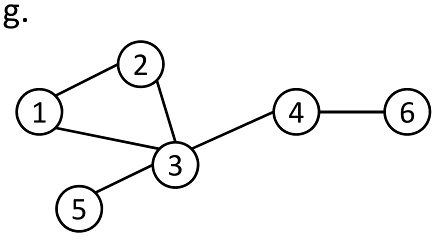
CS475/CM375
Lecture 5: Sept. 27

Ordering Methods

Reading: [Saad] Sect 3.3

Degree

- Definition: Degree of a node = # of nodes adjacent to a given node
  - E.g.



deg(1) = 2
deg(3) =
deg(5) =

Level Sets

- Envelope ordering strategies are often based on level sets S_i
  - $S_1 \rightarrow$ consists of a single node, the starting node
  - $S_2 \rightarrow$ all (graph) neighbours of the node in S_1
  - $S_3 \rightarrow$ all neighbours of nodes in S_2 that are not in S_1, S_2.
- In general, S_i consists of all neighbours of S_{i-1} that are not in $S_1, S_2, \ldots, S_{i-2}$.
- Ordering: nodes in S_1, nodes in S_2, etc.

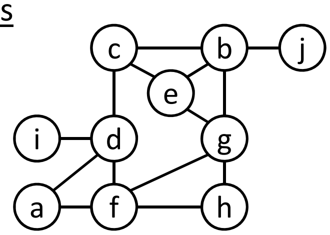
Cuthill-McKee Ordering (1969)

1. Determine starting node
2. For $i = 1, \ldots, n$ find all unnumbered neighbours of node i and number them in order of degree (smallest first). Surprisingly, the reverse ordering is better, so add 3)
3. Reverse Cuthill McKee (RCM, 1971, George)

$$node_i^{RCM} = node_{n-i+1}^{CM} \qquad i = 1,2, \ldots, n$$

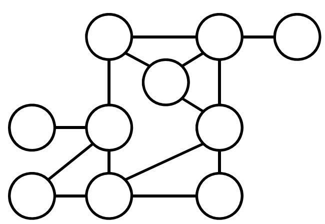
Example 1

Node #	node	unnumbered neighbours
1	g	
2		
3		
4		
5		
6		
7		
8		
9		
10		

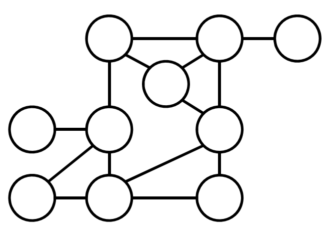


Example 1

- CM ordering

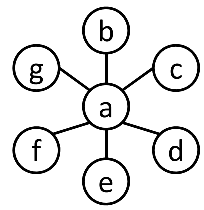


- RCM ordering



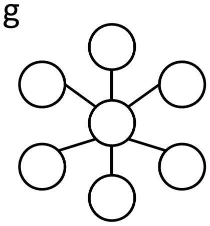
Example 2

Node #	node	unnumbered neighbours
1	b	
2		
3		
4		
5		
6		
7		

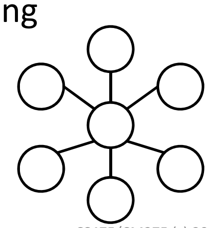


Example 2

- CM ordering matrix

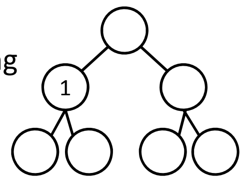


- RCM ordering matrix

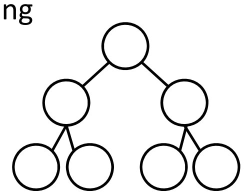


RCM Properties

- If graph is a tree, then no matter what node you start with, RCM ordering produces no fill (not true for CM)
- Example 3:
  - CM ordering matrix



  - RCM ordering matrix

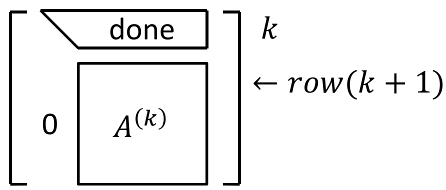


RCM properties

- Notes:
  1. RCM does not necessarily produce an optimal ordering (i.e., ordering which introduces least amount of fill)
  2. In general, NP-complete problem to find optimal ordering

Local Strategy (Markowitz 1957)

- Min fill-in only for the current step of GE
  - E.g., after k steps of GE:



Local Strategy (Markowitz 1957)

- Let $r_i^{(k)}$ = number of entries in row i of $A^{(k)}$
  $c_j^{(k)}$ = number of entries in col j of $A^{(k)}$
- Then the max possible amount of fill is:

$$\left(r_i^{(k)} - 1\right)\left(c_j^{(k)} - 1\right)$$

  - E.g.

Local Strategy (Markowitz 1957)

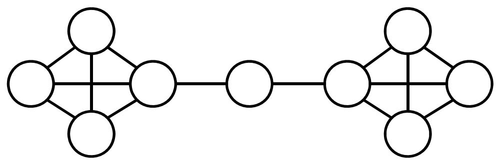
- Markowitz strategy: select $a_{ij}^{(k)}$ that minimizes
$$\left(r_i^{(k)} - 1\right)\left(c_j^{(k)} - 1\right)$$
- Note: different from $r_i^{(k)} c_j^{(k)}$,
which prefers $r_i = 1$ or $c_j = 1$.
- For symmetric structure, $\min_i r_i^{(k)} = \min_j c_j^{(k)}$
  - ∴ We find node i, $k + 1 \leq i \leq n$ such that $\min_i r_i^{(k)} - 1$
  - Then we use $a_{ii}^{(k)}$ as the pivot

Minimum degree ordering

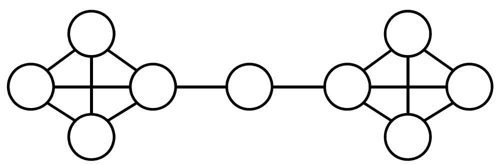
- In the matrix graph, choose the node with least edges connected to it.
- A tree graph will be ordered correctly.
- It is a local strategy; no guarantee it will produce the least amount of total fill

Example

- Minimum degree ordering amount of fill



- Optimal ordering amount of fill



Tie-breaking

1. Select the node that has the smallest node number in original order
2. RCM preordering: minimum degree
   Tie broken by selecting earlier RCM ordered node