Propositional logic

Readings: Sections 1.1 and 1.2 of Huth and Ryan.

In this module, we will consider propositional logic, which will look familiar to you from Math 135 and CS 251. The difference here is that we first define a formal proof system and practice its use before talking about a semantic interpretation (which will also be familiar) and showing that these two notions coincide.

Declarative sentences (1.1)

A **proposition** or **declarative sentence** is one that can, in principle, be argued as being true or false.

Examples: “My car is green” or “Susan was born in Canada”.

Many sentences are not declarative, such as “Help!”, “What time is it?”, or “Get me something to eat.”

The declarative sentences above are **atomic**; they cannot be decomposed further. A sentence like “My car is green AND you do not have a car” is a compound sentence or **compositional sentence**.

To clarify the manipulations we perform in logical proofs, we will represent declarative sentences symbolically by **atoms** such as p, q, r. (We avoid t, f, T, F for reasons which will become evident.)

Compositional sentences will be represented by **formulas**, which combine atoms with **connectives**. Formulas are intended to symbolically represent statements in the type of mathematical or logical reasoning we have done in the past.

Our standard set of connectives will be $\neg$, $\wedge$, $\vee$, and $\rightarrow$. (In Math 135, you also used $\leftrightarrow$, which we will not use.) Soon, we will describe the set of formulas as a formal language; for the time being, we use an informal description.

The set of connectives is due to the British mathematician George Boole, who described an algebra using them (now called Boolean algebra) in 1854.

We will introduce the connectives in an intuitive fashion, by describing their effect on declarative sentences. In doing so, we anticipate the semantics which we will use to decide if a sentence is true or false.

However, it’s important to keep in mind that our proof system is not concerned with true or false; it is concerned with what constitutes a legal proof. Each of the rules makes intuitive sense, and this is not surprising in light of our goal to show that provable equals true. But we maintain a distinction between semantics and syntax at this point.

The formal language of propositional logic

Let ϕ range over the set of propositional formulas. Then the following grammar specifies the set of possible values for ϕ:

$$\phi \quad ::= \quad p \mid \phi \wedge \phi \mid \phi \vee \phi \mid \neg\phi \mid \phi \rightarrow \phi$$

The identifier p ranges over atoms; as we noted previously, we may also use q, r, etc. to denote atoms.

This language includes four logical connectives: $\neg$ is a unary connective, and $\wedge$, $\vee$, and $\longrightarrow$ are binary connectives.

Abstract syntax

In CS241 we studied context-free grammars as a tool for parsing programs. There, our concern was in language recognition and parsing, and therefore our grammars were very precise. Such grammars exemplify what is called **concrete syntax**.

The grammar for propositional logic is typical of what is called **abstract syntax**—we don't worry too much about ambiguity or parsing practicalities (indeed, this grammar is ambiguous). Instead, we simply assume that the parsing has been taken care of. If we need to explicitly indicate order of operations, we will use parentheses.

Binding priorities

To keep from having to explicitly use parentheses to disambiguate every propositional formulat, we adopt conventions regarding which connectives “bind more tightly” than others.

In algebra, we have such conventions. We understand $3 + 4 \times 5$ to mean $3 + (4 \times 5)$.

We could say that $\times$ **binds** more tightly than $+$, or that $\times$ has precedence over $+$.

In a similar fashion, we say that $\neg$ binds more tightly than $\vee$ and $\wedge$, which bind more tightly than $\longrightarrow$.

Thus, if p, q, and r are atoms, then the formula

$$p \longrightarrow q \wedge r$$

is taken to mean $p \longrightarrow (q \wedge r)$, and *not* $(p \longrightarrow q) \wedge r$.

Similarly, we can simplify

$$(\neg p) \longrightarrow (q \vee r)$$

to $\neg p \longrightarrow q \vee r$.

The connective $\longrightarrow$ is right-associative; that is, $p \longrightarrow q \longrightarrow r$ means $p \longrightarrow (q \longrightarrow r)$.

When we use these rules, we should understand that our simplified formulas really represent properly parenthesized formulas (which is what our formal definitions will define).

Other texts might declare further conventions which we will not use, such as $\wedge$ binding more tightly than $\vee$.

The connectives, by example

Suppose we have the following statements:

p: Ling passed CS 245.

q: Ling fulfilled her breadth requirements.

r: Ling earned her B.CS degree.

The first connective, $\neg$ (pronounced "not"), intuitively expresses negation. $\neg p$ means "Ling did not pass CS 245." This is a **unary** connective, in that it applies to only one formula. The rest of the connectives are **binary**.

The connective $\vee$ (pronounced "or") intuitively expresses disjunction, or the sense that at least one of the two formulas it connects is true. $p \vee q$ means "Ling passed CS 245, or fulfilled her breadth requirements."

Note that the English word "or" sometimes has the sense that only one of the two things it connects can be true, not both. But our logical connective $\vee$ will permit both to be true, so it acts more like the English construct "and/or", as in "Ling passed CS 245 and/or fulfilled her breadth requirements."

There are phrases in English which give this sense without using the English word "or". Translation of an English sentence or paragraph into formal logic is a difficult art.

The connective $\wedge$ (pronounced "and") intuitively expresses conjunction, or the sense that both of the formulas it connects are true. $p \wedge q$ means "Ling passed CS 245 and fulfilled her breadth requirements."

