Lecture 4: Statistical Learning
CS480/680 Intro to Machine Learning

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Pascal Poupart
David R. Cheriton School of Computer Science

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Statistical Learning

- View: we have uncertain knowledge of the world
- Idea: **learning simply reduces this uncertainty**

Terminology

- **Probability distribution:**
  - A specification of a probability for each event in our sample space
  - Probabilities must sum to 1

- Assume the world is described by two (or more) random variables
  - **Joint probability distribution**
    - Specification of probabilities for all combinations of events

Joint distribution

- Given two random variables A and B:
- Joint distribution:

  $\Pr(A = a \wedge B = b)$ for all a, b

- **Marginalisation (sumout rule):**

  $\Pr(A = a) = \Sigma_b \Pr(A = a \wedge B = b)$

  $\Pr(B = b) = \Sigma_a \Pr(A = a \wedge B = b)$

Example: Joint Distribution

sunny

	cold	~cold
headache	0.108	0.012
~headache	0.016	0.064

~sunny

	cold	~cold
headache	0.072	0.008
~headache	0.144	0.576

P(headache∧sunny∧cold) = 0.108

P(~headache∧sunny∧~cold) = 0.064

P(headache) = 0.108 + 0.012 + 0.072 + 0.008 = 0.2

marginalization

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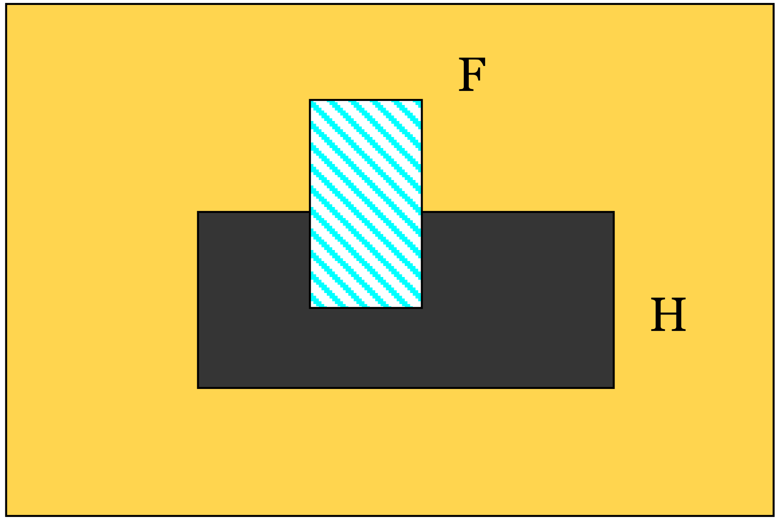
Conditional Probability

- $\Pr(A|B)$: fraction of worlds in which B is true that also have A true

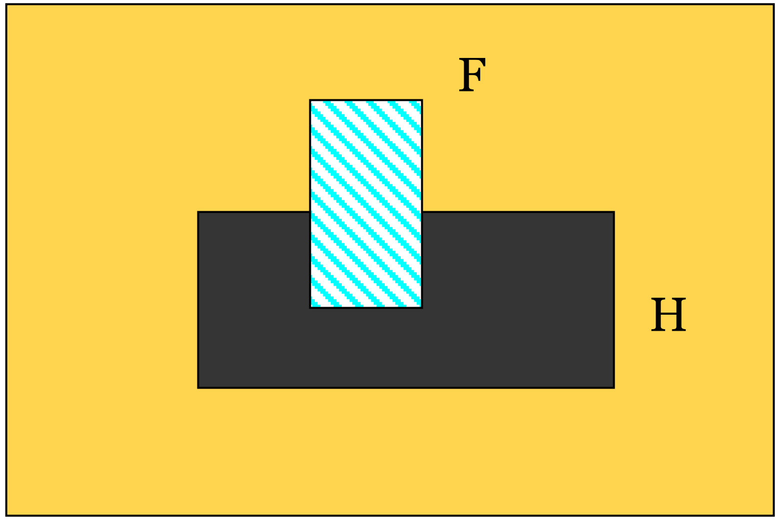


H="Have headache"
F="Have Flu"

$$\Pr(H) = 1/10$$
$$\Pr(F) = 1/40$$
$$\Pr(H|F) = 1/2$$

Headaches are rare and flu is rarer, but if you have the flu, then there is a 50-50 chance you will have a headache

