Bayes Nets (cont)

CS 486/686
University of Waterloo
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Outline

- Inference in Bayes Nets
- Variable Elimination

Inference in Bayes Nets

- The independence sanctioned by D-separation (and other methods) allows us to compute prior and posterior probabilities quite effectively.
- We'll look at a couple simple examples to illustrate. We'll focus on networks without *loops*. (A loop is a cycle in the underlying *undirected* graph. Recall the directed graph has no cycles.)

Simple Forward Inference (Chain)

- Computing marginal requires simple forward "propagation" of probabilities
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$P(J)=\Sigma_{M,ET}\, P(J,M,ET)$ (marginalization)

$P(J)=\Sigma_{M,ET}\, P(J|M,ET)P(M|ET)P(ET)$ (chain rule)

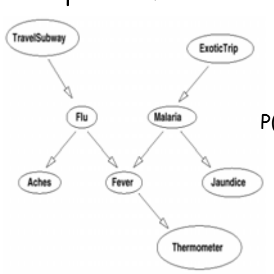
$P(J)=\Sigma_{M,ET}\, P(J|M)P(M|ET)P(ET)$ (conditional independence)

$P(J)=\Sigma_{M}P(J|M)\Sigma_{ET}P(M|ET)P(ET)$ (distribution of sum)

Note: all (final) terms are CPTs in the BN
Note: only ancestors of J considered

Simple Forward Inference (Chain)

- Same idea applies when we have "upstream" evidence



$P(J|ET) = \Sigma_M P(J,M|ET)$ (marginalisation)

$P(J|ET) = \Sigma_M P(J|M,ET)\, P(M|ET)$ (chain rule)

$P(J|ET) = \Sigma_M P(J|M)\, P(M|ET)$ (conditional independence)

Simple Forward Inference (Pooling)

- Same idea applies with multiple parents

$P(Fev) = \Sigma_{Flu,M,TS,ET}\, P(Fev,Flu,M,TS,ET)$

$= \Sigma_{Flu,M,TS,ET}\, P(Fev|Flu,M,TS,ET)\, P(Flu|M,TS,ET)\, P(M|TS,ET)\, P(TS|ET)\, P(ET)$

$= \Sigma_{Flu,M,TS,ET}\, P(Fev|Flu,M)\, P(Flu|TS)\, P(M|ET)\, P(TS)\, P(ET)$

$= \Sigma_{Flu,M}\, P(Fev|Flu,M)\, [\Sigma_{TS}\, P(Flu|TS)\, P(TS)]\, [\Sigma_{ET}\, P(M|ET)\, P(ET)]$

- (1) by marginalisation; (2) by the chain rule; (3) by conditional independence; (4) by distribution
  - note: all terms are CPTs in the Bayes net

Simple Forward Inference (Pooling)

- Same idea applies with evidence

$P(Fev|ts,\sim m) = \Sigma_{Flu} P(Fev,Flu|ts,\sim m)$

$= \Sigma_{Flu} P(Fev\ |Flu,ts,\sim m)\ P(Flu|ts,\sim m)$

$= \Sigma_{Flu} P(Fev|Flu,\sim m)\ P(Flu|ts)$

Simple Backward Inference

- When evidence is downstream of query variable, we must reason "backwards." This requires the use of Bayes rule:

$P(ET\ |\ j) = \alpha\ P(j\ |\ ET)\ P(ET)$

$= \alpha\ \Sigma_M P(j,M|ET)\ P(ET)$

$= \alpha\ \Sigma_M P(j|M,ET)\ P(M|ET)\ P(ET)$

$= \alpha\ \Sigma_M P(j|M)\ P(M|ET)\ P(ET)$

- First step is just Bayes rule
  - normalizing constant α is $1/P(j)$; but we needn't compute it explicitly if we compute $P(ET\ |\ j)$ for each value of ET: we just add up terms $P(j\ |\ ET)\ P(ET)$ for all values of ET (they sum to $P(j)$)

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Backward Inference (Pooling)

- Same ideas when several pieces of evidence lie "downstream"

$P(ET|j,fev) = \alpha\ P(j,fev|ET)\ P(ET)$

$= \alpha\ \Sigma_{M,Fl,TS} P(j,fev,M,Fl,TS|ET)\ P(ET)$

$= \alpha\ \Sigma_{M,Fl,TS} P(j|fev,M,Fl,TS,ET)\ P(fev|M,Fl,TS,ET)\ P(M|Fl,TS,ET)\ P(Fl|TS,ET)\ P(TS|ET)\ P(ET)$

$= \alpha\ P(ET)\ \Sigma_M P(j|M)\ \Sigma_{Fl} P(fev|M,Fl)\ \Sigma_{TS} P(Fl|TS)\ P(TS)$

  - Same steps as before; but now we compute prob of both pieces of evidence given hypothesis ET and combine them. Note: they are independent given M; but not given ET.