As with $\vee$, there are many phrases in English expressing conjunction. The sentence "Ali passed CS 245, although he failed the midterm" is of the form $p \wedge s$, where s represents "Ali failed the CS 245 midterm".

The final connective, $\longrightarrow$ (pronounced "implies") intuitively expresses implication. $r \longrightarrow (p \wedge q)$ means "If Ling earned her B.CS degree, then she passed CS 245 and fulfilled her breadth requirements."

Natural deduction (1.2)

A proof system is a mathematical formalization of a notion of proof. There are many proof systems; we will study one called natural deduction, invented by Gerhard Gentzen in the 1930's. This system nicely captures many of the aspects of mathematical proof we're familiar with, as well as having desirable technical qualities beyond the scope of this course.

Proof systems work on a syntactic level; they can be viewed as mechanical manipulations of formulas. Of course, we guide these manipulations towards a desired result.

As an example, let's go back to our earlier set of statements.

p: Ling passed CS 245.

q: Ling fulfilled her breadth requirements.

r: Ling earned her B.CS degree.

Consider the following argument: “If Ling earned her B.CS degree, she passed CS 245 and fulfilled her breadth requirements. Ling did not fulfil her breadth requirements. Therefore, Ling did not earn her B.CS degree.”

The three sentences formalize as $r \rightarrow (p \wedge q)$, $\neg q$, and $\neg r$, respectively. But the last one is derived in some fashion from the first two.

In a typical situation, we have a set of formulas $\phi_1, \phi_2, \ldots, \phi_n$, and we wish to apply proof rules to these to derive new formulas, among which is our desired conclusion ψ. We summarize this as:

$$\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$$

This is a **sequent**. The sequent for our example is:

$$r \rightarrow (p \wedge q), \neg q \vdash \neg r$$

We read the symbol $\vdash$ as “yields” or “proves”. A sequent has zero or more formulas on the left of $\vdash$, and one formula on the right. The order of the formulas on the left doesn’t matter.

A sequent such as $r \rightarrow (p \wedge q), \neg q \vdash \neg r$ is **valid** if we can find a proof for it in our proof system.

In what follows, we introduce the rules of the proof system of natural deduction one by one, with some intuition as to why they might be considered rules, and examples of their use. Along the way, we will accumulate notation to be used in the proof system.

Proof systems

A proof system consists of a set of **rules** dictating how a proof of some formula may be obtained from proofs some other formulas.

Typically, each connective $\Box$ in the language of formulas (in our case, $\vee$, $\wedge$, $\rightarrow$, and $\neg$) as associated with it at least one of each of the following types of rules:

- **introduction rules**—these are rules that allow you to **obtain** formulas containing $\Box$;
- **elimination rules**—these are rules that allow you to **use** formulas containing $\Box$ to derive other formulas.

The rules for conjunction

Conjunction has one introduction rule and two elimination rules.

And-introduction ($\wedge$i) says that if we have ϕ and ψ as formulas, we can conclude $\phi \wedge \psi$. We write this:

$$\frac{\phi \quad \psi}{\phi \wedge \psi} \wedge\text{i}$$

You can view this as a “before and after” view, with an abbreviated name of the rule on the side. If you can construct proofs of the formulas above the line, the rule says you then have a proof of the formula below the line.

For and-elimination, the intuition is that if you can prove $\phi \wedge \psi$, then you can prove each of ϕ and ψ individually:

$$\frac{\phi \wedge \psi}{\phi} \wedge\mathsf{e}_1 \qquad \frac{\phi \wedge \psi}{\psi} \wedge\mathsf{e}_2$$

Note that in natural deduction, these and all subsequent rules apply to whole formulas. They cannot be used to selectively rewrite subformulas. For instance, we cannot rewrite $(p \wedge q)$ as p if it appears buried within a larger formula.

More concretely, the rule **does not** permit the following deduction:

$$\frac{(p \wedge q) \rightarrow r}{p \rightarrow r} \wedge\mathsf{e}_1$$

In fact, as we shall eventually see, this is also an incorrect conclusion from a semantic point of view.

A proof

This proof shows the validity of the sequent $p \wedge q \vdash q \wedge p$.

1	$p \wedge q$	premise
2	p	$\wedge e_1 \quad 1$
3	q	$\wedge e_2 \quad 1$
4	$q \wedge p$	$\wedge i \quad 3, 2$

We have numbered each line on the left, and labelled the rules used on the right, with the corresponding line numbers. We also label as a **premise** anything that is on the left-hand side of the sequent. This is how we will be doing proofs.

This notation is a flattened version of the proof using the rules in the form we gave them, because that more resembles a tree.

$$\dfrac{\dfrac{p \wedge q}{q}\,\wedge\mathrm{e}_2 \qquad \dfrac{p \wedge q}{p}\,\wedge\mathrm{e}_1}{q \wedge p}\,\wedge\mathrm{i}$$

We can reconstruct such a tree from our version. Since the flattened version is much more convenient, we will not be using tree proofs.

In general, our proofs may not be unique. That is, there may be many different ways to demonstrate the validity of a particular sequent.

Another proof

Give a proof of the sequent $(p \land q) \land r \vdash q \land r$.

Note that although we applied our rules to atoms in the previous proof, the rules are stated in terms of general formulas. In fact, because the rules talk about the "top-level" structure of the formulas to which they are applied, we can get other proofs from this proof almost for free.

We can take our previous proof, mechanically substitute any formulas ϕ, ψ, and χ for p, q, and r, and get a new valid proof.

