Simulations For Hedging Financial Contracts With Optimal Decisions: A Case Study

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Abstract

Simulations are powerful techniques for quantifying risk exposures. This paper presents a methodology for simulating the performance of hedging strategies for financial contracts with embedded optimization features. As a case study, we provide simulations of mutual fund guarantees offering a reset provision. In Canada, these types of contracts are known as segregated funds. The optimization component of these contracts is that the holder can choose when to lock in market gains, typically up to two or four times per calendar year. Recently, Canadian regulators have imposed new capital requirements for firms selling these contracts. However, these requirements can be reduced if hedging strategies are put in place. The techniques presented here would allow companies to evaluate their proposed hedging strategies and to quantify their remaining risk exposures. We study the effect of non-optimal investor behaviour on the hedging of these contracts. In particular, we present results for the heuristic use of the reset feature; for example, locking in whenever the underlying asset value has risen by 15% as recently suggested by a Canadian Institute of Actuaries task force on segregated funds.

Keywords: stochastic simulation, mutual fund guarantees, hedging, segregated funds, variable annuities, investor behaviour modelling.

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1 Introduction and Motivation

Monte Carlo simulations are commonly used for quantifying risk exposures to derivative security contracts. The basic approach involves generating many possible paths for the underlying state variables, thereby obtaining a distribution of values of the contract, with one realized value for each path. There are several reasons why simulations are so popular in modern computational finance. First, the premise is very intuitive and the results can often be readily interpreted. Second, it is relatively easy to model the behaviour of complex contracts (e.g. path-dependent contracts such as Asian or lookback options) in a simulation model. Finally, these methods scale well to high dimensional problems.

One of the major limitations of current simulation methods is that it can be very difficult to incorporate optimization components which are included in many contracts. For individual realizations of the stock price path, as we march forward in time, we are unable to determine how one should utilize optimization features. There has been some recent progress in this area (see Broadie and Glasserman (1998); Boyle et al. (1999) and references therein), but such approaches assume that the contract holder's decisions occur only at discrete points in time rather than continuously (i.e. they approximate an American option by a Bermudan option). In this paper we utilize a previously computed solution to provide us with this information. We can then use Monte Carlo techniques to simulate the behaviour of these contracts under various scenarios.

As a case study to demonstrate these techniques, we provide simulations of a mutual fund guarantee which offers a reset feature. These contracts are known in Canada as segregated funds. They have been very popular in recent years, reaching a market size of about $31 billion CDN at the end of 1999.[1] Recently, variable annuities offering many similar features have become increasingly popular in the U.S., with an estimated market size of $1 trillion USD (Milevsky and Posner (2000)). Essentially, segregated funds consist of a mutual fund combined with a ten year maturity guarantee with complex features.[2] The *reset* feature allows the investor to lock in market gains as the value of the underlying mutual fund increases. When the investor resets, he exchanges his existing guarantee for a new ten year maturity guarantee set at the current value of the mutual fund. Typically, the investor is able to reset the contract up to a maximum of two to four times per calendar year. This introduces an optimization component to these contracts, where the investor must decide when he or she should reset and lock in at the higher guarantee level.

The complex optionality embedded in segregated funds and their long term nature mean that they provide a very interesting case study for simulation and hedging techniques. Moreover, this is a topic of considerable practical importance. Recently Windcliff et al. (2000, 2001), Falloon (1999) and others have raised concerns that some of these contracts have been under-priced by companies offering these products. Further, the Canadian regulatory board OSFI (Office of the Superintendent of Financial Institutions) has imposed new capital

[1]Source: *Canadian Life and Health Insurance Facts* (2000 Edition), published by the Canadian Life and Health Insurance Association, Inc. Note that although not all segregated funds include reset provisions, the vast majority of them do.

[2]Some of the available features and variations of these contracts are discussed in Windcliff et al. (2000, 2001).

requirements for these contracts. These capital requirements can be reduced if suitable hedging strategies have been implemented. The techniques presented here would allow companies to evaluate their proposed hedging strategies and to quantify their remaining risk exposure.

2 Contract Description: Segregated Fund Guarantees

More than one thousand Canadian mutual funds can be purchased with an accompanying maturity guarantee. Typically, after ten years the investor is entitled to receive the greater of either the current value of their account, or their initial deposit amount. As the value of their account rises, the guarantee becomes less valuable to the investor and they become more likely to lapse out of the contract, i.e. withdraw their funds. In part to avoid this, and for other reasons, many financial institutions offer more sophisticated guarantees, with optionality such as the reset feature. The reset feature allows the investor to lock in market gains. When the investor chooses to reset their guarantee, the investor is given a new guarantee which matures at ten years past the reset date with the guarantee level set to the current level of the underlying asset. The investor usually has a limited number of reset opportunities per year; typically up to two or four resets per year. These contracts are attractive to investors since they give the investor the benefits of the higher anticipated returns in equity markets while providing downside protection should markets fall.

In this paper, we study the effectiveness of a proposed hedging strategy for a prototypical segregated fund guarantee. A summary of the important structure of the contract is given in Table 1. The hedging strategy was derived from the solution of a partial differential equation model described in Windcliff et al. (2000). This model assumes continuous rehedging and optimal investor behaviour. Both of these assumptions are unrealistic and we would like to study their effect on the performance of the hedging strategy.

We will assume an initial investment of \$100. The investor initially receives a guarantee at this level which matures in 10 years. The guarantee level can be reset by the investor up to two times per calendar year, simply by notifying the insurer. Upon reset, the guarantee level is set to the value of the fund at reset, and the maturity date is extended to be 10 years from the reset date.

No initial fee is charged to enter into the contract. Instead, proportional fees are deducted over time to pay for the cost of providing the guarantee. In the simulations provided in this paper, a proportional fee of $r_m = 2\%$ (per annum) is paid for the management of the underlying mutual fund. In addition, a proportional fee of r_e is used to cover the cost of providing the guarantee. The total proportional fee of $r_m + r_e$ can be thought of as a standard management expense ratio (M.E.R.) on a mutual fund, except that some part of it is being used to fund the guarantee. A deferred sales charge (D.S.C.) is applied upon early redemption. In this work, we use a sliding scale from 5% in the first year to 0% in the sixth and further years. It is assumed that the D.S.C. is charged by the underlying mutual fund and that none of this fee is allocated to funding the guarantee portion of these contracts.

A standard mortality feature is provided. If the investor dies, the greater of the current fund value or its guaranteed level is provided immediately (at the time of death). In this work, we use mortality data for a Canadian female aged 50 years. In Windcliff et al. (2000), we show that the contribution of the mortality feature to the value of these contracts is

Investor profile	50 year old female.
Deterministic lapse rate	5% p.a.
Optimal lapsing	Yes.
Initial investment	\$100
Maturity term	10 years, maximum expiry on investor's 80^{th} birthday.
Resets	Two resets per year permitted until the investors 70^{th} birthday. Upon reset: Guarantee level = Asset level, Maturity extended by 10 years.
Mortality benefits	Guarantee paid out immediately upon the death of the investor.
M.E.R.	A proportional fee of $r_m = 2\%$ is allocated to fund manager of underlying fund. We will also charge additional proportional fees at a rate of r_e to cover the cost of providing the maturity guarantee. The total proportional fees are thus charged at a rate of $r_m + r_e$.
D.S.C.	A deferred sales charge is paid on early redemption using a sliding scale from 5% in first year to 0% after 5 years in fund.
Volatility	$\sigma = 20\%$
Interest rate	$r = 6\%$
Drift rate (before fees)	$\mu = 13\%$

TABLE 1: *Specification of the segregated fund guarantee contract and market information used during subsequent simulations.*

minimal for this demographic type.[3] The contract expires after the investor's 80^{th} birthday; i.e. the maximum duration of the contract is 30 years. The investor is not allowed to reset the guarantee level after her 70^{th} birthday.

The value of the guarantee will also depend upon market conditions. Here we assume that the guarantee is provided on a fund with a volatility of $\sigma = 20\%$. We also assume a spot interest rate of $r = 6\%$. Although we will not pursue this in the current paper, it would also be interesting to investigate the effect of deviations from these assumptions.

Finally, the level of investor optimality, i.e. how *efficiently* they use the reset feature, will affect the cost of providing the guarantee portion of these contracts. We begin by assuming optimal investor behaviour, which is clearly unrealistic.[4] We also study the effectiveness of using a heuristic rule for utilizing the reset feature. For example, in a report on segregated funds produced by a task force established by the Canadian Institute of Actuaries, it is suggested that issuers of these contracts should assume that investors will reset their guarantee whenever the value of the underlying mutual fund has increased by 15% over the investor's current guarantee level.

Investors can also lapse out of the contract. This can occur in one of two ways. First, for liquidity or other exogenous reasons, investors may simply choose to close their accounts. We assume that 5% of accounts are lapsed each year in this deterministic manner, i.e. independently of the value of the fund. Second, lapsing may in some cases be an optimal strategy for investors. As mentioned above, a D.S.C. is applied to the account during the first five years of the contract. Although this will tend to discourage lapsing, it is no longer relevant after this initial period, and if a high M.E.R. is being charged, the investors can lapse to avoid paying the remaining proportional fees. The investor should optimally choose to lapse out of the segregated fund contract if the value of the guarantee is less than the value of the proportional payments required to stay in the guarantee. This is particularly relevant during the closing decade where the investor has no more reset opportunities. If the asset value becomes much larger than the guarantee level, the investor should lapse out of the contract to avoid paying the remaining fees since the guarantee that they hold is unlikely to have any value at maturity.

3 A Mathematical Description of the Hedging Strategy

The guarantee embedded in a segregated fund can be thought of as a standard put option with additional optionality. In order to develop a hedging strategy for a segregated fund guarantee we follow a standard partial differential equation (PDE) approach as described in Merton (1973). For simplicity, suppose that we are modelling the value of a guarantee provided on a segregated fund which tracks an index which can be traded in a liquid market.

[3] This is consistent with work by Milevsky and Posner (2000) in the context of variable annuities in the U.S.

[4] Although the assumption of optimal investor behaviour is presently unrealistic due to a combination of lack of knowledge on the part of investors and the complex optionality embedded in the contracts, it is conceivable that in the future financial advisors could assist their clients in making more optimal decisions.

In general, we will not be able to hedge with such a perfectly correlated asset and it will be necessary to establish a minimum variance cross hedge.

We model the value of the index, S_I, as following the stochastic differential equation (SDE)

$$dS_I = \mu S_I dt + \sigma S_I dz, \tag{1}$$

where μ is the drift rate of the index, σ is the volatility and dz is an increment from a Wiener process. This expressions can be integrated to obtain

$$S_I(t) = S_I(0) e^{(r-\frac{1}{2}\sigma)t+\sigma\sqrt{t}\phi}, \tag{2}$$

where $S_I(0)$ is the initial value of the index and $\phi \sim N(0,1)$.

If S represents the value of the underlying segregated fund, then S follows the stochastic differential equation

$$dS = (\mu - (r_m + r_e))S dt + \sigma S dz, \tag{3}$$

where r_m is the management fee of the underlying index tracking mutual fund and r_e is the proportional fee charged to cover the cost of providing the guarantee. Again, we can integrate this SDE to obtain

$$\begin{aligned} S(t) &= S(0) e^{-(r_m+r_e)t} e^{(r-\frac{1}{2}\sigma)t+\sigma\sqrt{t}\phi} && (4) \\ &= e^{-(r_m+r_e)t} S_I(t). && (5) \end{aligned}$$