Conditional Probability



H="Have headache"
F="Have Flu"

$$\Pr(H) = 1/10$$
$$\Pr(F) = 1/40$$
$$\Pr(H|F) = 1/2$$

$\Pr(H|F) =$ Fraction of flu inflicted worlds in which you have a headache
= (# worlds with flu and headache)/(# worlds with flu)
= (Area of "H and F" region)/(Area of "F" region)
= $\Pr(H \wedge F) \,/\, \Pr(F)$

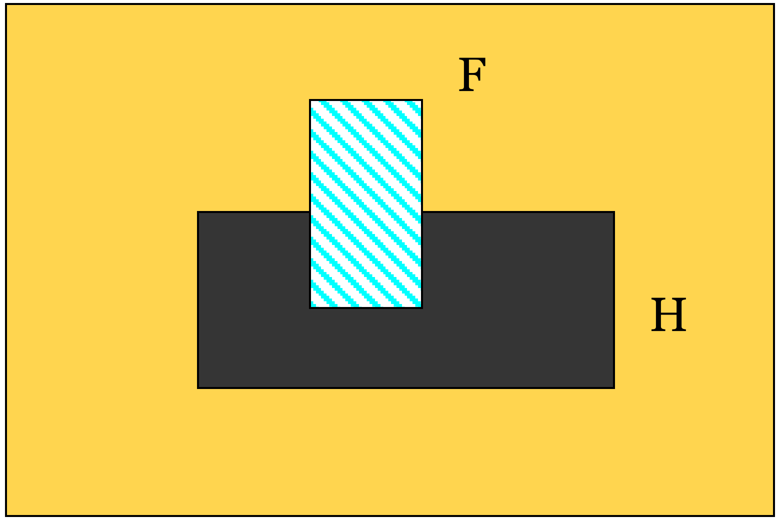
Conditional Probability

- Definition: $\Pr(A|B) = \Pr(A \wedge B) / \Pr(B)$

- Chain rule: $\Pr(A \wedge B) = \Pr(A|B)\Pr(B)$

Memorize these rules!

Inference



H="Have headache"
F="Have Flu"

$\Pr(H) = 1/10$
$\Pr(F) = 1/40$
$\Pr(H|F) = 1/2$

One day you wake up with a headache. You think "Drat! 50% of flues are associated with headaches so I must have a 50-50 chance of coming down with the flu"

Is your reasoning correct?

$\Pr(F \wedge H) = P(F)P(H|F) = \frac{1}{40}\left(\frac{1}{2}\right) = \frac{1}{80}$

$\Pr(F|H) = \frac{P(F,H)}{P(H)} = \frac{1/80}{1/10} = \frac{1}{8}$

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Example: Joint Distribution

sunny

	cold	~cold
headache	0.108	0.012
~headache	0.016	0.064

~sunny

	cold	~cold
headache	0.072	0.008
~headache	0.144	0.576

$$\Pr(headache \wedge cold \mid sunny) = \frac{P(h,c,s)}{P(s)} = \frac{0.108}{0.108+0.012+0.016+0.064} = 0.54$$

$$\Pr(headache \wedge cold \mid \sim sunny) = \frac{P(h,c,\sim s)}{P(\sim s)} = \frac{0.072}{0.072+0.008+0.144+0.576} = 0.09$$

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Bayes Rule

- Note: $\Pr(A|B)\Pr(B) = \Pr(A \Lambda B) = \Pr(B \Lambda A) = \Pr(B|A) Pr(A)$

- Bayes Rule: $\Pr(B|A) = \frac{\Pr(\mathrm{A}|\mathrm{B})\Pr(B)}{\Pr(A)}$

Memorize this!

Using Bayes Rule for inference

- Often, we want to form a hypothesis about the world based on what we have observed
- Bayes rule is vitally important when viewed in terms of stating the belief given to hypothesis H, given evidence e

Likelihood

Prior probability

$$P(H|e) = \frac{P(e|H)P(H)}{P(e)}$$

Posterior probability

Normalizing constant

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Bayesian Learning

- **Prior:** $\Pr(H)$
- **Likelihood:** $\Pr(e|H)$
- **Evidence:** $\boldsymbol{e} = < e_1, e_2, \ldots, e_N >$

- **Bayesian Learning** amounts to computing the posterior using Bayes' Theorem:

$$\Pr(H|\boldsymbol{e}) = k \Pr(\boldsymbol{e}|H)\Pr(H)$$

$1/P(e)$ normalization constant

Bayesian Prediction

- Suppose we want to make a prediction about an unknown quantity X

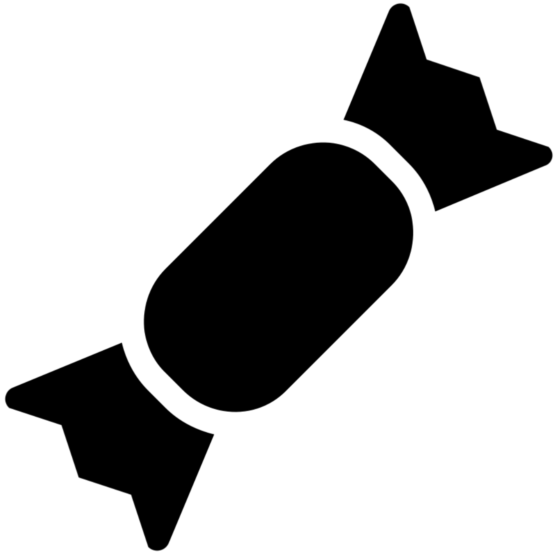
- $$\begin{aligned}\Pr(X|\boldsymbol{e}) &= \Sigma_i \Pr(X|\boldsymbol{e}, h_i)P(h_i|\boldsymbol{e}) \\ &= \Sigma_i \Pr(X|h_i)P(h_i|\boldsymbol{e})\end{aligned}$$

- Predictions are weighted averages of the predictions of the individual hypotheses

- Hypotheses serve as “intermediaries” between raw data and prediction

Candy Example

- Favorite candy sold in two flavors:
  - Lime (hugh)
  - Cherry (yum)
- Same wrapper for both flavors
- Sold in bags with different ratios:
  - 100% cherry
  - 75% cherry + 25% lime
  - 50% cherry + 50% lime
  - 25% cherry + 75% lime
  - 100% lime



Candy Example

- You bought a bag of candy but don't know its flavor ratio

- After eating k candies:
  - What's the flavor ratio of the bag?
  - What will be the flavor of the next candy?

Statistical Learning

- **Hypothesis H:** probabilistic theory of the world
  - h_1: 100% cherry
  - h_2: 75% cherry + 25% lime
  - h_3: 50% cherry + 50% lime
  - h_4: 25% cherry + 75% lime
  - h_5: 100% lime
- **Examples E:** evidence about the world
  - e_1: 1st candy is cherry
  - e_2: 2nd candy is lime
  - e_3: 3rd candy is lime
  - …

Candy Example

- Assume prior $\Pr(H) = < 0.1, 0.2, 0.4, 0.2, 0.1 >$
- Assume candies are **i.i.d. (identically and independently distributed)**

$$\Pr(\boldsymbol{e}|h) = \Pi_n P(e_n|h)$$

- Suppose first 10 candies all taste lime:

$$\Pr(\boldsymbol{e}|h_5) = 1^{10} = 1$$

$$\Pr(\boldsymbol{e}|h_3) = 0.5^{10} = 0.00097$$

$$\Pr(\boldsymbol{e}|h_1) = 0^{10} = 0$$

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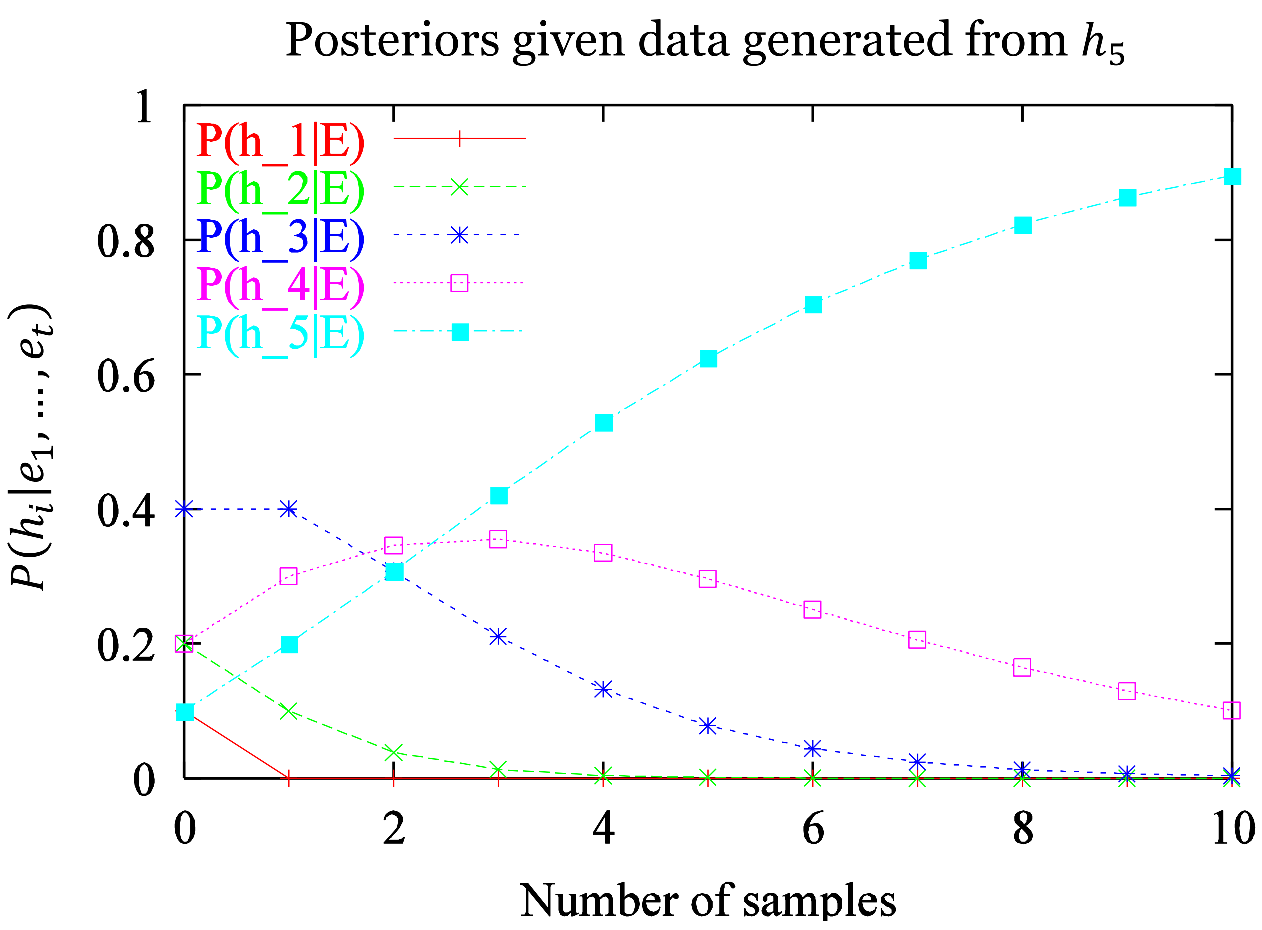
Posterior