Thus any proof we write can be used as a template (via consistent substitution of formulas for atoms) for an unbounded number of other proofs.

Let's do this by substituting $(x \wedge y)$ for p, $(x \vee y)$ for q, and $(\neg x)$ for r, yielding the sequent $((x \wedge y) \wedge (x \vee y)) \wedge (\neg x)) \vdash (x \vee y) \wedge (\neg x)$.

1	$((x \wedge y) \wedge (x \vee y)) \wedge (\neg x)$	premise
2	$((x \wedge y) \wedge (x \vee y))$	$\wedge\mathrm{e}_1 \quad 1$
3	$(x \vee y)$	$\wedge\mathrm{e}_2 \quad 2$
4	$(\neg x)$	$\wedge\mathrm{e}_2 \quad 1$
5	$(x \vee y) \wedge (\neg x)$	$\wedge\mathrm{i} \quad 3, 4$

This may not be useful, but it is correct.

Implication—the elimination rule

$$\frac{\phi \qquad \phi \rightarrow \psi}{\psi} \rightarrow\text{e}$$

This is a famous rule, called *modus ponens* (Latin for "mode that affirms"). In classical deduction, this is the only rule used. As you can imagine, this makes proofs longer, and less related to mathematical proofs we know and love.

Modus ponens captures the argument in the following paragraph.

"Ling earned her B.CS degree. If Ling earned her B.CS degree, Ling passed CS 245. Therefore, Ling passed CS 245."

In this example, p can stand for "Ling earned her B.CS degree", q for "Ling passed CS 245", and the sequent we have shown valid with one application of $\rightarrow$e is $p, p \rightarrow q \vdash q$.

In this simple application of $\rightarrow$e, the formulas ϕ and ψ are atomic. But in general, they may be compound formulas as well.

$$\frac{(p \wedge q) \quad (p \wedge q) \rightarrow (r \vee s)}{(r \vee s)} \rightarrow\text{e}$$

Implication—the introduction rule

Suppose we wanted to prove a sequent like $\vdash p \longrightarrow (q \longrightarrow p)$.

We would need an introduction rule for $\longrightarrow$, which would require us to show that $q \longrightarrow p$ is a theorem, if p is taken as a premise.

We show a proof of $p \longrightarrow (q \longrightarrow p)$ below:

1	p	assumption
2	q	assumption
3	p	copy of 1
4	$q \rightarrow p$	$\rightarrow$ i 2–3
5	$p \rightarrow (q \rightarrow p)$	$\rightarrow$i 1–4

What did we do? We introduced a proof box, which was a way of marking the fact that we had made an assumption p. This kept the assumption from "leaking" into the rest of the proof. But the premises (and presumably any derived formulas) were valid inside the proof box.

We used the assumption to conclude $q \longrightarrow p$. Then we introduced an implication connecting the assumption p and the conclusion $q \longrightarrow p$, and labelled it as having been derived using the rule $\longrightarrow$i.

As part of the process, we used the same technique to establish the theorem $q \longrightarrow p$ under a premise of p. We introduced a *nested* proof box, in which q is introduced as an assumption. We can then conclude p immediately, as p had already been assumed. An application of $\longrightarrow$i then allows us to conclude $q \longrightarrow p$.

Note that we used a “copy rule”, which allows us to do what we did in line 3. It lets us copy any formula whose proof box is still open. Without this, we’d have to complicate our description of the $\longrightarrow$i rule.

We need to describe this rule more generally.

Here is the general form of the rule $\rightarrow$i.

$$\frac{\boxed{\begin{array}{c}\phi\\ \vdots\\ \psi\end{array}}}{\phi \rightarrow \psi}\ \rightarrow\text{i}$$

The first line in the box is an arbitrary formula ϕ of our choice. In the box, we are allowed to use premises, plus any derived formula that is on a previous line, with the exception of those introduced within boxes that have since been closed. This allows us to nest boxes (and their corresponding assumptions).

This is reminiscent of the scope of local definitions in a programming language. We will see this notion again later in the course.

Implication Example

Let us try to prove the transitivity of implication:

$p \longrightarrow q, q \longrightarrow r \vdash p \longrightarrow r$.

By replicating this proof with the appropriate substitutions, we can show that for any formulas ϕ, ψ, χ, the sequent $\phi \longrightarrow \psi, \psi \longrightarrow \chi \vdash \phi \longrightarrow \chi$ is valid.

Disjunction—introduction rules

Introducing and eliminating conjunction was fairly straightforward.

Introducing disjunction is also straightforward.

$$\frac{\phi}{\phi \vee \psi} \vee\mathsf{i}_1 \qquad \frac{\psi}{\phi \vee \psi} \vee\mathsf{i}_2$$

Eliminating disjunction is more complicated. Intuitively, if we have a formula of the form $\phi \vee \psi$, we can't just conclude ϕ or ψ, because we don't know which of them is true. We need a way to get from this to some new conclusion χ.

Disjunction—elimination rule

If we know that $\phi \lor \psi$ is true, we know that at least one of the two is true. So a reasonable rule for concluding χ would require us to be able to prove χ from either one of ϕ or ψ. Since we can't be sure that either one is true, we must assume them in order to prove χ.

$$\frac{\phi \lor \psi \qquad \boxed{\begin{matrix}\phi \\ \vdots \\ \chi\end{matrix}} \qquad \boxed{\begin{matrix}\psi \\ \vdots \\ \chi\end{matrix}}}{\chi} \ \lor\mathrm{e}$$

We have already shown that $\wedge$ is commutative.