To hedge the segregated fund guarantee which has value V, we can create the following hedging portfolio Π as

$$\Pi = V - \Delta S_I \tag{6}$$

where Δ represents the number of shares of the index held. Note that in general, it is not possible for the insurer to hedge by directly trading in the underlying mutual fund. After standard analysis such as that found in Wilmott et al. (1993), one finds that the appropriate position in the index which eliminates risk over infinitesimal intervals is given by

$$\Delta = e^{-(r_m+r_e)t} \frac{\partial V}{\partial S}. \tag{7}$$

The value of the segregated fund guarantee, $V = V(S, K, U, T, t)$ depends upon the value of the underlying segregated fund S, the current guarantee level K, the number of reset opportunities used during the current calendar year U, the current maturity date of the guarantee T and the current time t. In Windcliff et al. (2000), the value is shown to satisfy a recurrence of linear complementarity problems,

$$\frac{\partial V}{\partial t} + (r - r_e - r_m)S\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rV - \mathcal{R}(t) r_e S + \mathcal{M}(t) \max(K - S, 0) \leq 0 \tag{8}$$

$$V^* \leq V \tag{9}$$

where one of (8,9) holds with equality. In these equations, V^* is the value of the contract which investors receive if they choose to reset, $\mathcal{R}(t)$ represents the fraction of the investors remaining in the contract (i.e. those who have neither lapsed nor died) at time t and $\mathcal{M}(t)$ is the rate of mortality at time t. In equation (8), the term $\mathcal{R}(t)r_eS$ represents the incoming proportional payments made by investors remaining in the contract towards the guarantee. The term $\mathcal{M}(t)\max(K-S,0)$ represents payments made to the estates of investors who pass away during the infinitesimal interval dt.

The mathematical complexity of this problem arises due to the minimum value constraint $V^* \leq V$. There are no known solutions to the linear complementarity problem (8,9) even in the highly simplified case where $r_m = r_e = \mathcal{M}(t) = 0$ and $V^* = \max(K-S,0)$, which is the standard American put option problem. In our situation, V^* represents the value of the contract the investor receives upon using the reset feature. If the investor has remaining reset opportunities left at time t, then

$$V^*(S,K,U,T,t) = V(S,S,U+1,t+10,t). \tag{10}$$

The solution to this linear complementarity problem is approximated using a penalty method as described in Windcliff et al. (2000); Zvan et al. (1998).

4 Simulating Contracts With Optimization Features

Standard Monte Carlo techniques are not well suited to simulate the behaviour of contracts which contain optimization features such as those contained in American put options and segregated fund guarantees with the reset feature. The reason is that we are unable to determine how the investor should utilize these features if we only use information available from the simulation. In this paper, we utilize a precomputed solution to the linear complementarity problem (8,9) to provide us with this information. Effectively, we solve backward from the terminal time to the present using our penalty method approach, storing information along the way regarding states at which it is optimal for investors to reset their guarantees or lapse out of the contract. We then simulate the performance of hedging strategies forward in time from the present, using the previously stored information to track actions taken by investors.

Other current research in applications of Monte Carlo techniques to optimization problems in finance has a significantly different focus. For example, Boyle et al. (1999) develop an application of the stochastic mesh method described in Broadie and Glasserman (1998) to value complex reset options. This method requires that the continuous early exercise feature of the contract can be well-approximated by exercise decisions at a set of specified discrete points in time. Further, the Green's function of the PDE must be easily computable. Essentially, in this application, Monte Carlo methods are used to stochastically integrate the Green's function.

It should be emphasized that we are interested in simulating the *real world* behaviour of these contracts. This is in contrast with Monte Carlo applications where simulation techniques are used to determine the *no-arbitrage* value of a contract. In other words, we are interested in simulations under the actual probability P-measure (as in "value at risk" type calculations), rather than the risk-neutral Q-measure used for pricing purposes. In our situation, we have already computed the no-arbitrage value by solving a recurrence of linear complementarity problems. The purpose of the simulations made here is to study the effects of some of the assumptions made during the development of our dynamic hedging strategy. Specifically, in this paper we address the rehedging interval and the degree of optimality displayed in investors' use of the reset feature. Of these, the second is novel as far as we are aware. The issue of discrete rehedging has been studied before (see e.g. Boyle and Emanuel (1980); Boyle and Hardy (1997)), but not in as complex a setting as here.

Since we do not adjust our hedge position continuously, and we do not always assume that investors will behave in an optimal fashion, the outcomes from our simulations will not be deterministic, and our hedged position is not going to be completely risk free. To assess tradeoffs between risk and return, we want to calculate quantities such as the present value of the expected profit, its standard deviation, and its value at risk.[5] As pointed out by Wirch and Hardy (1999), this latter risk measure can be quite misleading in our context. This is because a lot of the risk exposure can be in the extreme tail of the distribution (i.e. a small probability of a very large loss). Following Wirch and Hardy (1999), we will also compute a conditional tail expectation as an alternative to value at risk. All of this will give us quantitative information which will assist us in determining the necessary capital requirements for these contracts.

In Figure 1 we present a class diagram demonstrating a framework which can be used to simulate the performance of various hedging strategies for contracts which may involve optimization components such as segregated fund guarantees and American put options. This framework allows us to compare the performance of a standard actuarial reserve, where proportional fees are merely collected in a risk free account, with the performance of a dynamic delta hedging strategy which uses a standard finance no-arbitrage approach to replicating market outcomes. Further, we can study the effect of the assumption of optimal investor behaviour on the cost of providing the guarantee. Using the optimal exercise boundary computed during the solution of the linear complementarity problem (8,9), we can model the cost of providing the guarantee when the investor acts optimally. On the other hand, we can also impose a heuristic rule for utilizing the reset feature. In this work, we use the heuristic rule that the investor should reset the guarantee if the stock price rises 15% above the current guarantee level, as suggested by a task force on segregated funds established by the Canadian Institute of Actuaries.

4.1 Some Computational Details

The solution to the linear complementarity problem (8,9) was computed using techniques described in Windcliff et al. (2000). See Table 4.1 for details concerning the numerical PDE methods used to obtain the precomputed solution.

During the stochastic simulation, it was necessary to extract the hedging parameter $\Delta = \Delta(S, K, U, T, t)$ from our precomputed solution. In general, the value of the under-

[5] Note that we are using the term value at risk in the sense of a quantile reserve requirement. However, as is typical in insurance, this is over a much longer horizon (in our case up to 30 years) than is usually the case in finance, where the focus is on potential losses over a short time (e.g. over a 10 day period).