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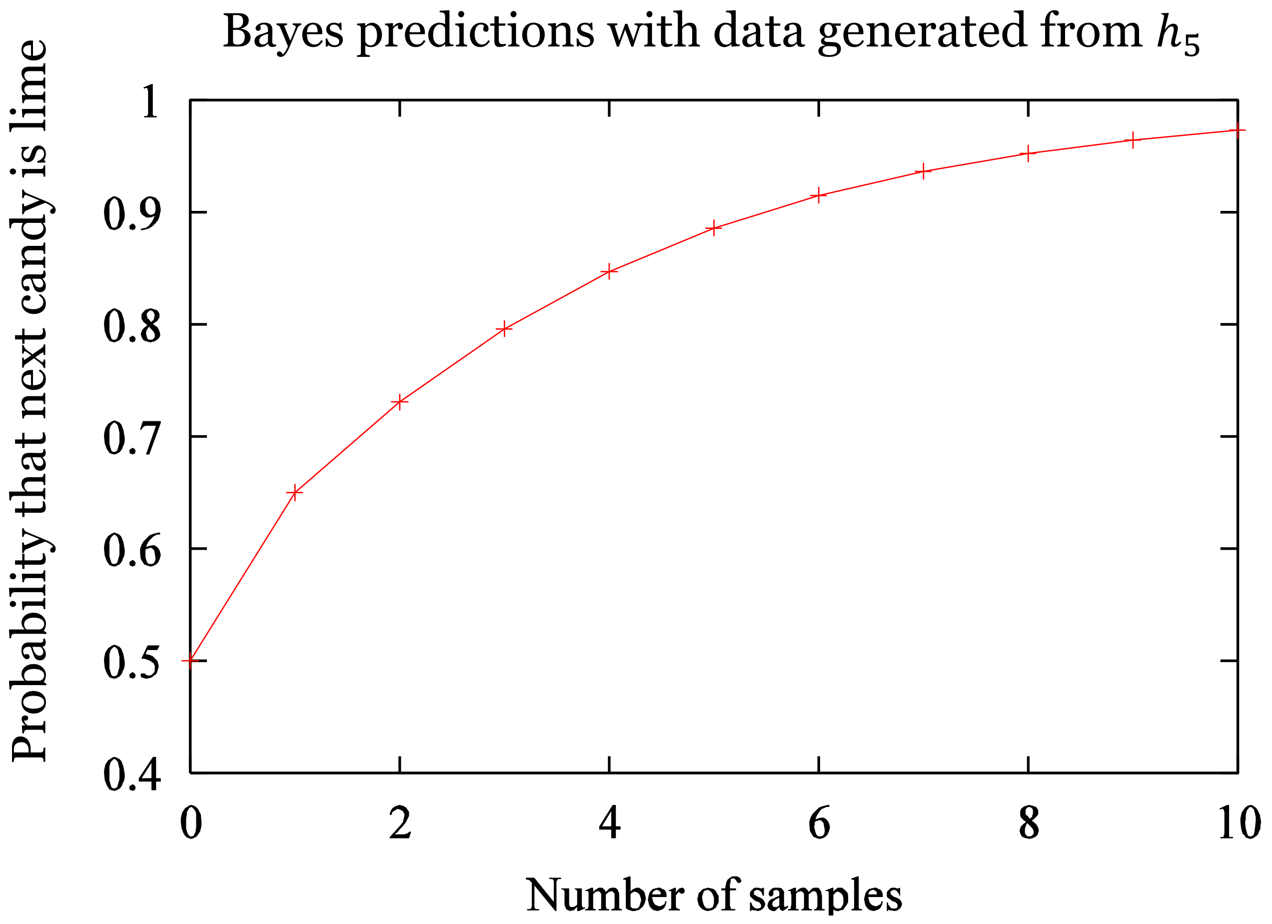
Prediction


Bayes predictions with data generated from h_5
Probability that next candy is lime
1
0.9
0.8
0.7
0.6
0.5
0.4
0
2
4
6
8
10
Number of samples

Bayesian Learning

- Bayesian learning properties:
  - **Optimal** (i.e., given prior, no other prediction is correct more often than the Bayesian one)
  - **No overfitting** (all hypotheses are considered and weighted)

- There is a price to pay:
  - When hypothesis space is large, Bayesian learning may be intractable
  - i.e., sum (or integral) over hypothesis often intractable
- Solution: approximate Bayesian learning

Maximum a posteriori (MAP)

- Idea: make prediction based on **most probable hypothesis** h_{MAP}

$$h_{MAP} = argmax_{h_i} \Pr(h_i|\boldsymbol{e})$$

$$\Pr(X|\boldsymbol{e}) \approx \Pr(X|h_{MAP})$$

- In contrast, Bayesian learning makes prediction based on **all** hypotheses weighted by their probability

MAP properties

- MAP prediction **less accurate** than Bayesian prediction since it relies only on **one** hypothesis h_{MAP}
- But MAP and Bayesian predictions converge as data increases
- **Controlled overfitting** (prior can be used to penalize complex hypotheses)

- **Finding h_{MAP} may be intractable:**
  - $h_{MAP} = argmax_h \Pr(h|\boldsymbol{e})$
  - Optimization may be difficult

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Maximum Likelihood (ML)

- Idea: simplify MAP by assuming uniform prior (i.e., $\Pr(h_i) = \Pr(h_j)\ \forall i, j$)

$$h_{MAP} = argmax_h \Pr(h) \Pr(\boldsymbol{e}|h)$$

$$h_{ML} = argmax_h \Pr(\boldsymbol{e}|h)$$

- Make prediction based on h_{ML} only:

$$\Pr(X|\boldsymbol{e}) \approx \Pr(X|h_{ML})$$

Maximum Likelihood (ML) properties

- ML prediction **less accurate** than Bayesian and MAP predictions since it ignores prior info and relies only on **one** hypothesis h_{ML}
- But ML, MAP and Bayesian predictions converge as data increases
- Subject to **overfitting** (no prior to penalize complex hypothesis that could exploit statistically insignificant data patterns)
- Finding h_{ML} is often easier than h_{MAP}

$$h_{ML} = argmax_h \Sigma_n \log \Pr(e_n|h)$$

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