CS475 / CS675
Lecture 6: May 19

Stability

Reading: [TB] Chapt 22, p. 163-170

Stability of Factorization

- Problem arises when
  1. Zero pivot: $a_{kk}^{(k-1)} = 0$ or
  2. Small pivot: $a_{kk}^{(k-1)} \approx 0$
- Pivoting (picture):
  - Complete pivoting: search for largest element in A_{22}
  - Partial pivoting: search for largest element in column k

Modified LU factorization

- Modified LU factorization: $PA = LU$ where P is a permutation matrix obtained from swapping during partial pivoting
  - E.g. permutation P when swapping rows 2 and 4
  - Pivoting is stable, but may introduce fill in sparse systems
  - Some conditions on A can ensure pivoting is not necessary

Stable GE when A is SPD

- Theorem: Suppose A is SPD. Then during GE, $a_{kk}^{(k-1)} > 0$.
- Proof:
  - For $n = 1$, obviously true.
  - In general, use inductive argument
  - Write $A = \begin{bmatrix} a_{11} & v^T \\ v & A_{22} \end{bmatrix}$
  - Since A is SPD, $a_{11} > 0$
  - Eliminate v using a_{11} as pivot: $\begin{bmatrix} a_{11} & v^T \\ v & A_{22} \end{bmatrix} \xrightarrow{\text{GE}} \begin{bmatrix} a_{11} & v^T \\ 0 & A_{22} - \frac{vv^T}{a_{11}} \end{bmatrix}$

Proof continued

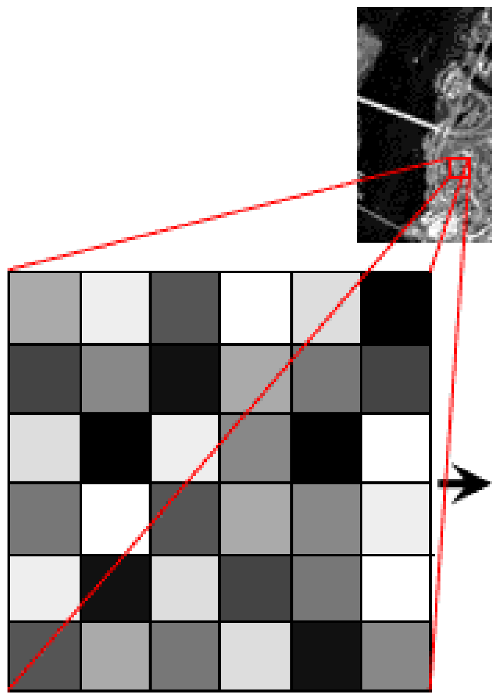
- Let $A_{22}^{(1)} = A_{22} - vv^T/a_{11}$. Note $A_{22}^{(1)}$ is symmetric.
- Need to show that $A_{22}^{(1)}$ is SPD.
- Let $x \in \Re^{n-1}, x \neq 0$. Consider $y = \begin{bmatrix} -x^T v/a_{11} \\ x \end{bmatrix} \in \Re^n$
- A SPD $\Rightarrow y^T A y > 0$
  - i.e. $0 < \begin{bmatrix} -\frac{x^T v}{a_{11}} & x^T \end{bmatrix} \begin{bmatrix} a_{11} & v^T \\ v & A_{22} \end{bmatrix} \begin{bmatrix} -\frac{x^T v}{a_{11}} \\ x \end{bmatrix}$
    $= \begin{bmatrix} 0 & x^T A_{22} - \frac{x^T v v^T}{a_{11}} \end{bmatrix} \begin{bmatrix} -\frac{x^T v}{a_{11}} \\ x \end{bmatrix}$
    $= x^T A_{22} x - x^T (vv^T) x/a_{11}$
    $= x^T (A_{22} - vv^T/a_{11}) x = x^T A_{22}^{(1)} x$
  - $\therefore A_{22}^{(1)}$ is SPD $\Longrightarrow \mathrm{a}_{22}^{(1)} > 0 \Longrightarrow$ pivot is positive
- Continuing this process, one can show that $a_{kk}^{(k-1)} > 0 \ \ \forall k$

Other conditions that ensure stable GE

- Other matrices for which pivoting is not necessary:
  - Row diagonally dominant
    i.e., $|a_{kk}| > \sum_{j \neq k} |a_{kj}| \quad k = 1, \ldots, n$
  - Column diagonally dominant
    i.e., $|a_{kk}| > \sum_{i \neq k} |a_{ik}| \quad k = 1, \ldots, n$

Images

- Images are treated as 2D functions:
  - u_{ij} = pixel value at row i, column j



170	238	85	255	221	0
68	136	17	170	119	68
221	0	238	136	0	255
119	255	85	170	136	238
238	17	221	68	119	255
85	170	119	221	17	136

