CS475/CS675
Lecture 4: May 12, 2016

Sparse Gaussian Elimination,

Graph Representation

Reading: [Saad] Sect 3.1-3.2

5-Point Stencil

- An easy way to denote 2D finite difference equations

$$\begin{bmatrix} 0 & -\frac{1}{h^2} & 0 \\ -\frac{1}{h^2} & \frac{4}{h^2} & -\frac{1}{h^2} \\ 0 & -\frac{1}{h^2} & 0 \end{bmatrix}$$

$$\begin{array}{ccccc} & & T_{i,j+1} & & \\ & & | & & \\ T_{i-1,j} & — & T_{i,j} & — & T_{i+1,j} \\ & & | & & \\ & & T_{i,j-1} & & \end{array}$$

Numbering of unknowns

- Picture:

- Note: the values on the boundary are zero

- The unknowns are: $$\begin{matrix} T_{1,1} & T_{2,1} & \cdots & T_{m,1} \\ T_{1,2} & T_{2,2} & \cdots & T_{m,2} \\ \vdots & \vdots & \ddots & \vdots \\ T_{1,m} & T_{2,m} & \cdots & T_{m,m} \end{matrix}$$

- Total number $= m \times m = m^2 \equiv n$

Natural ordering

- Ordering: first in the x-direction, then y-direction
  - i.e., $T_{1,1}, T_{2,1}, \dots, T_{m,1}; T_{1,2}, T_{2,2}, \dots$

- The system of linear equations

$$
\begin{aligned}
&i = 1, j = 1: \quad \frac{4}{h^2}T_{1,1} - \frac{1}{h^2}T_{2,1} - \frac{1}{h^2}T_{1,2} = f_{1,1} \\
&i = 2, j = 1: \; -\frac{1}{h^2}T_{1,1} + \frac{4}{h^2}T_{2,1} - \frac{1}{h^2}T_{3,1} - \frac{1}{h^2}T_{2,2} = f_{2,1} \\
&\vdots \\
&i = m, j = m: -\frac{1}{h^2}T_{m,m-1} - \frac{1}{h^2}T_{m-1,m} + \frac{4}{h^2}T_{m,m} = f_{m,m}
\end{aligned}
$$

Matrix Form

- Example $(m = 4, n = 16)$

Graph Representation of Matrices

- Given a sparse matrix A, a node is associated with each row.
- If $a_{i,j} \neq 0$, there exists an edge from node i to j

$$A = \begin{bmatrix} \times & \times & & \\ & \times & \times & \\ & \times & \times & \times \\ \times & & & \times \end{bmatrix} \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} \qquad G(A):$$

Graph for Symmetric Matrices

- For symmetric matrices, arrows can be dropped (as well as self loops)

$$A = \begin{bmatrix} \times & \times & & \times \\ \times & \times & \times & \\ & \times & \times & \times \\ \times & & \times & \times \end{bmatrix} \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} \qquad G(A):$$

Physical/Geometric Interpretation

- Graph of a matrix often has a simple physical/geometric interpretation
  - 1D Laplacian

    $A =$ $G(A)$:

  - 2D Laplacian

    $A =$ $G(A)$:

GE and Matrix Graph

- "Visualize" eliminations by matrix graph

e.g. $\begin{bmatrix} \times & \times & & \times \\ \times & \times & \times & \\ & \times & \times & \times \\ \times & & \times & \times \end{bmatrix} \xrightarrow{\text{GE}} \begin{bmatrix} \times & \times & & \times \\ 0 & \times & \times & \blacksquare \\ & \times & \times & \times \\ 0 & \blacksquare & \times & \times \end{bmatrix}$

fill-in

$G(A)$:

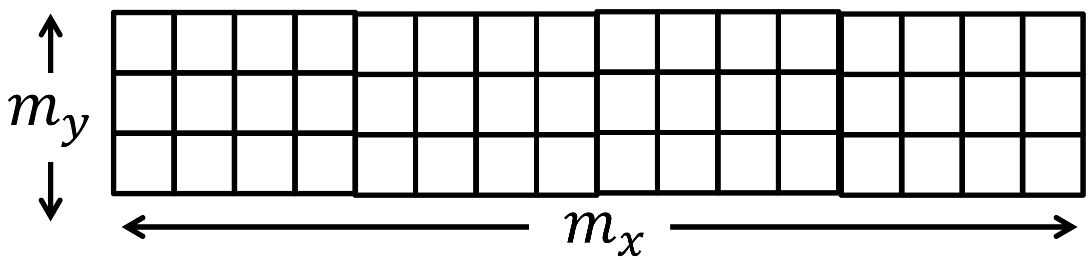
GE and Matrix Graph

- Elimination of node i produces a new graph with
  - Node i deleted, all edges containing node i deleted
  - New edge (j, k) added (fill-in) if there was an edge (i, j) & (i, k) in the old graph.

- Notes
  - Matrix (with symmetric structure) graph is unchanged by renumbering of the nodes
  - But orderings (which nodes to be removed first) may result in much less fill during GE.

Ordering Algorithms

- Consider the following matrix graph:



- Assume $m_x \gg m_y$. If we use the natural ordering, what would the matrix look like?

Ordering Algorithms

- If we had numbered along y-direction first, the matrix becomes:

- Which ordering results in less fill? Why?

Band Matrices

- Note: GE preserves band structure
  - Picture:

- Amount of work to factor a band matrix:
  - $O(m^2 n)$ where $m =$ bandwidth
  - x-first ordering → $flops(GE) = O(m_x^2 n)$
  - y-first ordering → $flops(GE) = O(m_y^2 n)$

Envelope Methods

- In general, bandwidth is not the same for each row
  - Example:

- In each row, fill can occur only between the 1^{st} nonzero entry and the diagonal.
- To limit the amount of fill, keep the envelope as close to the diagonal as possible

Envelope Methods

- Try to number nodes so that graph neighbours have numbers as close together as possible
  - Example: