Lecture 8: Perceptrons, Neural Networks
CS480/680 Intro to Machine Learning

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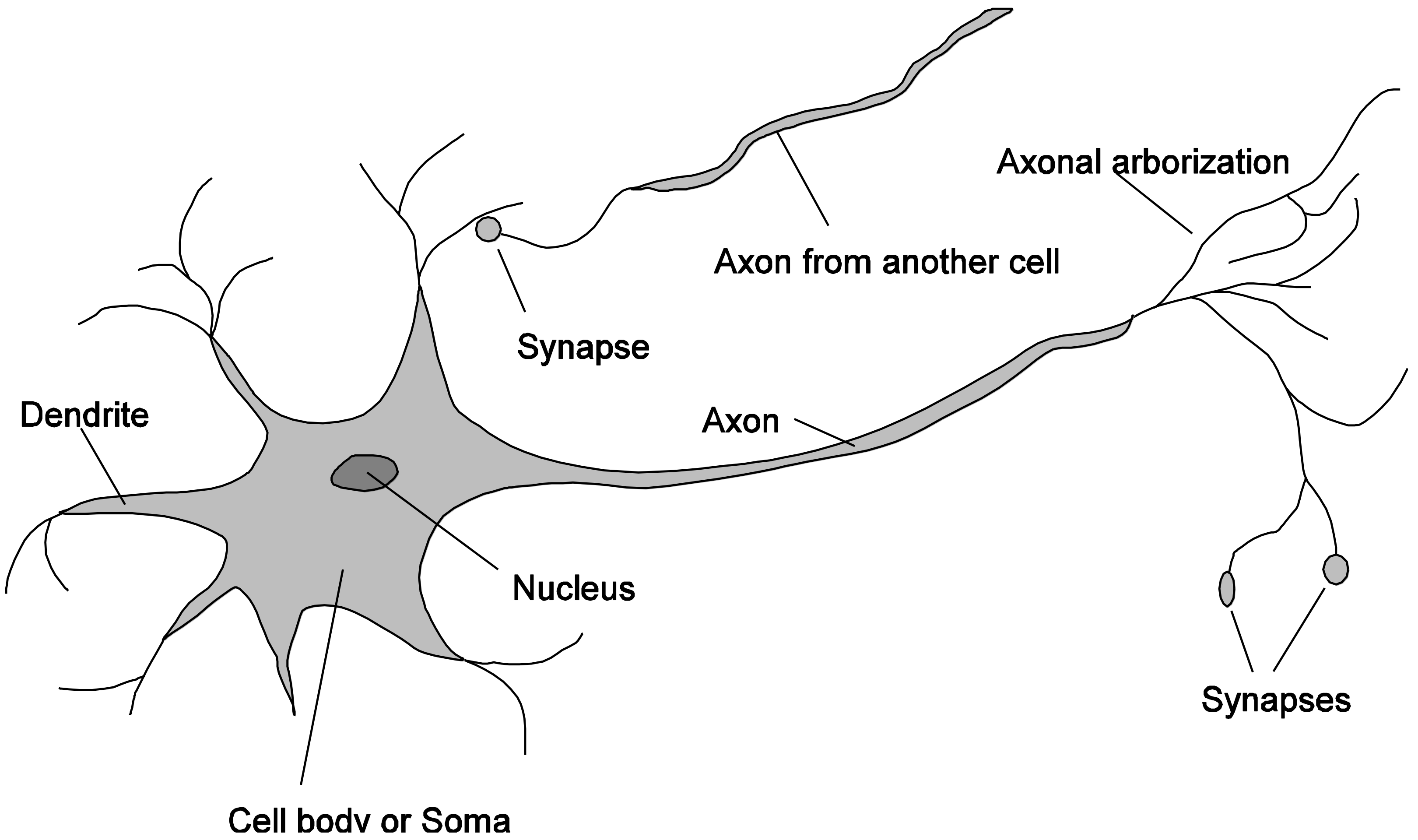
Outline

- Neural networks
  - Perceptron
  - Supervised learning algorithms for neural networks

Brain

- Seat of human intelligence
- Where memory/knowledge resides
- Responsible for thoughts and decisions
- Can learn
- Consists of nerve cells called **neurons**