We take the commutativity of $\vee$ for granted from English and experience, but the sequent $p \vee q \vdash q \vee p$ does require a proof.

1	$p \vee q$	premise
2	p	assumption
3	$q \vee p$	$\vee$i$_2$ 2
4	q	assumption
5	$q \vee p$	$\vee$i$_1$ 4
6	$q \vee p$	$\vee$e 1, 2–3,4–5

We now have introduction and elimination rules for all three binary connectives. It remains to consider negation...

But what is negation?

What is negation?

What is it that we would like $\neg\phi$ to mean, for a formula ϕ?

This question is related to the central notion in logic of the meaning of truth.

If we conclude a formula ϕ in our proof system, do we mean that ϕ is true, or do we mean that ϕ has a proof?

Certainly, if the logic is consistent, a proof of ϕ implies the truth of ϕ.

What about the reverse? If ϕ is true, can ϕ be proved? Maybe not!

So perhaps the real meaning of the conclusion ϕ is simply that ϕ can be proved.

But if that's true, then maybe $\neg\phi$ doesn't actually mean that ϕ is false, but simply that ϕ has no proof!

A parting of ways

The meaning of proof-theoretic negation represents a major point of departure between schools of logical thought, and the choice we make fundamentally affects the properties of the resulting logic.

If we believe that $\neg\phi$ means that ϕ is false, then we are **classicists** and our proof theory becomes a proof theory for **classical logic**. We will then handle negation in a way that hopefully leads to a semantics of interchanging T and F.

If we believe that $\neg\phi$ means that ϕ simply has no proof, then we are **intuitionists** and our proof theory becomes that of **intuitionist logic**. To fully understand the intuitionist perspective, we must re-evaluate the rules we have presented so far as being statements about *proofs*, rather than logical formulas.

Intuitionism vs. Classicism

We will see that both perspectives on the proof theory of negation have strong arguments in their favour. Although we are generally more used to classical reasoning, we will see that there are theorems that arise as a result of the classical interpretation of negation that appear nonsensical.

On the other hand, intuitionism prevents us from making some deductions we truly believe are valid.

Thus, it is important to understand the difference between the two perspectives, and the consequences of choosing one over another.

We will first complete our study of propositional logic under the classical interpretation, and then re-examine our proof rules from an intuitionist perspective.

Note that all of the rules presented so far are valid under both classicism and intuitionism.

Rules for double negation (classical)

The simplest way in which a negation may arise in a formula is in the form of *double negation*. The rules for double negation are straightforward:

$$\frac{\neg\neg\phi}{\phi}\ \neg\neg\text{e} \qquad \frac{\phi}{\neg\neg\phi}\ \neg\neg\text{i}$$

We are maintaining the naming convention that “e” refers to elimination and “i” to introduction.

A double negation in English looks like “It is not true that Ling did not pass CS 245.”

It turns out that the $\neg\neg$i rule can be derived from other rules; the textbook demonstrates this, and then moves it into the category of derived rules.

Negation—elimination rule (classical)

In order to introduce or eliminate negation, we have to talk about contradictions. A contradiction is any formula of the form $\phi \wedge \neg\phi$ or $\neg\phi \wedge \phi$. We introduce the symbol $\bot$ (pronounced "bottom") into our proofs to represent a contradiction. This naturally leads to the not-elimination or $\neg$e rule:

$$\frac{\phi \quad \neg\phi}{\bot}\ \neg\text{e}$$

We could also call this rule bottom-introduction, since it plays that role. But this is not commonly done.

Bottom elimination (classical)

The rule for bottom elimination may be a little surprising.

$$\frac{\bot}{\phi}\ \bot\text{e}$$

This says that with $\bot$ as a premise, one can conclude anything. In other words, $\bot \to \phi$ is a theorem for all ϕ.

The rule is reminiscent of statements like “If 2+2=5, then I’m a monkey’s uncle.” We need this rule in order to get our notion of proof to coincide with the still-to-be-introduced notion of semantic interpretation of formulas.

This rule is an example of the important notion of **vacuous truth**: any statement based on false premises is true.

Negation—introduction rule (classical)

Introducing negation is our last rule that requires the use of proof boxes. If we make an assumption ϕ and end up with a contradiction, then ϕ should not be true, so we must be able to conclude that $\neg\phi$ is true.

$$\frac{\boxed{\begin{array}{c}\phi \\ \vdots \\ \bot\end{array}}}{\neg\phi}\ \neg\text{i}$$

We motivated formal propositional logic with the following example: "If Ling earned her B.CS degree, she passed CS 245 and fulfilled her breadth requirements. Ling did not fulfil her breadth requirements. Therefore, Ling did not earn her B.CS degree."

We can now show the corresponding sequent to be valid.

Here is the proof of validity of $r \rightarrow (p \wedge q), \neg q \vdash \neg r$.

1	r	assumption
2	$r \rightarrow (p \wedge q)$	premise
3	$p \wedge q$	$\rightarrow$e 1–2
4	q	$\wedge$e$_2$ 3
5	$\neg q$	premise
6	$\bot$	$\neg$e 4,5
7	$\neg r$	$\neg$i 1–6

The derived rule *modus tollens* (classical)

We now establish the following rule, which comes as a consequence of the rules we already have:

$$\frac{\phi \rightarrow \psi \quad \neg\psi}{\neg\phi}\ \text{MT}$$

Modus tollens (Latin for “mode that denies”) captures the argument “If Ling earned her B.CS degree, then Ling passed CS 245. Ling did not pass CS 245. Therefore, Ling did not earn her B.CS degree.”