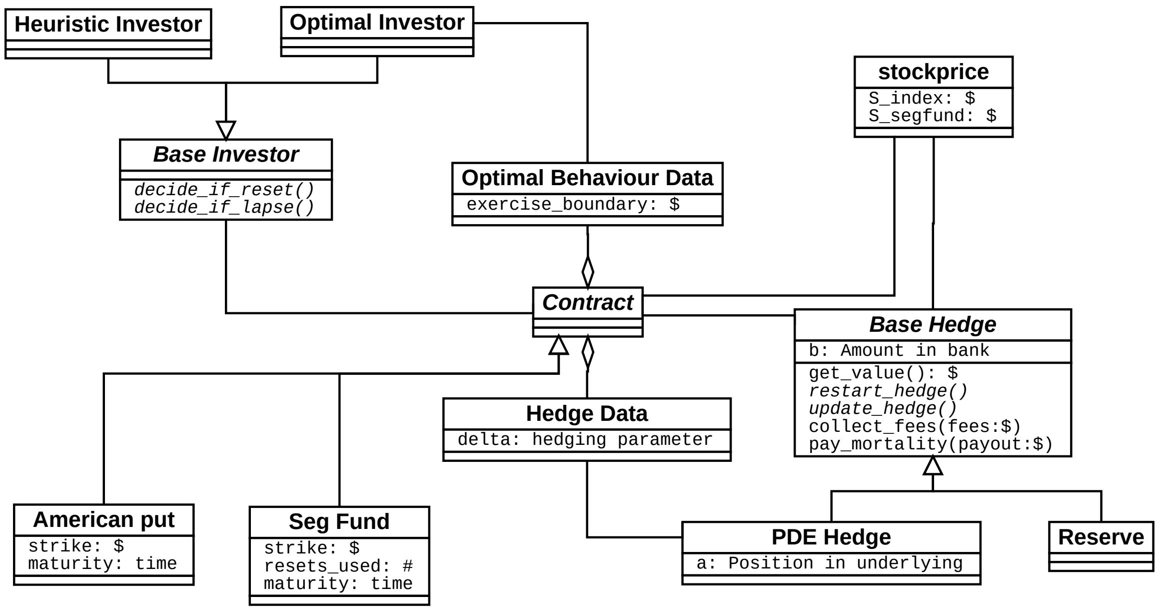


FIGURE 1: *A class diagram demonstrating a framework which can be used to simulate the performance of various hedging strategies for segregated fund guarantee. In this model, the investor can impose several different rules in order to decide when she should utilize the reset feature.*

Variable	Number of nodes	Notes
S	100	Irregular mesh, refined at strike.
K	1	Similarity transformation used (Windcliff et al. (1999)).
U	3	Discrete variable.
T	201	Number of active nodes at any instant (except during closing decade).
t	601	Timesteps

TABLE 2: *Details about the numerical PDE method used to obtain the precomputed solution.*

lying segregated fund S was not contained in the S-grid from the precomputed solution, necessitating in linear interpolation in the S-direction. The guarantee level was computed using a similarity transformation described in Windcliff et al. (1999) and as a result, no interpolation was necessary in the K-direction. All necessary values of the discrete variable $0 \leq U \leq U_{\max}$ were available in the precomputed solution, so no interpolation was necessary in the U-direction. Finally, the data in the precomputed solution was constructed so that there was no interpolation in the T-grid (for the expiry times) or in the t-grid (for the timesteps). This approach was chosen to avoid the extrapolation in these dimensions due to the dynamic nature of these grids; see Windcliff et al. (2000) for details about problems arising due to the dynamic discretization. It should be noted that we also ran simulations using a precomputed solution which used twice as many nodes for each variable. The effects of discrete rehedging were orders of magnitude larger than the errors introduced by the interpolation methods used.

When performing the Monte Carlo simulations, the integrated solution of the SDE (4) was used. Note that we did not employ any variance reduction techniques. This is because we wanted to assess *both* the mean and the variance of our hedging strategy: adopting a strategy to improve the precision of our estimate of the mean would have meant that our assessment of the risk of a strategy was biased downwards.

5 Results

In this section we compare the distribution of outcomes which are realized when a hedging strategy is implemented versus a reserve where fees are collected in a risk free account. We have noted elsewhere (Windcliff et al. (2000, 2001)) that in practice many of these contracts are under-priced from a no-arbitrage point of view. In other words, with the proportional fees which are being charged, it is not possible for institutions offering these products to hedge their risk exposure. Many firms have decided to not hedge these products and simply collect the fees without explicitly allocating any resources in the event of a market downturn.

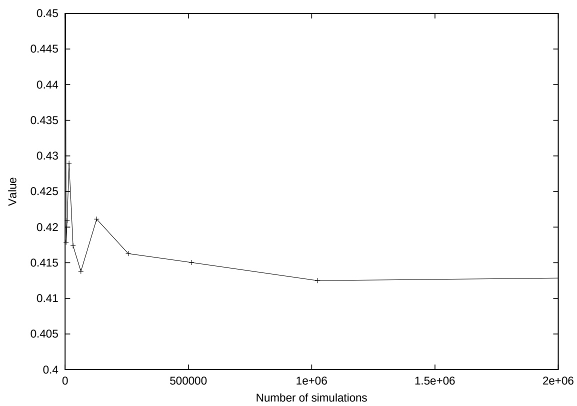
Since there is no initial charge for entering into a segregated fund guarantee, the relevant task when pricing these contracts is to determine the proportional fee r_e which must be charged in order to make the initial value of the guarantee zero. As described in Windcliff et al. (2000), the valuation is performed for various values of r_e until the initial value of the contract is smaller than some given tolerance. For the contract described in Section 2 the appropriate fee is $r_e \approx 1.52\%$. For this proportional fee the initial value is less than $\pm\$0.05$.

In Table 3 we can assess the effects of incorporating a dynamic hedging strategy to manage the risk associated with selling a segregated fund guarantee. The standard deviation has been reduced by a factor of approximately six in exchange for the expected profit in the unhedged case which is approximately \$8.[6] The convergence of these simulations is shown in Figure 2. Note that the y-axes in the two panels have radically different scales. Of course, given the

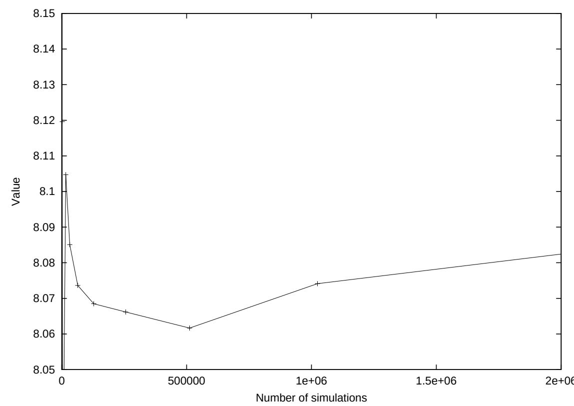
[6] Note that the numbers reported in Table 3 are standard deviations, not standard errors. The latter are usually reported in Monte Carlo studies to provide evidence regarding the precision of the estimate of the mean. In this case we are interested in the tradeoff between risk (as measured by standard deviation) and return, and so we report standard deviations. Of course, the standard errors can easily be computed from the standard deviations by dividing by the square root of the number of simulation runs.

