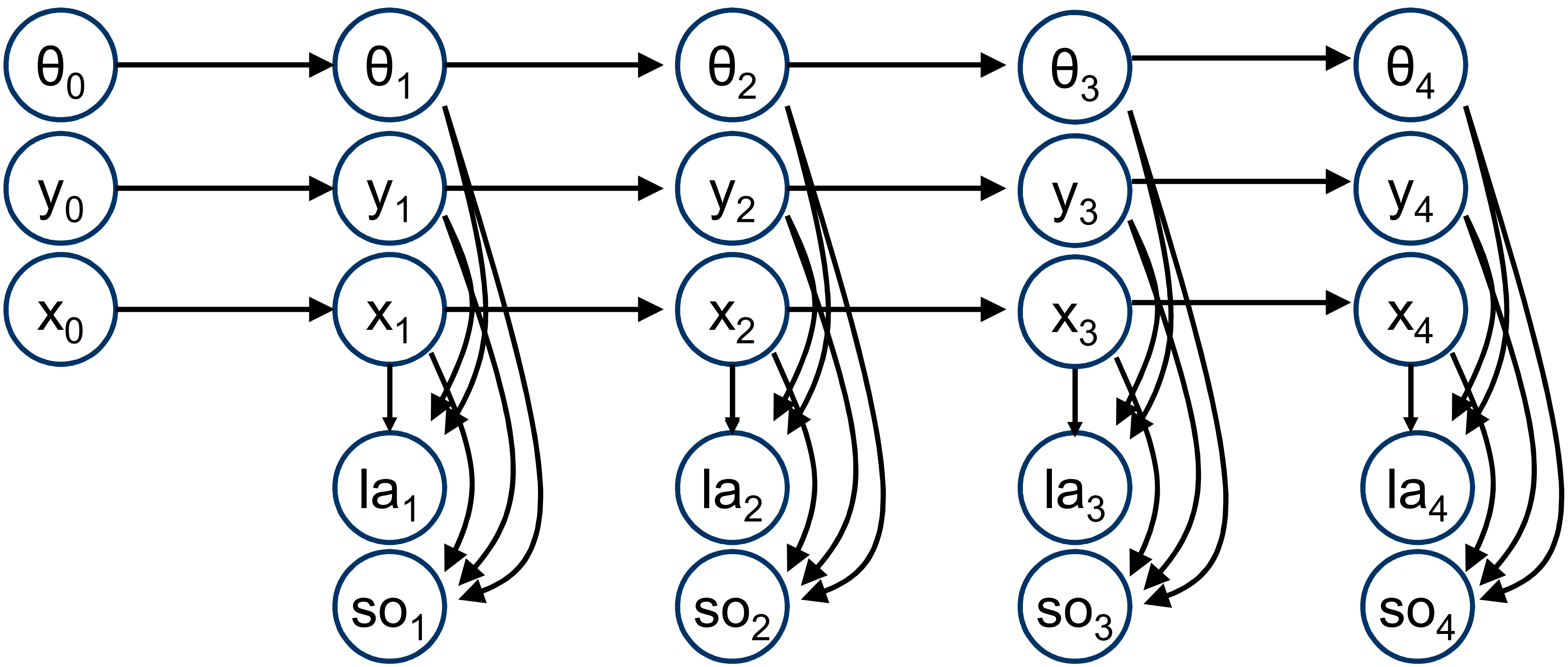
- What if # of states or observations are exponential?

Dynamic Bayesian Networks

- Idea: encode states and observations with several random variables
- Advantage: exploit conditional independence to save time and space
- HMMs are just DBNs with one state variable and one observation variable

Mobile Robot Localisation

- States: (x,y) coordinates and heading θ
- Observations: laser and sonar



DBN complexity

- Conditional independence allows us to write transition and observation models very compactly!
- Time and space of inference: conditional independence rarely helps…
  - inference tends to be exponential in the number of state variables
  - Intuition: all state variables eventually get correlated
  - No better than with HMMs ☹

Non-Stationary Process

- What if the process is not stationary?
- Solution: add new state components until dynamics are stationary
- Example:
  - Robot navigation based on (x,y,θ) is non-stationary when velocity varies…
  - Solution: add velocity to state description e.g. (x,y,v,θ)
  - If velocity varies… then add acceleration
  - Where do we stop?

Non-Markovian Process

- What if the process is not Markovian?
- Solution: add new state components until dynamics are Markovian
- Example:
  - Robot navigation based on (x,y,θ) is non-Markovian when influenced by battery level…
  - Solution: add battery level to state description e.g. (x,y,θ,b)

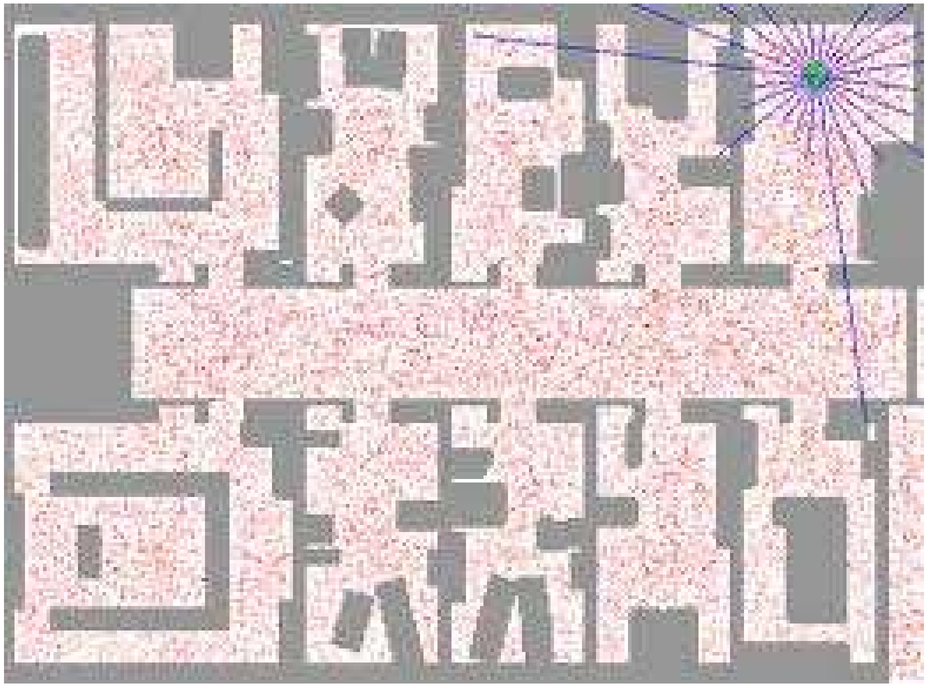
Markovian Stationary Process

- Problem: adding components to the state description to force a process to be Markovian and stationary may significantly increase computational complexity
- Solution: try to find the smallest state description that is self-sufficient (i.e., Markovian and stationary)

Probabilistic Inference

- Applications of static and temporal inference are virtually limitless
- Some examples:
  - mobile robot navigation
  - speech recognition
  - patient monitoring
  - help system under Windows
  - fault diagnosis in Mars rovers
  - etc.

Robot localisation



- University of Washington robotics and State Estimation
- http://www.cs.washington.edu/ai/Mobile_Robotics/mcl/

Neato Robotics

- Robotic Vacuum Cleaners by Neato Robotics
- Use particle filtering (approximate inference technique based on sampling) for simultaneous localisation and mapping

- See patent:
  http://www.faqs.org/patents/assignee/neato-robotics-inc/