The validity of the English argument we just made suggests that the following English argument is also valid: “If Ling earned her B.CS degree, then Ling passed CS 245. Therefore, if Ling did not pass CS 245, Ling did not earn her B.CS degree.”

As a sequent, this has the form $p \rightarrow q \vdash \neg q \rightarrow \neg p$. Recall that in Math 135, $\neg q \rightarrow \neg p$ was defined as the “contrapositive” of $p \rightarrow q$.

Proof of modus tollens

We can now prove *modus tollens*, or $\phi \to \psi, \neg\psi \vdash \neg\phi$. But there is a slight problem. This is not really a sequent. It is a template or framework for creating sequents. If we substitute any specific formulas for ϕ and ψ, we will get a sequent.

The textbook does not discuss this, and it is a source of possible confusion. We will call this a **sequent schema**.

In order to prove it, we will create a **proof schema**, which has the same parameters, ϕ and ψ. Any substitution of specific formulas in the proof schema yields a valid proof.

1	$\phi \rightarrow \psi$	premise
2	$\neg\psi$	premise
3	ϕ	assumption
4	ψ	$\rightarrow$e 1,3
5	$\bot$	$\neg$e 2,4
6	$\neg\phi$	$\neg$i 3–5

Note that everywhere we used MT, we could fill in these lines instead, with the appropriate substitutions. Using MT lets us shorten proofs. We will keep it as a **derived rule** (not a primitive rule).

The derived rule *reductio ad absurdum* (classical)

Reductio ab absurdum (Latin for “reduction to the absurd”) is often called proof by contradiction, and that is how we will abbreviate it (PBC). As a rule, it looks like this:

$$\dfrac{\boxed{\begin{array}{c}\neg\phi \\ \vdots \\ \bot\end{array}}}{\phi}\ \text{PBC}$$

This looks like not-introduction turned upside down, so it is not surprising that we can use not-introduction and double-negation-elimination to prove this rule.

The textbook writes PBC as $(\neg\phi \to \bot) \vdash \phi$, and provides a five-line proof schema. However, there is a problem with this: $\bot$ is not part of our language of formulas. We can make a rule that in sequent schema, we can substitute any formula of the form $\psi \wedge \neg\psi$ for $\bot$, but doing this to the five-line proof schema in the book does not yield a valid proof.

Instead, we will reason about why PBC should be valid, by viewing it as shorthand for a longer sequence. Any application of PBC starts with a proof box with $\neg\phi$ at the top and $\bot$ at the bottom, followed by ϕ. If we substitute an application of $\neg$i, we can put $\neg\neg\phi$ after the box. Then it is simple, using $\neg\neg$e, to conclude ϕ.

In other words, one application of PBC can be replaced by one application of $\neg$i and one application of $\neg\neg$e.

The derived rule *tertium non datur* (classical)

Our last derived rule is the law of the excluded middle, or LEM. (The Latin name of the rule translates as “a third [thing] is not given”). This simply states that $\phi \vee \neg\phi$ is a theorem for any ϕ. Intuitively, it means that either ϕ is true or $\neg\phi$ is true.

This is a useful rule because its proof is longer than the others, and so it saves much space when it is used. On the other hand, it is less obvious how to use it than some of the other rules we have. It can be a source of disjunctions that can be used in or-elimination (see example 1.24 in the textbook). It plays an important role in a mathematical proof in the next module.

Once again, we use a proof schema.

1	$\neg(\phi \vee \neg\phi)$	assumption
2	ϕ	assumption
3	$\phi \vee \neg\phi$	$\vee$i$_1$ 2
4	$\bot$	$\neg$e 1,3
5	$\neg\phi$	$\neg$i 2–4
6	$\phi \vee \neg\phi$	$\vee$i$_2$ 5
7	$\bot$	$\neg$e 1,6
8	$\neg\neg(\phi \vee \neg\phi)$	$\neg$i 1–7
9	$\phi \vee \neg\phi$	$\neg\neg$e 8

As an aside, note that the proofs of both PBC and LEM make use of double negation elimination. This will be important later.

We could continue creating as many derived rules as we like, but these will suffice for our purposes.

Theorems

Consider the following proof of $p \vdash q \rightarrow p$.

1	q	assumption
2	p	premise
3	$q \rightarrow p$	$\rightarrow$i 1–2

We could "move p to the right" by wrapping this whole proof in a proof box that starts by assuming p and ends by proving $p \rightarrow (q \rightarrow p)$.

In other words, we can show validity of the sequent

$\vdash p \rightarrow (q \rightarrow p)$.

1	p	assumption
2	q	assumption
3	p	copy of 1
4	$q \rightarrow p$	$\rightarrow$i 2–3
5	$p \rightarrow (q \rightarrow p)$	$\rightarrow$i 1–4

We call a formula ϕ for which the sequent $\vdash \phi$ is valid a **theorem** within our proof system.

We are still going to use the word “theorem” for things we prove, using “Math 135”-style proofs, about our proof systems. Please make sure you always understand in which sense the word is being used.