Hedging strategy	Mean ($)	Standard Deviation
PDE hedge	0.41	1.55
No hedge	8.08	9.51

TABLE 3: *Statistics on the distribution of the present value of the returns realized when using a hedge computed from the solution of a PDE model and when using a standard actuarial reserve after 2048000 simulations when a fee of $r_e = 1.52\%$ is charged to pay for the guarantee. The PDE model was rehedged 20 times per year.*



(a) *Dynamic hedging strategy.*



(b) *No hedging plan.*

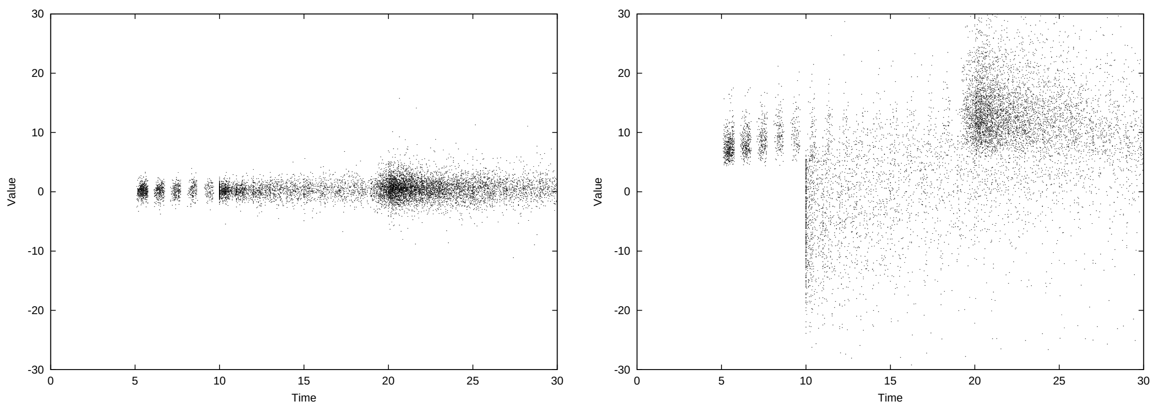
FIGURE 2: *Convergence of the expected present value of the outcomes as a function of the number of simulations performed. Note that different scales are used for the y-axes.*

reduced variance for the hedged position, much greater computational effort is required to reach a specific confidence level for the mean of the unhedged position.

The distribution of the present values of outcomes is shown in Figure 3 as well as the time at which the outcome is actually realized. Each outcome represents the maturity of an aggregate contract.[7] We emphasize that for these contracts the maturity date, T, is unknown in advance since it depends upon the investor's exercise strategy. Reported present values are computed by discounting payouts from T back to $t = 0$ at the risk free rate r. Therefore, present values should be interpreted as the amount which should be set aside at $t = 0$ in a risk free account in order to cover expected payouts on the contract.[8] Since the proportional

[7]The contracts have been aggregated over a population type during our modelling of mortality benefits (see Windcliff et al. (2000) for a complete discussion). The contract maturity dates plotted in figures such as Figure 3 are for the termination of an individual simulation, which occurs either when the guarantee matures or when it becomes optimal for investors to lapse. The plotted values in these figures are the difference between the present values of the hedge position and the value of the payout to the investors remaining when the simulation ends. At that time, the aggregate hedge position has already had payments deducted from it in the form of mortality benefits to investors who died earlier. Similarly, the values of the payout to investors remaining at the end of the simulation do not incorporate investors who lapse deterministically out of the contract before then.

[8]For valuation purposes, the use of the risk free rate can be at least approximately justified for the dynamic PDE hedge. One might reasonably argue that payoffs to unhedged positions should be discounted at a somewhat higher rate. As our focus here is on hedging, as opposed to valuation, we will not investigate this issue further here, except to note that discounting at a higher rate would clearly reduce the computed



(a) *Dynamic hedging strategy.* (b) *No hedging plan.*

FIGURE 3: *Present values of realized net positions $V_{net} = V_{hedge} - V_{payout}$ for the segregated fund guarantee described in Section 2 when the correct proportional fee $r_e = 1.52\%$ is charged. See accompanying text for a description of the behaviour illustrated in these figures. These plots are based on a sample of 10,000 simulations.*

fee being charged represents the fair value for the contract, the mean value of the position is approximately zero when the hedging strategy from the PDE model is implemented as seen in Figure 3(a). Ideally the value of the hedged position should always be identically zero. The variance arises due to the fact that the hedge is re-computed twenty times per year discretely rather than in a continuous fashion. An interesting point to note is that despite the fact that these contracts have very long maturities, the performance of the dynamic hedge does not degrade nearly as much as might have been expected.

Figure 3(b) shows the case where the insurer does not establish a dynamic hedge but rather simply collects the proportional fees in a reserve. We can see several distinct features:

- During the first five years there are no terminations of the aggregated contract. Due to the deferred sales charge (D.S.C.), during these years it is not optimal to lapse out of the contract. However, during years six through ten, it may be optimal for investors to lapse and leave the contract to avoid paying the proportional fees. In particular, this is the case if investors have already used their reset opportunities. The vertical bands which can be noticed in the figure correspond to the beginning of each year, where the investor receives two new reset opportunities. As a result, at these times the investors will not lapse out of the contract. It should be noted that if no hedging plan has been established, outcomes where the investor lapses result in a profit for the insurer since fees have been collected and no payments are made out of the fund. Of course, by not having a hedging plan the insurer is exposed to much more downside risk later on.

