CS 475/CM 375 - Fall 2011: Assignment 4

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Due: December 1, Thursday (in class)

1. (10 marks) Determine SVDs of the following matrices (by hand calculation):

   (a) $\begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}$, (b) $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$, (c) $\begin{bmatrix} 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$, (d) $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, (e) $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

2. (5 marks) Suppose A is an $m \times n$ matrix and B is the $n \times m$ matrix obtained by rotating A 90° clockwise on paper (not exactly a standard mathematical transformation!). Do A and B have the same singular values? Prove that the answer is yes or give a counterexample. (Hint: Consider A^T. Then consider P = rotating the identity matrix 90° clockwise. Let v be any vector. Compare v^T and $v^T P$.)

   (Note: The answer to this question is important when one wants to use SVD to compress an image. Would the compression be affected if one applies SVD to the image versus one applies SVD to the rotated image?)

3. (10 marks) Let A be a matrix of size $m \times n$, $m \geq n$, not necessarily square. Consider

$$H = \begin{bmatrix} 0 & A^T \\ A & 0 \end{bmatrix}.$$

   Compute the eigenvalue decomposition of H; i.e. find Q such that $HQ = Q\Lambda$, Λ is a diagonal matrix. How are Q and Λ related to the singular vectors and singular values of A? Explain.

4. (10 marks) Let A be a matrix of size $m \times n$ and w is an $n \times 1$ vector. Define

$$B = \begin{bmatrix} A \\ w^T \end{bmatrix}.$$

   (a) Show that $\sigma_1(B) \leq \sqrt{\|A\|_2^2 + \|w\|_2^2}$. (Hint: By definition, $\|B\|_2 = \max_{\|x\|_2=1} \|Bx\|_2$.)

   (b) Show that $\sigma_n(B) \geq \sigma_n(A)$. (Hint: Consider the best rank $n-1$ approximation, $\tilde{B}$, to B, compute $B - \tilde{B}$, and then apply the low rank approximation theorem.)

5. (10 marks) A grey scale image is produced by the following MATLAB commands:

```
load detail
subplot(2,2,1)
image(X)
colormap(gray(64))
axis image, axis off
```

Write a MATLAB script, `Compress.m`, which will

1. Compute the singular value decomposition of X.
2. Compress the image by keeping the k largest singular values.
3. Reconstruct the compressed image using the truncated singular values.
4. Compute and display the compression ratio, which is defined to be $2k/n$, n is the matrix size. Also, compute the relative error of approximation, σ_{k+1}/σ_1.
5. Display the original and compressed images on the same figure using `subplot`.

Try k = 1, 20, 100. For the 3 compressed images, show the values of k, compression ratio, and the relative error on the title.