CS475/CS675
Lecture 8: May 26, 2016

Iterative Methods

Reading: [Saad] Chapt 4

SOR Iteration

- Successive Over Relaxation (SOR)
- Weighted average of x_i^k and $x_i^{k+1} = GS\left(x_i^k\right)$

$$x_i^{k+1} = (1-w)x_i^k + \frac{w\left(b_i - \sum_{j<i} a_{ij}x_j^{k+1} - \sum_{j>i} a_{ij}x_j^k\right)}{a_{ii}}$$

$$\text{SOR} = \begin{cases} GS & w = 1 \\ underrelaxation & w < 1 \\ overrelaxation & w > 1 \end{cases}$$

- Select $M = \frac{1}{w}D - L$
- For a suitably chosen w (>1), SOR can be much better than GS

Convergence Analysis

- Q1: Under what conditions does the iteration converge?
- Q2: If the iteration converges, how fast is it?

- <u>Def:</u> λ is an eigenvalue and v an eigenvector of A if

$$Av = \lambda v \qquad (v \neq 0)$$

- <u>Def:</u> the spectral radius of A, $\rho(A)$, is the largest absolute value of the eigenvalues of A

Convergence Analysis

- Theorem: the iterative method
$$x^{k+1} = x^k + M^{-1}\left(b - Ax^k\right)$$
is convergent for any x_0 and b if and only if
$$\rho(I - M^{-1}A) < 1$$

- $I - M^{-1}A$ is called the iteration matrix
- $\rho(I - M^{-1}A)$ is called the rate of convergence

Minimization Formulation

- Assume A is SPD. Consider the functional:
  $F(x) \equiv \frac{1}{2} x^T A x - b^T x \qquad x \in \Re^n$

- <u>Theorem:</u> The solution of $Ax = b$ is equivalent to the solution of the minimization problem:
  $$\min_x F(x)$$

Minimization Formulation

- Proof: the minimizer satisfies $F'(x) = 0$ i.e. $\frac{\partial F}{\partial x_k} = 0$
  - Note $F(x) = \frac{1}{2}\sum_{ij} a_{ij}x_i x_j - \sum_i b_i x_i$
  - Hence $\frac{\partial F}{\partial x_k} =$

Minimization Formulation

- Since $F(x)$ is convex, then local min = global min
- Picture:

Search Directions

- Idea: minimize $F(x)$ along the direction $p \neq 0$
- Let x^k = current approximation
- Define $x^{k+1} = x^k + \alpha p$

- Determine α by $\min F(x^{k+1})$ along p
  - i.e., $\min_{\alpha} F(x^k + \alpha p)$

Search Directions

- Let $f(\alpha) \equiv F\left(x^k + \alpha p\right)$
  $=$

- Hence $0 = f'(\alpha) =$
  and $\alpha =$

Search Directions

- Notes
  1. A is SPD $\Longrightarrow p^T A p > 0$
  2. What is the optimal search direction p?
- Picture:

Steepest Descent

- Local optimal direction
- Consider $f'(\alpha) = p^T\left(Ax^k - b\right) + \alpha p^T A p$
- Then $f'(0) = p^T F'\left(x^k\right)$
  $= \text{changes in } F \text{ at } x^k \text{ in the direction } p$
- Idea: make $f'(0)$ as negative as possible by varying p

Steepest Descent

- Assume $||p||_2 = 1$. Then

$$f'(0) \text{ is max if } p = \frac{F'(x^k)}{||F'(x^k)||} = \text{steepest ascent}$$

$$f'(0) \text{ is min if } p = -\frac{F'(x^k)}{||F'(x^k)||} = \text{steepest descent}$$

$$= \frac{r^k}{||r^k||} \qquad \text{(where } F'(x) = -r\text{)}$$

Steepest Descent

- Steepest descent method:
  $x^{k+1} = x^k + \alpha_k r^k$ $\qquad\qquad$ α_k = step length
- The optimal $\alpha_k = \frac{(r^k)^T r^k}{(r^k)^T A r^k}$ $\qquad\qquad$ $(\alpha = p^T r^k / p^T A p)$
- Also $r^{k+1} =$

Steepest Descent

- <u>Algorithm</u>

  Given x^0, compute $r^0 = b - Ax^0$

  for $k = 0,1,2, \ldots$

  $$\alpha_k = \frac{\left(r^k\right)^T r^k}{\left(r^k\right)^T A r^k}$$

  $$x^{k+1} = x^k + \alpha_k r^k$$

  $$r^{k+1} = r^k - \alpha_k A r^k$$

  end

Steepest Descent

- Notes
  1. Only 1 matrix-vector product (Ar^k) per iteration
  2. “Nonlinear” iterative method:

$$x^{k+1} = x^k + \alpha_k(b - Ax^k)$$

i.e., $M = M^k = \frac{1}{\alpha_k} I$