By iterating the process we just used, we can convert any proof of validity of a sequent $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$ into a proof of validity of $\vdash \phi_1 \rightarrow (\phi_2 \rightarrow \ldots (\phi_n \rightarrow \psi) \ldots)$. In fact, we can put the formulas ϕ_i into such a theorem in any order.

Provable equivalence

Two formulas ϕ and ψ are provably equivalent if $\phi \vdash \psi$ and $\psi \vdash \phi$ are valid. We sometimes write this $\phi \dashv\vdash \psi$.

As an example, we proved $p \rightarrow q \vdash \neg q \rightarrow \neg p$ earlier. If we prove $\neg q \rightarrow \neg p \vdash p \rightarrow q$, it will show that the formulas $p \rightarrow q$ and $\neg q \rightarrow \neg p$ are provably equivalent.

1	$\neg q \rightarrow \neg p$	premise
2	p	assumption
3	$\neg\neg p$	$\neg\neg$i 2
4	$\neg\neg q$	MT 1,3
5	q	$\neg\neg$e 4
6	$p \rightarrow q$	$\rightarrow$i 2–5

Just as with mathematical proofs, there is a certain art to constructing proofs in natural deduction. There is no simple recipe or algorithm to create a proof of a sequent. You should study the more complicated examples in the textbook, which also gives some guidelines as to what to try. Do as many proofs as you can manage.

Intuitionism

Having fully developed the proof theory of propositional logic, let us now go back and consider the consequences of adopting the intuitionist point of view.

As intuitionists, we observe that a derivation of some formula ϕ is a statement about both ϕ's truth and its provability.

As we noted before, we can then choose to interpret $\neg\phi$ as meaning “not true” or “not provable”. As intuitionists, we choose the latter.

This one choice dramatically influences how we view the formulas we derive. A formula ϕ is now a statement about a proof of ϕ, rather than the mere truth of ϕ. Consequently, the rules for working with the logical connectives must be viewed in this new light.

Intuitionism and conjunctions

If we interpret a formula ϕ as meaning “I can construct a proof of ϕ,” then the formula $\phi \wedge \psi$ means “I can construct a proof of ϕ and a proof of ψ.”

Equivalently, it means “I can construct a proof of $\phi \wedge \psi$.”

The intuitionistic proof rules for conjunction are the same as the classical ones.

Intuitionism and implication

The formula $\phi \rightarrow \psi$ now means “If I have a proof of ϕ, I can transform it into a proof of ψ.”

The usual rules for $\rightarrow$-elimination (i.e., modus ponens) and $\rightarrow$-introduction still apply under intuitionism.

In the case of modus ponens, the reasoning is that if we have a proof of ϕ and a way of transforming a proof of ϕ into a proof of ψ, then we have a proof of ψ.

The derived rule of modus tollens also remains valid — if we can translate any proof of ϕ to a proof of ψ, and yet we cannot prove ψ, then ϕ must not be provable.

Intuitionism and disjunction

Consider the following two statements:

“I have a proof of $\phi \vee \psi$.”

“I have a proof of ϕ or I have a proof of ψ.”

Are these statements equivalent?

“I have a proof of $\phi \vee \psi$.”

“I have a proof of ϕ or I have a proof of ψ.”

Are these statements equivalent?

No! The second statement is stronger than the first.

Certainly, if we possess a proof of ϕ or a proof of ψ, then either proof would suffice as a proof of $\phi \vee \psi$. But given a proof of $\phi \vee \psi$, depending on how the proof was obtained, we may not be able to decompose it into a proof of ϕ alone or ψ alone.

Consider the law of the excluded middle from classical logic. According to LEM, the formula $p \vee \neg p$ is a provable theorem. But neither p on its own nor $\neg p$ on its own is a theorem; hence neither of these has a proof!

So which interpretation of $\phi \vee \psi$ is appropriate for intuitionism?

As we shall see, it turns out that the only way a formula $\phi \vee \psi$ can arise intuitionistically is through one of the two $\vee$-introduction rules. Therefore, we always either have on hand a proof of ϕ or a proof of ψ.

Put more strongly, whenever we arive at a disjunctive formula $\phi \vee \psi$ in an intuitionist proof, *we always know which of ϕ and ψ is true (proved)!*

The rules for $\vee$-introduction and -elimination are the same as for classical logic; the difference in interpretation forced upon us by intuitionism arises from our treatment of negation.

Intuitionism and negation

By definition, the intuitionist interpretation of the formula $\neg\phi$ is "There is no proof of ϕ," which is not necessarily equivalent to saying that ϕ is false.

Suppose we prove a formula ϕ. Then is not the case that ϕ is unprovable. Hence $\neg\phi$ is not a true statement, and therefore must have no proof. Put again, if ϕ is provable, the statement that ϕ is unprovable must be unprovable.

Therefore, from ϕ, we can deduce $\neg\neg\phi$, and we have the rule for double negation introduction:

$$\frac{\phi}{\neg\neg\phi}\ \neg\neg\mathrm{i}$$

Intuitionism and negation cont'd.

Can we carry out the same reasoning in reverse?

Suppose we have the formula $\neg\neg\phi$. The intuitionistic interpretation of this formula is "There is no proof that there is no proof of ϕ," or equivalently, "There is no proof that ϕ cannot be proved."

Is this statement equivalent to simply ϕ? It could be, if we knew all "true" statements were provable. In many logics, this is not the case (as we shall see later). Thus, we **cannot** conclude ϕ from $\neg\neg\phi$ under intuitionism!