- After the tenth year, the maturity guarantees begin to expire. In Figure 3(b) we see that if the insurer has not hedged their position the outcomes display a great deal of variance with an expected loss. In fact, one can infer that quite large additional reserves would be required to cover a 95% value at risk measure.

mean and standard deviation of the unhedged strategy.

- During the closing decade (after the twentieth year), the investors have no more reset opportunities. Again, if the stock price rises above the current guarantee level it becomes optimal for the investors to lapse out of the contract to avoid paying the remaining proportional fees. In this situation, the unhedged position realizes an expected profit since fees have been collected and no payments are made to the investor upon lapsing.

We reiterate that the results shown here demonstrate that it is possible to considerably reduce the variance of the realized profits or losses by establishing a dynamic hedge. In the case that the fair value is charged for the contract, this reduction in uncertainty is achieved at the expense of the expected profit of the insurer. In fact, the fees charged for these contracts in the market are frequently *lower* than the fair value for these contracts. We believe that these prices reflect the view by insurers that most investors act with varying degrees of non-optimality. This issue will be dealt with in Section 5.2. Despite the reduction in the expected profit, it may be desirable to establish some hedging strategy in order to lessen the impact of new capital requirements which are being imposed due to regulatory concerns regarding these contracts.

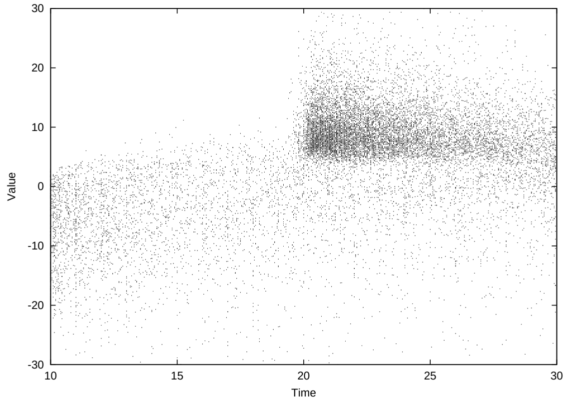
It has been observed in Windcliff et al. (2000), Windcliff et al. (2001) and Falloon (1999) that the fees being charged for these contracts are often insufficient to cover the required hedging costs to eliminate risk when optimal investor behaviour is assumed. If a proportional fee of $r_e = 1\%$ is charged to fund the guarantee, then from the viewpoint of the insurer, the initial cost of creating a hedge for this guarantee which is not covered by incoming payments is \$3.58. For the remainder of the simulations in this paper we assume that the proportional fee allocated to fund the guarantee is $r_e = 1\%$. The majority of segregated fund guarantees which are currently available are offered at fees which are less than this.

5.1 Discrete Rehedging Interval

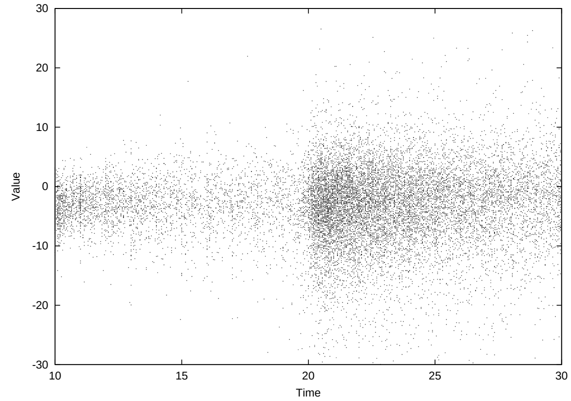
The theory used to derive the PDE hedging model assumes that the hedging position is updated continuously. This is clearly not possible in practice. Even high frequency (e.g. daily) hedging may not be possible, for two reasons. First, if rehedging is performed frequently, there will be substantial accumulated losses involved due to transaction costs. Second, for institutions offering many of these contracts to investors, it will not be possible to re-evaluate the positions required in a timely fashion.

In this section we use simulations to study the performance of the hedge if the hedging position is updated at a periodic interval. In the limit as the interval between rehedging times becomes *small* the variance of the outcomes should reduce to zero. Existing PDE methods do not give us a way to get a tight bound on the effect of varying the rehedging interval. It is our goal to quantify what we mean by a *small* rehedging interval so that we can understand the tradeoffs between frequent rehedging and reducing variance in the outcomes.

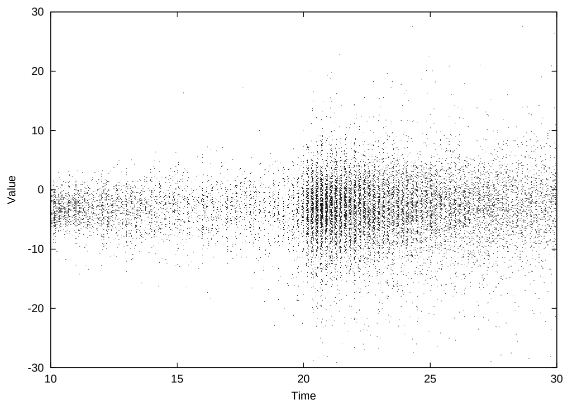
In Figure 4 we demonstrate the reduction in variance as the rehedging interval is decreased. Based on these results, it appears as though rehedging should performed approximately monthly. Notice that, in contrast with Figure 3, since the fees being charged are too low, it is not optimal to lapse during the first ten years of the contract (thus the x-axes of