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Variable Elimination

- The intuitions in the above examples give us a simple inference algorithm for networks without loops: the *polytree* algorithm.
- Instead we'll look at a more general algorithm that works for general BNs; but the polytree algorithm will be a special case.
- The algorithm, ***variable elimination***, simply applies the summing out rule repeatedly.
  - To keep computation simple, it exploits the independence in the network and the ability to distribute sums inward

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Factors

- A function $f(X_1, X_2,\ldots, X_k)$ is also called a *factor*. We can view this as a table of numbers, one for each instantiation of the variables $X_1, X_2,\ldots, X_k$.
  - A tabular rep'n of a factor is exponential in k
- Each CPT in a Bayes net is a factor:
  - e.g., $Pr(C|A,B)$ is a function of three variables, A, B, C
- Notation: $f(\mathbf{X},\mathbf{Y})$ denotes a factor over the variables $\mathbf{X} \cup \mathbf{Y}$. (Here $\mathbf{X}$, $\mathbf{Y}$ are *sets* of variables.)

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The Product of Two Factors

- Let $f(\mathbf{X},\mathbf{Y})$ & $g(\mathbf{Y},\mathbf{Z})$ be two factors with variables $\mathbf{Y}$ in common
- The *product* of f and g, denoted $h = f \times g$ (or sometimes just $h = fg$), is defined:

$$h(\mathbf{X},\mathbf{Y},\mathbf{Z}) = f(\mathbf{X},\mathbf{Y}) \times g(\mathbf{Y},\mathbf{Z})$$

f(A,B)		g(B,C)		h(A,B,C)			
ab	0.9	bc	0.7	abc	0.63	ab~c	0.27
a~b	0.1	b~c	0.3	a~bc	0.08	a~b~c	0.02
~ab	0.4	~bc	0.8	~abc	0.28	~ab~c	0.12
~a~b	0.6	~b~c	0.2	~a~bc	0.48	~a~b~c	0.12

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Summing a Variable Out of a Factor

- Let $f(X,\mathbf{Y})$ be a factor with variable X ($\mathbf{Y}$ is a set)
- We *sum out* variable X from f to produce a new factor $h = \Sigma_X f$, which is defined:

$$h(\mathbf{Y}) = \Sigma_{x \in Dom(X)} f(x,\mathbf{Y})$$

f(A,B)		h(B)	
ab	0.9	b	1.3
a~b	0.1	~b	0.7
~ab	0.4		
~a~b	0.6		

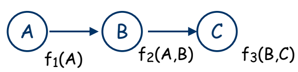
Restricting a Factor

- Let $f(X,\mathbf{Y})$ be a factor with variable X ($\mathbf{Y}$ is a set)
- We *restrict* factor f *to* X=x by setting X to the value x and "deleting". Define $h = f_{X=x}$ as: $h(\mathbf{Y}) = f(x,\mathbf{Y})$

f(A,B)		$h(B) = f_{A=a}$	
ab	0.9	b	0.9
a~b	0.1	~b	0.1
~ab	0.4		
~a~b	0.6		

Variable Elimination: No Evidence

- Computing prior probability of query var X can be seen as applying these operations on factors

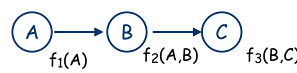


- $P(C) = \Sigma_{A,B} P(C|B) P(B|A) P(A)$
  $= \Sigma_B P(C|B) \Sigma_A P(B|A) P(A)$
  $= \Sigma_B f_3(B,C) \Sigma_A f_2(A,B) f_1(A)$
  $= \Sigma_B f_3(B,C) f_4(B) = f_5(C)$

Define new factors: $f_4(B)= \Sigma_A f_2(A,B) f_1(A)$ and $f_5(C)= \Sigma_B f_3(B,C) f_4(B)$