To reiterate, intuitionism **rejects** the rule for double negation elimination.

Intuitionism and negation cont'd.

On the other hand, a formula and its negation cannot simultaneously be provable; hence the (single) negation elimination rule still holds.

The "monkey's uncle rule" (i.e., $\perp$-elimination) also remains valid.

Moreover, if the assumption of a formula ϕ (specifically, the assumption that ϕ has a proof) leads to $\perp$, then we have a proof that ϕ is not provable, and we can conclude $\neg\phi$. Hence negation introduction also holds under intuitionism.

Intuitionism cont'd.

The only core deduction rule that intuitionism rejects is the rule for double negation elimination. What are the consequences of rejecting just this single rule?

If we review the proofs of our derived rules, we see that two of their proofs made use of double negation elimination. These are the derived rules PBC and LEM. By rejecting this rule, we also reject the proofs of these rules.

Can these two derived rules be established in some other way, without using double negation elimination?

Intuitionism cont'd.

The answer, it turns out, is no. As you will see in part on your assignment, the three rules, $\neg\neg$e, PBC, and LEM, are all equivalent to each other.

If we add any one of these rules to the core logic, the remaining rules follow as well. Thus, of the deduction rules we have seen in this course, intuitionism rejects the following three:

- double negation elimination
- proof by contradiction
- the law of the excluded middle.

It should not be surprising, given our interpretation of disjunction, that we reject the law of the excluded middle. Since we do not know, from the formula $\phi \vee \neg\phi$, which of ϕ and $\neg\phi$ (or both) is the provable one, we cannot accept the disjunction as a proved theorem.

The rejection of PBC is perhaps more surprising. Since we permit negation introduction, we can conclude $\neg\phi$ from a deduction that ϕ leads to $\perp$; however we **cannot** conclude ϕ from a deduction that $\neg\phi$ leads to $\perp$. All we can conclude from this deduction is $\neg\neg\phi$, which, as we have already argued, is **not** equivalent to ϕ.

So which logic is right?

Should we be classicists or intuitionists? Consider the following atomic sentences:

p: it is raining outside

q: Ling will pass CS245

Now consider the two statements, $p \longrightarrow q$ and $q \longrightarrow p$:

"If it is raining outside, then Ling will pass CS245."

"If Ling will pass CS245, then it is raining outside."

Do we have any reason to believe that either of these implications, i.e., $p \longrightarrow q$ or $q \longrightarrow p$, is true?

Probably not.

Which logic is right?

But it turns out that their disjunction, i.e., $(p \longrightarrow q) \vee (q \longrightarrow p)$, is a theorem of classical logic! Consider:

1	$q \lor \neg q$	LEM
2	q	assumption
3	p	assumption
4	q	copy of 2
5	$p \rightarrow q$	$\rightarrow$i 3–4
6	$(p \rightarrow q) \lor (q \rightarrow p)$	$\lor$i$_1$ 5
7	$\neg q$	assumption
8	q	assumption
9	$\bot$	$\neg$e 8, 7
10	p	$\bot$e 9
11	$q \rightarrow p$	$\rightarrow$i 8–10
12	$(p \rightarrow q) \lor (q \rightarrow p)$	$\lor$i$_2$ 11
13	$(p \rightarrow q) \lor (q \rightarrow p)$	$\lor$e 1, 2–6, 7–12

Which logic is right?

Thus, under classical reasoning, “If it is raining outside then Ling will pass CS245, or if Ling will pass CS245, then it is raining outside.”

More generally, if ϕ and ψ are propositional formulas, then the formula $(\phi \rightarrow \psi) \vee (\psi \rightarrow \phi)$ is a theorem.

Does this make sense? Is this really the kind of statement we want to be able to prove??

Which logic is right?

Can we prove the same theorem in intuitionist logic?

If $(\phi \rightarrow \psi) \vee (\psi \rightarrow \phi)$ is a theorem, then so is the result of replacing ψ by $\neg\phi$, i.e., $(\phi \rightarrow \neg\phi) \vee (\neg\phi \rightarrow \phi)$ is a theorem.

Then, under intuitionist reasoning, we must have a proof of either $\phi \rightarrow \neg\phi$ or $\neg\phi \rightarrow \phi$.

Suppose we have a proof of the latter. Then by instantiating ϕ to χ and $\neg\chi$, we obtain proofs of $\neg\chi \rightarrow \chi$ and $\neg\neg\chi \rightarrow \neg\chi$. From these two, we can obtain the theorem $\neg\neg\chi \rightarrow \chi$, which is equivalent to double negation elimination! Since we reject $\neg\neg$e under intuitionist logic, we must not have a proof of $\neg\phi \rightarrow \phi$.

Then we must have a proof of $\phi \to \neg\phi$. But then consider the following proof schema:

1	$\phi \to \neg\phi$	premise
2	ϕ	assumption
3	$\neg\phi$	$\to$ e 2, 1
4	$\bot$	$\neg$e 2, 3
5	χ	$\bot$e 4
6	$\phi \to \chi$	$\to$ i 2–5

What does this result mean?

If $\phi \longrightarrow \neg\phi$ has a proof, then $\phi \longrightarrow \chi$ has a proof for all ϕ, χ. Hence everything follows from everything, and all formulas turn out to be provable!