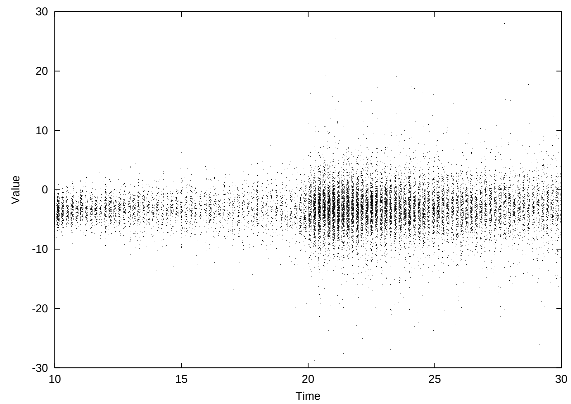
(a) *No hedging plan.*



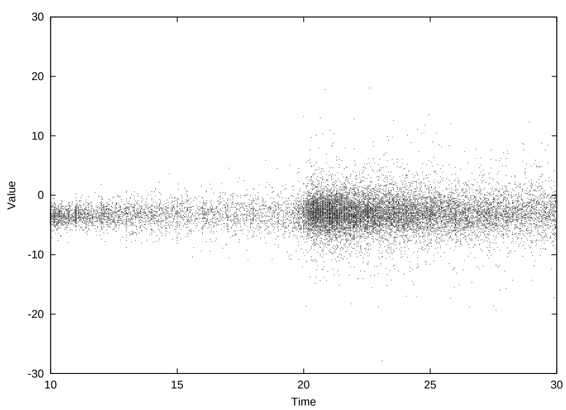
(b) *Rehedge once per year.*



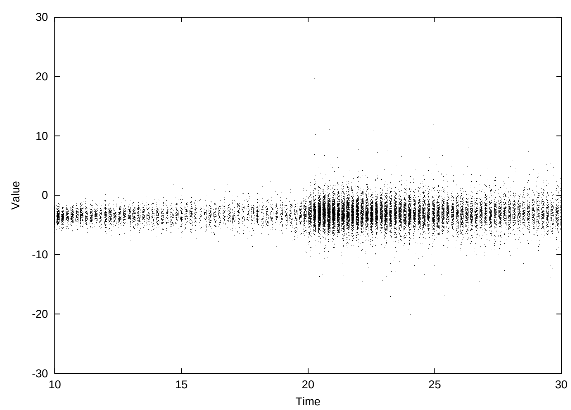
(c) *Rehedge twice per year.*



(d) *Rehedge five times per year.*



(e) *Rehedge ten times per year.*

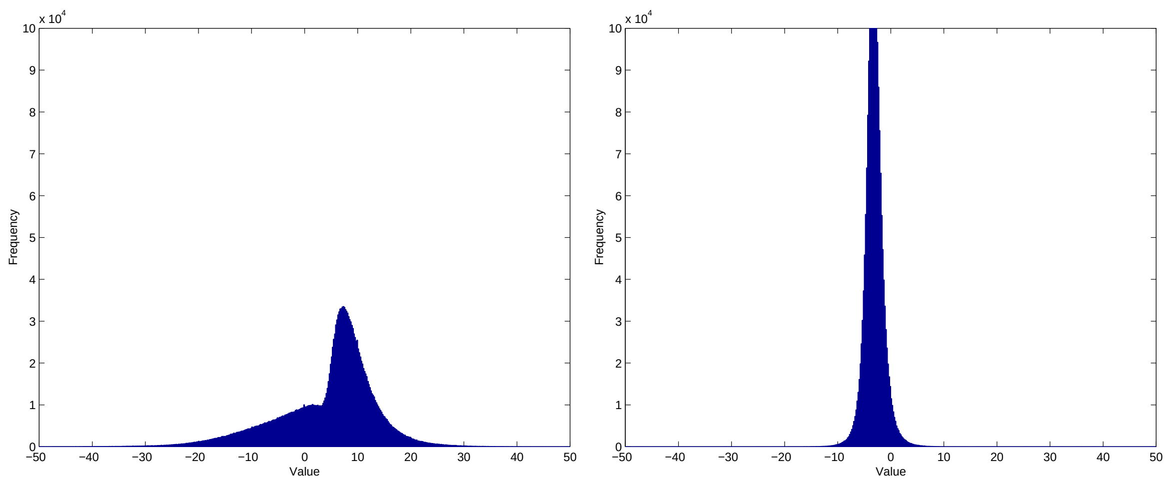


(f) *Rehedge twenty times per year.*

FIGURE 4: *The effect of the discrete rehedging interval on the distribution of outcomes. Points represent outcomes of individual price path simulations indicating the time at which the contract is terminated and the present value of the net position* V_{net} *of the insurer;* $V_{net} = V_{hedge} - V_{payout}$ *where* V_{payout} *is the value paid out the investor at contract maturity and* V_{hedge} *is the value of the hedging plan used by the insurer. These plots are based on a sample of 10,000 simulations.*

Rehedges per year	Expected value ($)	Standard deviation ($)
20	-3.12	1.83
10	-3.11	2.62
5	-3.16	3.55
2	-3.28	5.26
1	-3.33	7.01
No hedging	4.38	9.86

TABLE 4: *The expected present value and standard deviation of the returns on the sale of a segregated fund guarantee when a proportional fee of $r_e = 1\%$ is charged. This data was generated using 2048000 simulations.*



(a) *No hedging plan.* (b) *Rehedge twenty times per year.*

FIGURE 5: *Histogram of the distribution for the present value of the returns from the view of the provider of a segregated fund guarantee. A proportional fee of $r_e = 1\%$ is charged. These plots are based on 2048000 simulations.*

the figures start at the tenth year). Table 4 demonstrates the reduction in variance as the interval between updates to the hedging position is decreased. Histograms of the distributions of returns for the unhedged and twenty rehedges per year cases can be seen in Figure 5.

5.2 Heuristic Investor Behaviour

One of the most important applications of the simulation techniques described in this paper is the study of the effect of various heuristic investor behaviours on the distribution of returns on these contracts. The value and hedging strategy computed using the PDE model assumes optimal behaviour by the investors. By optimal behaviour, we mean that the investors' use of the reset feature results in the most expensive possible hedging strategy from the viewpoint of the insurer. Clearly this is not the way that most investors will utilize the reset feature.