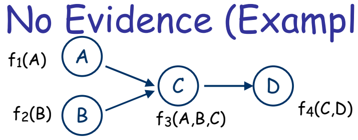
Variable Elimination: No Evidence

- Here's the example with some numbers



$f_1(A)$		$f_2(A,B)$		$f_3(B,C)$		$f_4(B)$		$f_5(C)$	
a	0.9	ab	0.9	bc	0.7	b	0.85	c	0.625
~a	0.1	a~b	0.1	b~c	0.3	~b	0.15	~c	0.375
		~ab	0.4	~bc	0.2				
		~a~b	0.6	~b~c	0.8				

VE: No Evidence (Example 2)



$P(D) = \Sigma_{A,B,C} P(D|C) P(C|B,A) P(B) P(A)$
$= \Sigma_C P(D|C) \Sigma_B P(B) \Sigma_A P(C|B,A) P(A)$
$= \Sigma_C f_4(C,D) \Sigma_B f_2(B) \Sigma_A f_3(A,B,C) f_1(A)$
$= \Sigma_C f_4(C,D) \Sigma_B f_2(B) f_5(B,C)$
$= \Sigma_C f_4(C,D) f_6(C)$
$= f_7(D)$

Define new factors: $f_5(B,C)$, $f_6(C)$, $f_7(D)$, in the obvious way

Variable Elimination: One View

- One way to think of variable elimination:
  - write out desired computation using the chain rule, exploiting the independence relations in the network
  - arrange the terms in a convenient fashion
  - distribute each sum (over each variable) in as far as it will go
    - i.e., the sum over variable X can be "pushed in" as far as the "first" factor mentioning X
  - apply operations "inside out", repeatedly eliminating and creating new factors (note that each step/removal of a sum eliminates one variable)

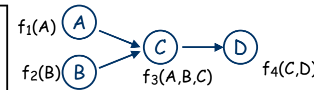
Variable Elimination Algorithm

- Given query var Q, remaining vars **Z**. Let F be the set of factors corresponding to CPTs for $\{Q\} \cup \mathbf{Z}$.

1. Choose an elimination ordering $Z_1, \ldots, Z_n$ of variables in **Z**.
2. For each Z_j -- in the order given -- eliminate $Z_j \in \mathbf{Z}$ as follows:
   (a) Compute new factor $g_j = \Sigma_{Z_j} f_1 \times f_2 \times \ldots \times f_k$, where the f_i are the factors in F that include Z_j
   (b) Remove the factors f_i (that mention Z_j) from F and add new factor g_j to F
3. The remaining factors refer only to the query variable Q. Take their product and normalize to produce P(Q)

VE: Example 2 again

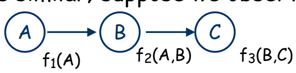
Factors: $f_1(A)\ f_2(B)\ f_3(A,B,C)\ f_4(C,D)$
Query: P(D)?
Elim. Order: A, B, C



Step 1: Add $f_5(B,C) = \Sigma_A f_3(A,B,C)\ f_1(A)$
Remove: $f_1(A)$, $f_3(A,B,C)$
Step 2: Add $f_6(C) = \Sigma_B f_2(B)\ f_5(B,C)$
Remove: $f_2(B)$, $f_5(B,C)$
Step 3: Add $f_7(D) = \Sigma_C f_4(C,D)\ f_6(C)$
Remove: $f_4(C,D)$, $f_6(C)$
Last factor $f_7(D)$ is (possibly unnormalized) probability P(D)

Variable Elimination: Evidence

- Computing posterior of query variable given evidence is similar; suppose we observe C=c:



$$\begin{aligned} P(A|c) &= \alpha\, P(A)\, P(c|A) \\ &= \alpha\, P(A)\, \Sigma_B P(c|B)\, P(B|A) \\ &= \alpha\, f_1(A)\, \Sigma_B f_3(B,c)\, f_2(A,B) \\ &= \alpha\, f_1(A)\, \Sigma_B f_4(B)\, f_2(A,B) \\ &= \alpha\, f_1(A)\, f_5(A) \\ &= \alpha\, f_6(A) \end{aligned}$$

New factors: $f_4(B) = f_3(B,c)$; $f_5(A) = \Sigma_B f_2(A,B)\ f_4(B)$; $f_6(A) = f_1(A)\ f_5(A)$

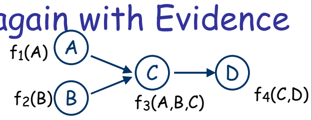
Variable Elimination with Evidence

Given query var Q, evidence vars **E** (observed to be **e**), remaining vars **Z**. Let F be set of factors involving CPTs for $\{Q\} \cup \mathbf{Z}$.