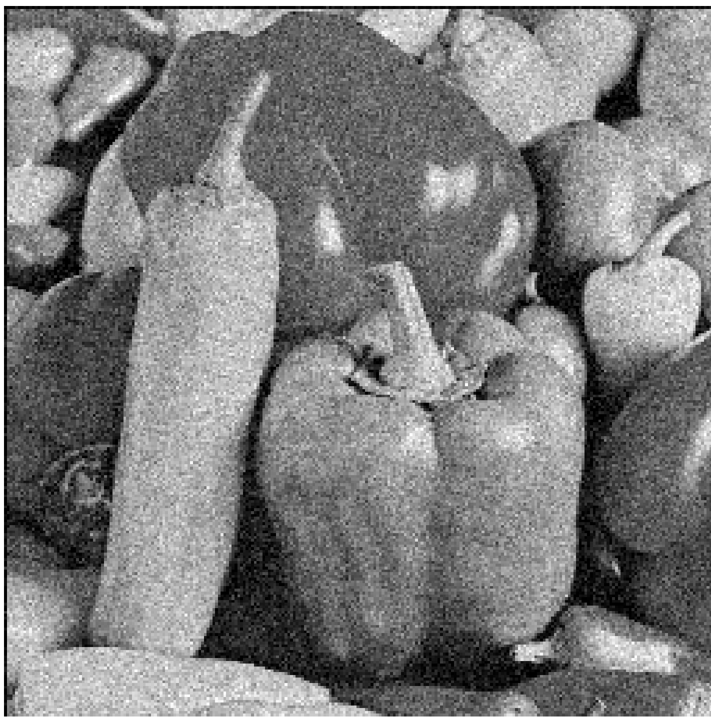
Image Denoising

- Inverse problem: given
  1. The observed image: $u^0 = u^* + n$ ($n =$ noise)
  2. Estimate of variance of noise: $||n||_2^2 = \sigma^2$

- Find an approximation to the original image u^*

Example

Noisy image



Denoised image

Optimization Approach

- Solve min "fluctuation of pixel values" subject to "noise constraint level"
- The objective function is to get rid of the noise
- The constraint is necessary to ensure that you don't get an image of constant pixel values
  - i.e., $||u - u^0||_2^2 \approx \sigma^2 = ||u^* - u^0||_2^2$
  - The noise level of u^0 w.r.t. u is the same as that w.r.t. u^*
- Ill-posed problem: many images u satisfy constraint
  - Need a selection criterion → regularization

Regularization model

- Noise level constraint optimization problem:
  $$\min_u R(u)$$
  subject to $\left||u - u^0|\right|_2^2 = \sigma^2$
- Equivalently: $\min_u f(u) \equiv \alpha R(u) + \left||u - u^0|\right|_2^2$
  – α measures tradeoff between fit and regularity
  – If $\alpha \approx 0$, then $u \approx u^0$
  – If $\alpha \approx \infty$, then $u \approx$ constant
- What is $R(u)$?
  – The idea is to minimize the fluctuation of u

Tikhonov regularization

- Let $R(u) = \int_\Omega |u(x)|^2 dx = ||u||_2^2$
- Then $\min_u \alpha ||u||_2^2 + ||u - u^0||_2^2$
- Solution: Euler-Lagrange equation

$$\alpha u + (u - u^0) = 0$$
$$(\alpha + 1)u = u^0$$
$$u = \frac{1}{\alpha+1} u^0$$

- Hence

$$\alpha \approx 0 \rightarrow u \approx u^0$$
$$\alpha \approx \infty \rightarrow u \approx 0$$

Laplacian Regularization

- Let $R(u) = \int_\Omega |\nabla u|^2 dx = ||\nabla u||_2^2$ $\quad$ (∇ $\rightarrow$ gradient)
- Then $\min_u \alpha ||\nabla u||_2^2 + ||u - u^0||_2^2$
- The idea is to have small slopes, not small pixel values
  - For noisy images, slopes are large

Laplacian Regularization

- Solution: Euler-Lagrange equation

  $-\alpha\Delta u + (u - u^0) = 0$ (Δ → Hessian)

  $-\alpha\Delta u + u = u^0$

- Finite difference approximation

  $\frac{\alpha}{h^2}\left(4u_{ij} - u_{i-1,j} - u_{i+1,j} - u_{i,j-1} - u_{i,j+1}\right) + u_{ij} = u^0_{ij}$

  matrix form: $\alpha A u + u = u^0$

  $(\alpha A + I)u = u^0$

Laplacian Regularization

- If the solution u is still too noisy, repeat the procedure:

  for $k = 0,1, \ldots, K$

  solve $(\alpha A + I)u^{k+1} = u^k$

  end

- Drawback: it tends to smear edges

Example