CS475/CM375
Lecture 2: Sept 15, 2011

Cholesky factorization, tridiagonal, band matrices

Reading: [TB] Chapt. 23 p. 172-176

Special Linear Systems

- Exploit special structures of linear systems
- More efficient LU factorization

- Symmetric systems
  - LDM^T factorization (variant of LU)
- Symmetric positive definite systems
  - GG^T factorization (a.k.a. Cholesky factorization)

LDM^T factorization

- Theorem: If all the leading principal submatrices of A are nonsingular, then there exist unique unit lower Δ matrices L and M, and a unique diagonal matrix D such that $A = LDM^T$.
- Partial Proof:
  - Factor $A = LU$
  - Define $D = diag(d_1, \ldots, d_n)$, $d_i = u_{ii}$ $\quad i = 1, \ldots, n$
  - Let $M^T = D^{-1}U$ = unit upper Δ $\quad$ (M = unit lower Δ)
  - Thus $A = LU = LD(D^{-1}U) = LDM^T$
- Note: $flops(LU) = flops(LDM^T)$

Symmetric systems

- Theorem: If A is symmetric, then $A = LDL^T$
- Proof:
  - By previous result, $A = LDM^T$
    $\Longrightarrow M^{-1}AM^{-T} = M^{-1}LDM^TM^{-T} = M^{-1}LD$
  - Since $M^{-1}AM^{-T}$ is symmetric, so is $M^{-1}LD$
  - Also, $M^{-1}L$ is lower Δ $\Longrightarrow$ $M^{-1}LD$ is lower Δ
  - So $M^{-1}LD$ is both lower Δ and symmetric
    $\Longrightarrow M^{-1}LD$ is diag $\Longrightarrow M^{-1}L$ is diag
  - Since $M^{-1}L$ is also unit lower Δ,
    then $M^{-1}L = I \Longrightarrow M = L$

Symmetric systems

- Notes
  1. We can save about half the work by computing L and D only.
  2. One way is to compute the U factor only during the LU factorization.

Positive definite systems

- Definition: A is positive definite iff $x^T Ax > 0$ for all $x \neq 0$.
- Properties of positive definite matrices:

Positive definite systems

- Theorem: If $A \in \Re^{n\times n}$ is PD and $X \in \Re^{n\times k}$ has rank k, then $B = X^T AX \in \Re^{k\times k}$ is also PD
- Proof:
  - Let z $\in \Re^{k\times 1}$. Then $z^T Bz = z^T X^T AXz = (Xz)^T A(Xz)$
  - If $Xz = 0$, then X is not rank k.
  - Hence $z^T Bz > 0$.

- Corollary: If A is PD, then all its principal submatrices are PD. In particular, all diag entries are positive.

Positive definite systems

- Corollary: If A is PD, then $A = LDM^T$ and D has positive diag entries.
- Proof:
  - Let $X = L^{-T}$.
    Then $X^T AX = L^{-1}(LDM^T)L^{-T} = DM^T L^{-T}$ is PD.
  - By previous corollary, $diag(DM^T L^{-T})$ has positive entries.
  - Note that M^T and L^{-T} are unit upper Δ.
    $\Longrightarrow M^T L^{-T}$ is also unit upper Δ
    $\Longrightarrow diag(DM^T L^{-T}) = D$

Symmetric positive definite systems

- Theorem: If A is SPD, then there exists unique lower Δ G such that

$$A = GG^T$$

- Proof:
  - $A = LDL^T$ and $D = diag(d_1, \dots, d_n)$, $d_i > 0$.
  - Define $D^{\frac{1}{2}} \equiv diag\left(\sqrt{d_1}, \dots, \sqrt{d_n}\right)$
  - Let $G = LD^{\frac{1}{2}}$. Then G is lower Δ

    $\Rightarrow GG^T = LD^{\frac{1}{2}}\left(LD^{\frac{1}{2}}\right)^T = LD^{\frac{1}{2}}D^{\frac{1}{2}}L^T = LDL^T = A$

Symmetric positive definite systems

- Examples

Cholesky factorization

- $A = GG^T$ is called the Cholesky factorization of A and the lower Δ G is called the Cholesky factor.

Cholesky factorization

- Algorithm big picture

Cholesky factorization

For $k = 1,2,\ldots,n$
    $a_{kk} = \sqrt{a_{kk}}$
    For $j = k+1,\ldots,n$
        $a_{jk} = a_{jk}/a_{kk}$
    End
    For $j = k+1,\ldots,n$
        for $i = j,\ldots,n$
            $a_{ij} = a_{ij} - a_{ik}a_{jk}$
        end
    End
End

$flops(Cholesky) \approx \frac{n^3}{3}$

Banded systems

- Definition: A has upper bandwidth q if $a_{ij} = 0\ \forall j > i + q$ and lower bandwidth p if $a_{ij} = 0\ \forall i > j + p$.
- Picture

Banded systems

- If A is banded, so are LU, GG^T, LDM^T
- Theorem: Let $A = LU$. If A has upper bandwidth q and lower bandwidth p, then U has upper bandwidth q and L has lower bandwidth p.
- Picture

Band Gaussian Elimination

For $k = 1,2, \ldots, n-1$
    For $i = k+1, \ldots, \min(k+p, n)$
        $a_{ik} = a_{ik}/a_{kk}$
    end
    for $i = k+1, \ldots, \min(k+p, n)$
        for $j = k+1, \ldots, \min(k+q, n)$
            $a_{ij} = a_{ij} - a_{ik}a_{kj}$
        end
    end
End

If $n \gg p$ and $n \gg q$, then $flops(band\ GE) \approx 2npq$

Tridiagonal systems

- Assume A is tridiagonal and symmetric
- Then
  - $L =$
  - $D =$

Tridiagonal System

- $A = LDL^T$ implies

$$a_{kk} = (LDL^T)_{kk}$$
$$= \sum_i \sum_j l_{ki} d_{ij} l^T_{jk}$$
$$= \sum_i l_{ki} d_{ii} l^T_{ik}$$
$$= \sum_i l^2_{ki} d_{ii}$$
$$= l^2_{k,k-1} d_{k-1,k-1} + l^2_{kk} d_{kk}$$
$$(i = k-1)$$
$$= l^2_{k-1} d_{k-1} + d_k$$

$$a_{k,k-1} = \sum_i \sum_j l_{ki} d_{ij} l^T_{j,k-1}$$
$$= \sum_i l_{ki} d_{ii} l^T_{i,k-1}$$
$$= \sum_i l_{ki} d_{ii}\, l_{k-1,i}$$
$$= l_{k,k-1} d_{k-1,k-1} l_{k-1,k-1}$$
$$(i = k-1)$$
$$= l_{k-1} d_{k-1}$$

Tridiagonal Factorization

- Algorithm

  $d_1 = a_{11}$

  for $k = 2, \ldots, n$

  $\quad l_{k-1} = a_{k,k-1}/d_{k-1}$

  $\quad d_k = a_{kk} - l_{k-1}a_{k,k-1} \quad (l_{k-1}d_{k-1} = a_{k,k-1})$

  end

- $flops(tridiag) = O(n)$