1. Replace each factor $f \in F$ that mentions a variable(s) in **E** with its restriction $f_{E=e}$ (somewhat abusing notation)
2. Choose an elimination ordering $Z_1, \ldots, Z_n$ of variables in **Z**.
3. Run variable elimination as above.
4. The remaining factors refer only to the query variable Q. Take their product and normalize to produce P(Q)

VE: Example 2 again with Evidence

Factors: $f_1(A)\ f_2(B)\ f_3(A,B,C)\ f_4(C,D)$
Query: P(A)?
Evidence: D = d
Elim. Order: C, B



Restriction: replace $f_4(C,D)$ with $f_5(C) = f_4(C,d)$
Step 1: Add $f_6(A,B) = \Sigma_C f_5(C)\ f_3(A,B,C)$
Remove: $f_3(A,B,C)$, $f_5(C)$
Step 2: Add $f_7(A) = \Sigma_B f_6(A,B)\ f_2(B)$
Remove: $f_6(A,B)$, $f_2(B)$
Last factors: $f_7(A)$, $f_1(A)$. The product $f_1(A) \times f_7(A)$ is (possibly unnormalized) posterior. So… $P(A|d) = \alpha\, f_1(A) \times f_7(A)$.

Some Notes on the VE Algorithm

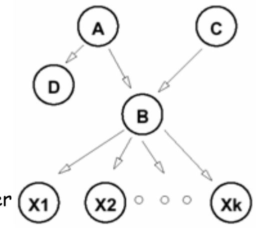
- After iteration j (elimination of Z_j), factors remaining in set F refer only to variables $X_{j+1}, \ldots Z_n$ and Q. No factor mentions an evidence variable E after the initial restriction.
- Number of iterations: linear in number of variables
- Complexity is linear in number of vars and exponential in size of the largest factor.
  - Recall each factor has exponential size in its number of variables
  - Can't do any better than size of BN (since its original factors are part of the factor set)
  - When we create new factors, we might make a set of variables larger.

Some Notes on the VE Algorithm

- The size of the resulting factors is determined by elimination ordering! (We'll see this in detail)
- For *polytrees*, easy to find good ordering (e.g., work outside in).
- For general BNs, sometimes good orderings exist, sometimes they don't (then inference is exponential in number of vars).
  - Simply *finding* the optimal elimination ordering for general BNs is NP-hard.
  - Inference in general is NP-hard in general BNs

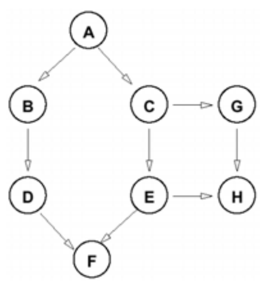
Elimination Ordering: Polytrees

- Inference is linear in size of network
  - ordering: eliminate only "singly-connected" nodes
  - e.g., in this network, eliminate D, A, C, X1,…; or eliminate X1,… Xk, D, A, C; or mix up…
  - result: no factor ever larger than original CPTs
  - eliminating B before these gives factors that include all of A,C, X1,… Xk !!!



Effect of Different Orderings

- Suppose query variable is D. Consider different orderings for this network
  - A,F,H,G,B,C,E:
    - good: why?
  - E,C,A,B,G,H,F:
    - bad: why?
- Which ordering creates smallest factors?
  - either max size or total
- which creates largest factors?



Relevance

- Certain variables have no impact on the query.
  - In ABC network, computing Pr(A) with no evidence requires elimination of B and C.
    - But when you sum out these vars, you compute a trivial factor (whose value are all ones); for example:
    - eliminating C: $f_4(B) = \Sigma_C f_3(B,C) = \Sigma_C \Pr(C|B)$
    - 1 for any value of B (e.g., $\Pr(c|b) + \Pr(\sim c|b) = 1$)
- No need to think about B or C for this query

Relevance: A Sound Approximation

- Can restrict attention to *relevant* variables. Given query Q, evidence **E**:
  - Q is relevant
  - if any node Z is relevant, its parents are relevant
  - if $E \in \mathbf{E}$ is a descendent of a relevant node, then E is relevant
- We can restrict our attention to the *subnetwork comprising only relevant variables* when evaluating a query Q

Next Class

- Decision making
  - Utility Theory
  - Decision Trees
- Russell & Norvig: Chapter 16