CS475 / CM375
Lecture 23: Nov 29, 2011

Convergence of Iterative Methods

Richardson Convergence

- The iteration matrix is given by:
  $$G^{Rich} \equiv I - M_{Rich}^{-1} A = I - (\theta I) A = I - \theta A$$
- Suppose (λ, v) is an eigenpair of A. Then
  $$G^{Rich} v = (I - \theta A) v = v - \theta \lambda v = (1 - \theta \lambda) v$$
- Hence $\mu \equiv 1 - \theta \lambda$ is an eigenvalue of G^{Rich}

Richardson Convergence

- Lemma: Let λ_{min} and λ_{max} be the smallest and largest eigenvalue of A. Then
  $$\rho\left(G^{Rich}\right) = \max\ \{|1 - \theta\lambda_{min}|, |1 - \theta\lambda_{max}|\}$$
- Proof:

Richardson Convergence

- Notes:
  - If $\lambda_{min} < 0$ and $\lambda_{max} > 0$, then either
    $$1 - \theta\lambda_{min} > 1 \quad (\theta > 0) \quad \text{or}$$
    $$1 - \theta\lambda_{max} > 1 \quad (\theta < 0)$$

  $\Longrightarrow \rho\left(G^{Rich}\right) > 1$

  $\Longrightarrow$ Richardson method diverges

Richardson Convergence

- <u>Theorem:</u> Assume all eigenvalues of A are positive. Then Richardson converges if and only if
$$0 < \theta < 2/\lambda_{max}$$
- <u>Proof:</u> if $0 < \theta < 2/\lambda_{max}$, then

Richardson Convergence

- <u>Proof continued:</u> Assume $\rho\left(G^{Rich}\right) < 1$. Then

Richardson Convergence

- $\theta_{opt}: -(1-\theta\lambda_{max}) = 1-\theta\lambda_{min}$
  $$\theta_{opt} = 2/(\lambda_{min}+\lambda_{\max})$$
- $\rho_{opt} = 1-\theta_{opt}\lambda_{min} = \frac{\lambda_{max}-\lambda_{min}}{\lambda_{max}+\lambda_{min}}$
  $$= \frac{\lambda_{max}/\lambda_{min}-1}{\lambda_{max}/\lambda_{min}+1} = \frac{\kappa(A)-1}{\kappa(A)+1}$$
- Picture:

Jacobi Convergence

- <u>Theorem:</u> If A and $2D-A$ are SPD, then Jacobi converges
- <u>Proof:</u> Let μ be an eigenval. of $I-M_J^{-1}A = I-D^{-1}A$

Jacobi Convergence

- Proof continued: Since $2D - A$ is SPD,

Gauss-Seidel & SOR Convergence

- Theorem: If A is SPD, then GS & SOR $(0 < w < 2)$ converge.
- Definition: A is an M-matrix if
  i. $a_{ii} > 0$
  ii. $a_{ij} < 0$
  iii. A^{-1} exists and $(A^{-1})_{ij} > 0 \quad \forall_{ij}$
- Theorem: If A is an M-matrix, Jacobi and GS converge. Moreover

$$\rho(I - M_{GS}^{-1}A) \leq \rho\left(I - M_J^{-1}A\right) < 1$$

  i.e. the convergence rate of GS is better than that of Jacobi

Example: Poisson Equation

- Recall: $(-u_{xx} - u_{yy} = f)$
- Theorem: Let A be the 2D Laplacian matrix. The eigenvalues of A are given by:

$$\lambda_{ij} = \frac{4}{h^2}\left[\sin^2\left(\frac{i\pi h}{2}\right) + \sin^2\left(\frac{j\pi h}{2}\right)\right] \quad 1 \le i, j \le m$$

Example: Poisson Equation

- The smallest eigenvalue is attained for $i = j = 1$

$$\lambda_{min} = \frac{8}{h^2}\sin^2\left(\frac{\pi h}{2}\right)$$

- The largest eigenvalue is attained for $i = j = m$

$$\lambda_{max} = \frac{8}{h^2}\sin^2\left(\frac{m\pi h}{2}\right)$$

$$= \frac{8}{h^2}\sin^2\left(\frac{\pi}{2}(1-h)\right) \quad \left(h = \frac{1}{m+1}, mh = 1 - h\right)$$

$$= \frac{8}{h^2}\cos^2\left(\frac{\pi h}{2}\right)$$

- A is SPD and an M-matrix

Richardson

- $\rho(I - \theta A) =$
$$\max\left\{\left|1 - \theta \frac{8}{h^2} \sin^2\left(\frac{\pi h}{2}\right)\right|, \left|1 - \theta \frac{8}{h^2} \cos^2\left(\frac{\pi h}{2}\right)\right|\right\}$$
- Convergence holds for $0 < \theta < \frac{h^2}{4\cos^2\left(\frac{\pi h}{2}\right)}$ and
$$\theta_{opt} = \frac{2}{\lambda_{min} + \lambda_{max}} = \frac{h^2}{4}$$
$$\rho_{opt} = \frac{\lambda_{max} - \lambda_{min}}{\lambda_{max} + \lambda_{min}} = 1 - 2\sin^2\left(\frac{\pi h}{2}\right)$$

Jacobi

- $D = \frac{4}{h^2} I = \theta_{opt}^{-1} I =$ optimal Richardson
$$\therefore \rho(I - D^{-1}A) = \rho_{opt} = 1 - 2\sin^2\left(\frac{\pi h}{2}\right) = \cos(\pi h)$$
- By Taylor expansion: $\cos(x) = 1 - \frac{x^2}{2} + \frac{x^4}{4!} + \cdots$
$$\therefore \rho(I - D^{-1}A) = \cos(\pi h) = 1 - \frac{\pi^2}{2} h^2 + O(h^4)$$
- For small mesh size h, $\rho(G^J) \approx 1 \rightarrow$ slow convergence

Gauss Seidel (GS)

- $\rho(I - M_{GS}^{-1}A) = \rho\left(I - M_J^{-1}A\right)^2$

$$= \cos^2(nh)$$
$$= 1 - \sin^2(\pi h)$$
$$= 1 - \pi^2 h^2 + O(h^4)$$

- For small h, slow convergence for GS
- Convergence rate $= 2 \times$ convergence rate of Jacobi

Successive Over Relaxation (SOR)

- For SOR:

$$w_{opt} = \frac{2}{1+\sin(\pi h)}$$
$$\rho_{opt}^{SOR} = w_{opt} - 1$$
$$= \frac{1-\sin(\pi h)}{1+\sin(\pi h)}$$
$$= 1 - 2\pi h + O(h^2)$$

- Optimal SOR is an order of magnitude better than GS and Jacobi