CS475/CS675
Lecture 3: May 10, 2016

1D, 2D Laplacian

Reading: [Saad] chapt 2

Sparse Matrices

1. Usually a constant number of nonzeros per row
   i.e., O(n) number of nonzero entries
   - Store only the nonzero entries

2. In GE/LU, the main computation:
   $a_{ij} = a_{ij} - a_{ik}a_{kj}/a_{kk}$
   $= 0 - 0 \times 0$ (most entries are 0)
   - Never operate on zeros

Sparse Matrices

3. A is sparse, but L and U can be dense (e.g., "Arrow" matrix)

$A =$

- The storage for L & U is $O(n^2)$
- Computation of LU is $O(n^3)$
- Use a different ordering of unknowns: $\tilde{x}_1 = x_2, \tilde{x}_2 = x_3, \ldots, \tilde{x}_n = x_1$

then $A =$

Application: heat conduction

- Heat conduction can be modeled by a partial differential equation (PDE):

$$-\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) = f(x, y, z)$$

where $x, y, z =$ coordinates over known range
$T(x, y, z) =$ temperature at (x, y, z)
$f(x, y, z) =$ source function

One dimension: heat conduction

- PDE: $-\frac{\partial^2 T}{\partial x^2} = f(x)$
- Picture:

How to Compute T?

- Divide interval $[0,1]$ into subintervals:
$$0 = x_0 < x_1 < x_2 \ldots < x_{n+1} = 1$$
$\{x_i\}$ are called grid points.
- Approximate temperature T at x_i : $T_i \approx T(x_i)$
  - Picture:

Assumptions

1. We assume temp $= 0$ at both ends
   – i.e., $T_0 = T_{n+1} = 0$
   – Thus the unknowns are $T_1, T_2, \ldots, T_n$

2. Uniform spacing:

   $$h = x_i - x_{i-1} = \frac{1}{n+1}$$

   $=$ grid size / mesh size

Finite Difference Approximation

- $\frac{\partial T}{\partial x}(x_i^-) \approx \frac{T_i - T_{i-1}}{h}$ (backward difference)
- $\frac{\partial T}{\partial x}(x_i^+) \approx \frac{T_{i+1} - T_i}{h}$ (forward difference)
- $\frac{\partial^2 T}{\partial x^2}(x_i) \approx \frac{\frac{\partial T}{\partial x}(x_i^+) - \frac{\partial T}{\partial x}(x_i^-)}{h}$

$$= \frac{\frac{T_{i+1} - T_i}{h} - \frac{T_i - T_{i-1}}{h}}{h}$$

$$= \frac{T_{i-1} - 2T_i + T_{i+1}}{h^2}$$ (central difference)

Finite Difference Approximation

- For each $x_i, i = 1,2, \ldots, n$, we have one equation:

$$-\frac{T_{i-1} - 2T_i + T_{i+1}}{h^2} = f_i \qquad (f_i = f(x_i))$$

i.e., $$-\frac{1}{h^2}T_{i-1} + \frac{2}{h^2}T_i - \frac{1}{h^2}T_{i+1} = f_i \qquad i = 1, \ldots, n$$

- This is a system of linear equations

Example

- For $n = 4$

$$
\begin{aligned}
i = 1 \quad & -\frac{1}{h^2}T_0 + \frac{2}{h^2}T_1 - \frac{1}{h^2}T_2 + 0\ T_3 + 0\ T_4 && = f_1 \\
i = 2 \quad & -\frac{1}{h^2}T_1 + \frac{2}{h^2}T_2 - \frac{1}{h^2}T_3 + 0\ T_4 && = f_2 \\
i = 3 \quad & 0\ T_1 - \frac{1}{h^2}T_2 + \frac{2}{h^2}T_3 - \frac{1}{h^2}T_4 && = f_3 \\
i = 4 \quad & 0\ T_1 + 0\ T_2 - \frac{1}{h^2}T_3 + \frac{2}{h^2}T_4 - \frac{1}{h^2}T_5 && = f_4
\end{aligned}
$$

- Matrix form:

Example

- General matrix form:

- This is the 1D Laplacian matrix
  - It is tridiagonal

Two dimensional heat conduction

- PDE: $-\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) = f(x, y)$
  - Picture:

- 2D Computational grid:

2D Finite Difference Approximation

- Approximate temp $T(x_i, y_j)$ by T_{ij}

$$-\frac{T_{i-1,j}-2T_{i,j}+T_{i+1,j}}{h^2}-\frac{T_{i,j-1}-2T_{i,j}+T_{i,j+1}}{h^2}=f_{i,j}$$

$$i,j=1,2,\ldots,m$$

$$\frac{4}{h^2}T_{ij}-\frac{1}{h^2}T_{i-1,j}-\frac{1}{h^2}T_{i+1,j}-\frac{1}{h^2}T_{i,j-1}-\frac{1}{h^2}T_{i,j+1}=f_{i,j}$$

$$i,j=1,2,\ldots,m$$