Neuron


Axonal arborization
Axon from another cell
Synapse
Dendrite
Axon
Nucleus
Synapses
Cell bodv or Soma

Comparison

- Brain
  - Network of neurons
  - Nerve signals propagate in a neural network
  - Parallel computation
  - **Robust (neurons die everyday without much impact)**

- Computer
  - Bunch of gates
  - Electrical signals directed by gates
  - Sequential and parallel computation
  - **Fragile (if a gate stops working, computer crashes)**

Artificial Neural Networks

- Idea: **mimic the brain to do computation**

- Artificial neural network:
  - Nodes (a.k.a. units) correspond to neurons
  - Links correspond to synapses

- Computation:
  - Numerical signal transmitted between nodes corresponds to chemical signals between neurons
  - Nodes modifying numerical signal corresponds to neurons firing rate

ANN Unit

For each unit i:

- **Weights: $\boldsymbol{W}$**
  - Strength of the link from unit i to unit j
  - Input signals x_i weighted by W_{ji} and linearly combined:

$$a_j = \sum_i W_{ji} x_i + W_{j0} = \boldsymbol{W_j} \, \bar{\boldsymbol{x}}$$

- **Activation function: $\boldsymbol{h}$**
  - Numerical signal produced: $y_j = h(a_j)$

ANN Unit

- Picture

Activation Function

- Should be nonlinear
  - Otherwise, network is just a linear function

- Often chosen to mimic firing in neurons
  - Unit should be “active” (output near 1) when fed with the “right” inputs
  - Unit should be “inactive” (output near 0) when fed with the “wrong” inputs

Common Activation Functions

Threshold

Sigmoid

Logic Gates

- McCulloch and Pitts (1943)
  - Design ANNs to represent Boolean functions
- What should be the weights of the following units to code AND, OR, NOT ?

AND

OR

NOT

Network Structures

- **Feed-forward network**
  - Directed **acyclic** graph
  - No internal state
  - Simply computes outputs from inputs

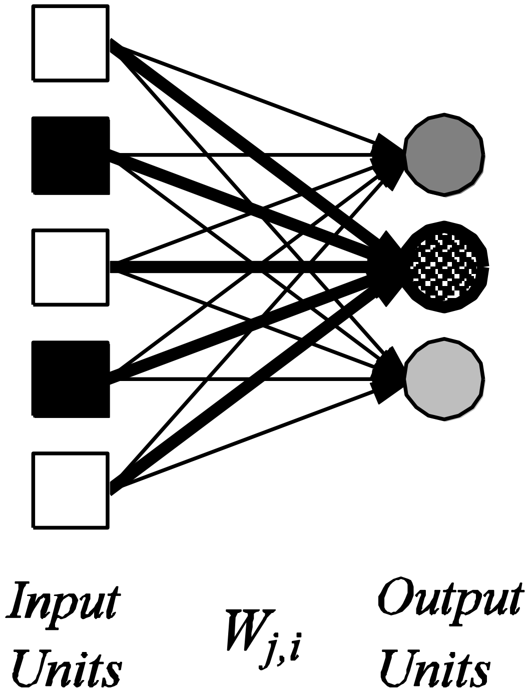
- **Recurrent network**
  - Directed **cyclic** graph
  - Dynamical system with internal states
  - Can memorize information

Feed-forward network

- Simple network with two inputs, one hidden layer of two units, one output unit

Perceptron

- Single layer feed-forward network



Supervised Learning

- Given list of $(\boldsymbol{x}, \boldsymbol{y})$ pairs
- Train feed-forward ANN
  - To compute proper outputs $\boldsymbol{y}$ when fed with inputs $\boldsymbol{x}$
  - Consists of adjusting weights W_{ji}
- **Simple learning algorithm for threshold perceptrons**

Threshold Perceptron Learning

- Learning is done separately for each unit j
  - Since units do not share weights

Perceptron learning for unit j:

- For each $(\boldsymbol{x}, y)$ pair do:
  - Case 1 (correct output produced): $\forall_i \; W_{ji} \leftarrow W_{ji}$
  - Case 2 (output produced is 0 instead of 1): $\forall_i \; W_{ji} \leftarrow W_{ji} + x_i$
  - Case 3 (output produced is 1 instead of 0): $\forall_i \; W_{ji} \leftarrow W_{ji} - x_i$
- Until correct output for all training instances

Threshold Perceptron Learning

- Dot products: $\bar{\boldsymbol{x}}^{\boldsymbol{T}}\bar{\boldsymbol{x}} \geq 0$ and $-\bar{\boldsymbol{x}}^{\boldsymbol{T}}\bar{\boldsymbol{x}} \leq 0$

- Perceptron computes

  1 when $\boldsymbol{w}^{\boldsymbol{T}}\bar{\boldsymbol{x}} = \sum_i x_i w_i + w_0 > 0$

  0 when $\boldsymbol{w}^{\boldsymbol{T}}\bar{\boldsymbol{x}} = \sum_i x_i w_i + w_0 < 0$

- If output should be 1 instead of 0 then

  $\boldsymbol{w} \leftarrow \boldsymbol{w} + \bar{\boldsymbol{x}}$ since $(\boldsymbol{w} + \bar{\boldsymbol{x}})^T\bar{\boldsymbol{x}} \geq \boldsymbol{w}^{\boldsymbol{T}}\bar{\boldsymbol{x}}$

- If output should be 0 instead of 1 then

  $\boldsymbol{w} \leftarrow \boldsymbol{w} - \bar{\boldsymbol{x}}$ since $(\boldsymbol{w} - \bar{\boldsymbol{x}})^T\bar{\boldsymbol{x}} \leq \boldsymbol{w}^{\boldsymbol{T}}\bar{\boldsymbol{x}}$

Threshold Perceptron Hypothesis Space

- Hypothesis space $h_{\boldsymbol{w}}$:
  - All binary classifications with parameters $\boldsymbol{w}$ s.t.

$$\boldsymbol{w}^T\overline{\boldsymbol{x}} > 0 \rightarrow +1$$
$$\boldsymbol{w}^T\overline{\boldsymbol{x}} < 0 \rightarrow -1$$

- Since $\boldsymbol{w}^T\overline{\boldsymbol{x}}$ is linear in $\boldsymbol{w}$, perceptron is called a **linear separator**

- **Theorem:** Threshold perceptron learning converges if and only if the data is linearly separable

Linear Separability

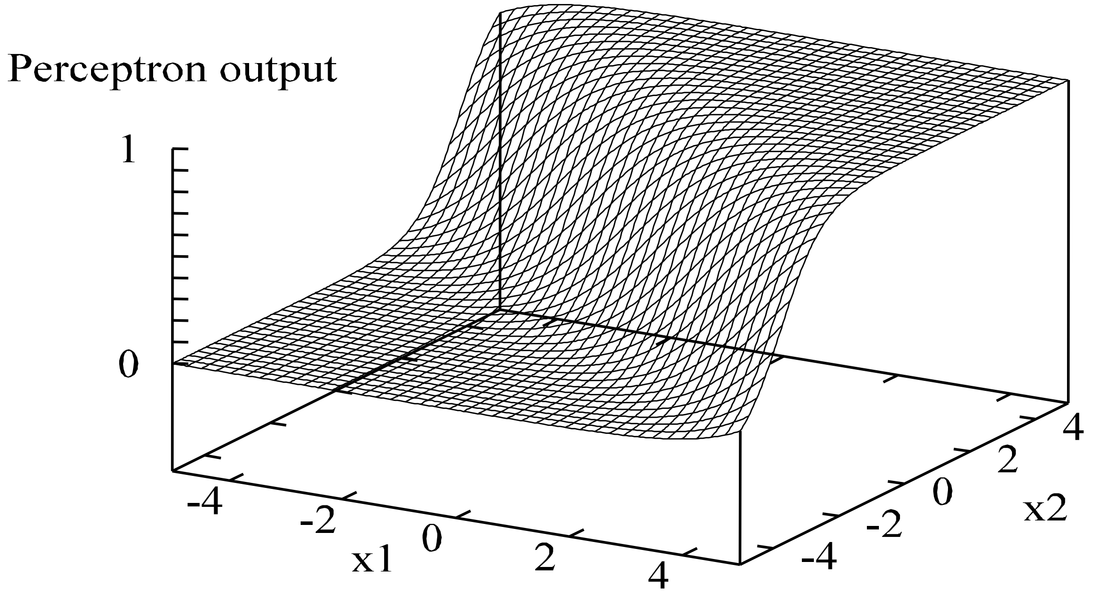
- Examples:

Linearly separable

Non-linearly separable

Sigmoid Perceptron

- Represent “soft” linear separators
- **Same hypothesis space as logistic regression**



Sigmoid Perceptron Learning

- Possible objectives
  - **Minimum squared error**

$$E(\boldsymbol{w}) = \frac{1}{2}\sum_n E_n(\boldsymbol{w})^2 = \frac{1}{2}\sum_n \left(y_n - \sigma\left(\boldsymbol{w}^T\bar{\boldsymbol{x}}_{\boldsymbol{n}}\right)\right)^2$$

  - **Maximum likelihood**
    - Same algorithm as for logistic regression
  - Maximum a posteriori hypothesis
  - Bayesian Learning

Gradient of Squared Error

- Gradient:

$$\frac{\partial E}{\partial w_i} = \sum_n E_n(w) \frac{\partial E_n}{\partial w_i}$$

$$= -\sum_n E_n(w) \sigma'(w^T \bar{x}_n) x_i$$

Recall that $\sigma' = \sigma(1-\sigma)$

$$= -\sum_n E_n(w) \sigma(w^T \bar{x}_n)\big(1 - \sigma(w^T \bar{x}_n)\big) x_i$$

Sequential Gradient Descent

Perceptron-Learning(training_set, network)

- Repeat
  - For each $(\boldsymbol{x_n}, y_n)$ in training_set do

$$E_n \leftarrow y_n - \sigma(\boldsymbol{w}^{\boldsymbol{T}}\bar{\boldsymbol{x}}_{\boldsymbol{n}})$$

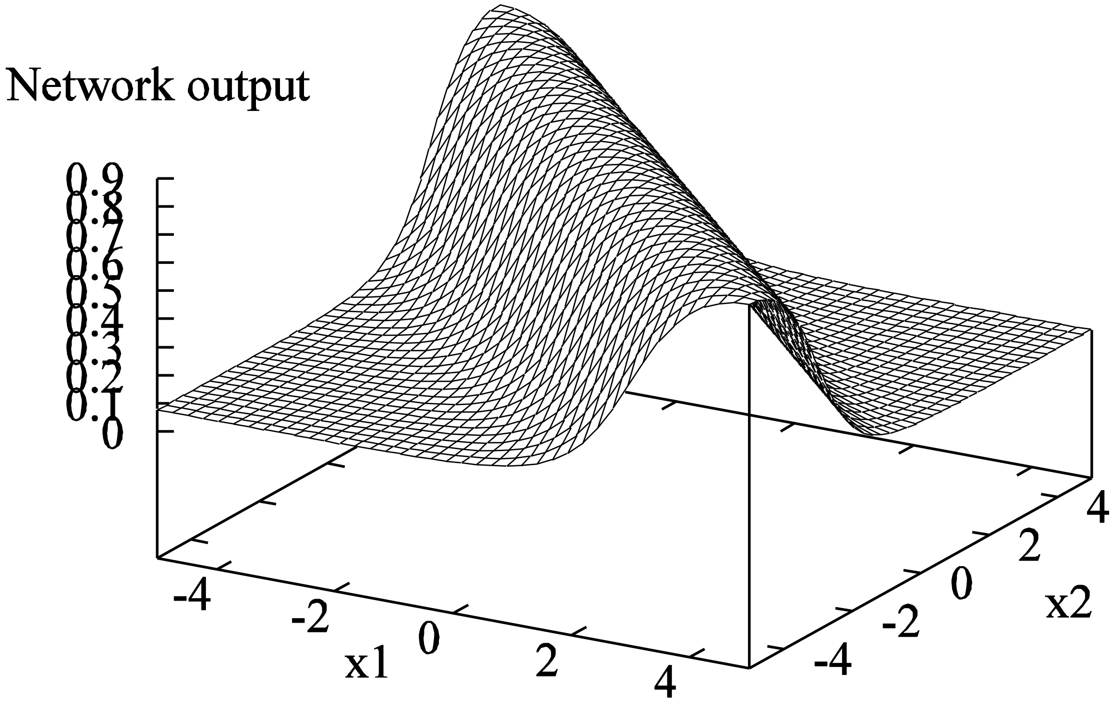
$$\boldsymbol{w} \leftarrow \boldsymbol{w} + \eta\, E_n\, \sigma\left(\boldsymbol{w}^{\boldsymbol{T}}\bar{\boldsymbol{x}}_{\boldsymbol{n}}\right)\left(1 - \sigma\left(\boldsymbol{w}^{\boldsymbol{T}}\bar{\boldsymbol{x}}_{\boldsymbol{n}}\right)\right)\, \bar{\boldsymbol{x}}_{\boldsymbol{n}}$$

- Until some stopping criterion satisfied
- Return learnt network

- N.B. η is a learning rate corresponding to the step size in gradient descent

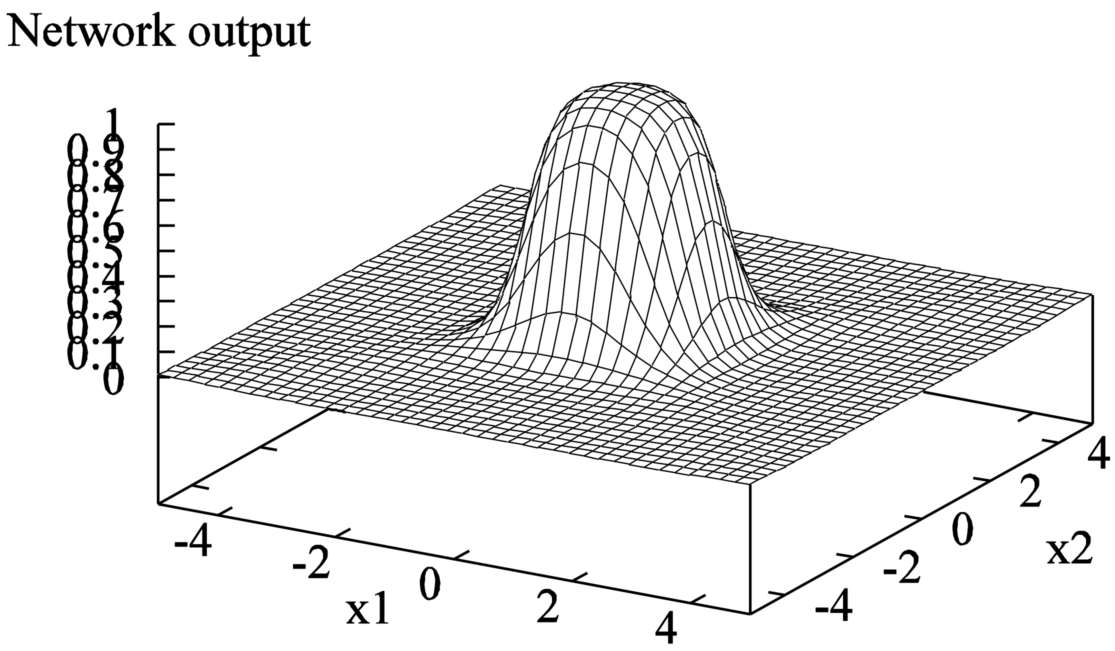
Multilayer Networks

- Adding two sigmoid units with parallel but opposite “cliffs” produces a ridge



Multilayer Networks

- Adding two intersecting ridges (and thresholding) produces a bump



Multilayer Networks

- By tiling bumps of various heights together, we can approximate any function.

- Training algorithm:
  - **Back-propagation** (gradient descent performed by propagating errors backward into the network)
  - Derivation next class

Neural Net Applications

- Neural nets can approximate any function, hence millions of applications
  - Speech recognition
  - Word embeddings
  - Machine translation
  - Vision-based object recognition
  - Vision-based autonomous driving
  - Etc.