Since this is not the case, we know that $\phi \longrightarrow \neg\phi$ has no proof. Since neither $\phi \longrightarrow \neg\phi$ nor $\neg\phi \longrightarrow \phi$ can have a proof, the statement $(\phi \longrightarrow \neg\phi) \vee (\neg\phi \longrightarrow \phi)$ cannot be a theorem of intuitionist logic.

Therefore, $(\phi \longrightarrow \psi) \vee (\psi \longrightarrow \phi)$ cannot be an intuitionist theorem either.

What we see, then, is an example where a result that seems "counter**intuitive**" is accepted by classical logic, but rejected by intuitionist logic.

So perhaps intuitionism allows us to prove only "sensible" theorems.

On the other hand....

Consider the following mathematical theorem and proof:

Theorem There exist irrational numbers a and b such that a^b is rational.

Proof Let $c = (\sqrt{2})^{\sqrt{2}}$. Either c is rational or c is irrational.

- if c is rational, take $a = b = \sqrt{2}$.
- if c is irrational, take $a = c$, $b = \sqrt{2}$, so that $a^b = ((\sqrt{2})^{\sqrt{2}})^{\sqrt{2}} = (\sqrt{2})^2 = 2$.

Either way, a^b is rational.

Do we believe this argument to be correct? Would an intuitionist?

An intuitionist would reject the proof of this theorem.

The statement “Either c is rational or c is irrational,” is essentially an appeal to LEM, which intuitionists do not accept.

As intuitionists, we cannot accept the disjunction “c is rational $\vee$ $\neg$(c is rational)” without either proof that c is rational, or proof that c is irrational. In other words, the proof, as laid out, **cannot proceed** without knowing whether c is rational or irrational.

Because we have not proved whether or not c is rational, we cannot exhibit specific irrational numbers a and b such that a^b is rational. We know that an answer is either $a = b = \sqrt{2}$ or $a = c, b = \sqrt{2}$. But we don’t know which it is!

On the other hand, if we had furnished a either a proof that c was rational or a proof that c was irrational, we would have known which branch of the proof to take, and we would have specific irrational numbers a and b that satisfy the theorem.

For this reason, intuitionist reasoning is often called **constructive**—it demands an explicit construction of any entity whose existence is asserted.

So should we, like the intuitionists, throw out the theorem? Are we really unconvinced that such numbers a and b exist?

Probably not. If we do not care to exhibit the numbers a and b explicitly, we may be perfectly happy to accept their mere existence, which is guaranteed by the classical argument.

Therefore, although intuitionism rejects certain nonsensical theorems, it may also prevent us from using modes of reasoning that we believe to be valid.

It seems, then, that there is no clearly superior choice between the two logics. We must simply be aware of the difference and understand the consequences of whatever choice we make.

The distinction between classicicm and intuitionism mirrors, in a way, the same distinction between mathematics and computer science.

Mathematical reasoning often involves hypothetical reasoning over incredibly large spaces, which can have no physical realization. Mathematicians are, in general, perfectly content to work with quantities they cannot construct; hence, classical reasoning comes naturally to the mathematician.

Computer science, however, is highly constructive. Although every computer algorithm is an implementation of a mathematical function, not every mathematical function has an implementation on a computer. The central notion of **implementation** in computer science forces us to restrict ourselves to those mathematical entities that have a physical realization. Hence, computer science is inherently constructive, and intuitionism therefore seems like a natural choice.

On the other hand, very few computer scientists seem to be true intuitionists; most will employ classical reasoning if it suits their needs.

However, there is at least one discipline within computer science in which intuitionism plays a central role: the theory of types. In the introductory module, we mentioned a fundamental and deep connection between types in certain programming languages and theorems of a particular logic. The connection is called the Curry-Howard isomorphism, and the logic in question is, in fact, intuitionist logic.

The typed programming languages in question are, in general, the functional languages, although strictly speaking, the types involved do not critically depend on the language being functional; the phenomenon is simply more prevalent among the functional languages.

What's coming next?

We have explored two proof theories for propositional logic: classicism and intuitionism. We have seen how they arise from an awareness of the different possible understandings of "truth", and which of these understandings the negation operator is intended to negate.

We have not yet shown that the deduction rules we presented match our semantic notion of "truth" or indeed, how much semantic truth is captured by these proof rules. We discuss these issues in the next module.

Furthermore, our language of propositional formulas is not adequate to capture the reasoning in the examples we initially considered, such as "Every even number is the sum of two odd numbers whose difference is at most 2." We need to extend it to capture the notion that the word "every" makes this a compositional statement. That will complicate both our proof system and our semantics.

Goals of this module

At the end of each lecture module (which may take up several days of lecture) we will review its goals.

You may not have achieved these goals just by listening to the lecture. Some will require reading in the text, doing assignment questions, and working through examples of your own choosing.

The list of goals is a guide to the level of understanding we expect from you, eventually.

We introduced a fair amount of terminology (formulas, sequents, conjunction, disjunction, etc.) with which you need to be comfortable.

You should have all of the rules of natural deduction in Figure 1.2 in the book (twelve rules, plus the copy rule, plus four derived rules) committed to memory. You should understand the intuitionist interpretations of propositional formulas. You should memorize which three proof rules are rejected by intuitionism. You should also understand how the rules of propositional logic correspond to elements of mathematical proof with which you are familiar from other math and CS courses.

You should be able to use the systems of classical and intuitionist natural deduction to demonstrate the validity of sequents. In addition to assignment questions, there are many exercises in the book for practice.