One heuristic rule which has been suggested by a Canadian Institute of Actuaries task force on segregated funds is that investors should reset their guarantee if the asset price rises to be 15% greater than the current guarantee level. If there were no volatility, given a drift

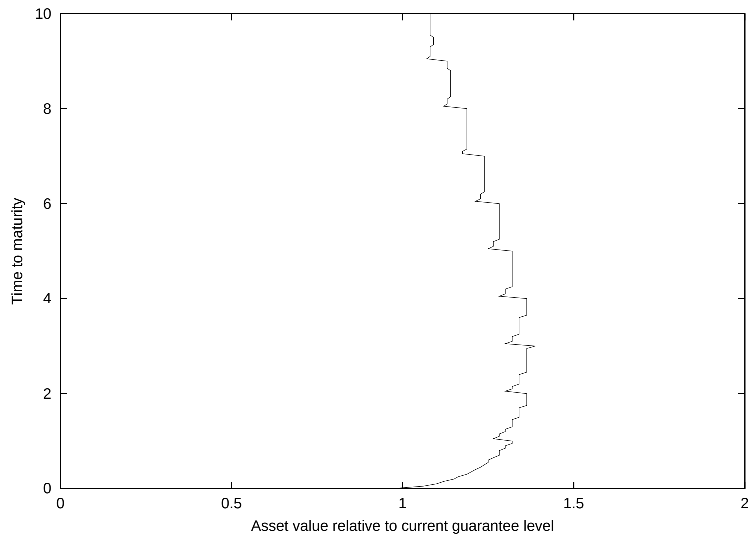


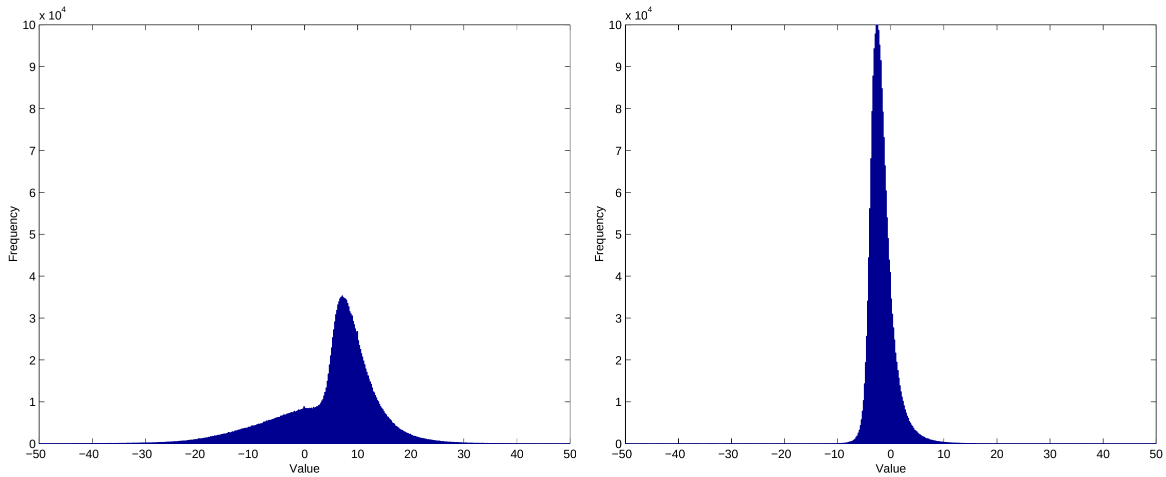
FIGURE 6: *The optimal exercise boundary, S^*, for the initial guarantee sold to the investor. The investor should optimally reset if $S > S^*K$ where S is the current asset value and K is the current guarantee level. Note that this is for the first use of the reset provision only (i.e. $U = 0, T = 10$). Upon resetting, U and T change and a different optimal exercise boundary applies.*

rate of $\mu = 13\%$, one would expect to use the reset feature approximately once per year. In Figure 6 we see that using 15% as a heuristic rule may be a reasonable approximation for the optimal exercise boundary of the initial guarantee during the first few years of the contract (i.e. when the time to maturity is near ten years), at least for the first use of the reset provision.

Figure 7 shows the distribution of returns realized when investors use a heuristic rule to determine their use of the reset feature. When compared with the results obtained when investors act optimally as in Figure 5, the mean value has increased and there is more positive skew in the case where the heuristic rule is applied.

A summary of the statistics of these distributions is given in Table 5. It is important to notice that the impact of heuristic investor behaviour is more favorable to the expected value of the PDE hedge than it is to the expected value when no hedging strategy is implemented. As a result, it may be possible to properly hedge the downside risk and still maintain an expected profit based on the assumption that investors will act non-optimally.

Another point to note is that although the standard deviation of the hedged position is larger in the case of heuristic investor behaviour, this is a result of the positive skew in the distribution. This can be observed in Figure 7(b), where the downside risk has been mostly removed using the dynamic hedge from the PDE model. As investors use the reset feature non-optimally, more positive skew is introduced. This is because the hedge from the PDE model always has enough resources to cover the most expensive sequence of actions that can be performed by the investor (at least in theory, if the hedged position is continuously adjusted). If the investor chooses to use the reset feature in some different fashion, the PDE hedge will still have sufficient resources and some of the excess that has been charged to the



(a) *No hedging plan.*

(b) *Rehedge twenty times per year.*

FIGURE 7: *Histogram of the distribution for the present value of the returns from the view of the provider of a segregated fund guarantee when the investor utilizes a heuristic rule for using the reset feature. The investor heuristically resets the guarantee if $S \geq 1.15K$, where S is the value of the segregated fund and K is the current guarantee level. During the final 10 years when the investor has no more reset opportunities, the investor heuristically lapses if $S \geq 1.25K$. A proportional fee of $r_e = 1\%$ is charged. These plots are based on 2048000 simulations.*

investor can be recovered.

This has very important financial implications regarding the hedging of these contracts. It may make sense for the insurer to hedge the most expensive possible outcome assuming optimal investor behaviour rather than assuming a degree of investor non-optimality as in Windcliff et al. (2001). In the event that the investor does not act optimally, the insurer will realize some additional profit. In this way, the insurer will always have sufficient resources to cover the guarantees which have been offered, at least under the idealized conditions of the model.

	Investor behaviour			
	Optimal		Heuristic	
Hedging strategy	Expected value ($)	Standard deviation ($)	Expected value ($)	Standard deviation ($)
PDE hedge	-3.12	1.83	-1.68	2.35
No hedge	4.38	9.86	4.82	9.66

TABLE 5: *The comparison of mean and standard deviation of the returns on the sale of a segregated fund guarantee when a proportional fee of $r_e = 1\%$ is charged when heuristic and optimal investor behaviours are applied. The PDE hedging strategy used involved rehedging twenty times per year. This data was generated using 2048000 simulations.*

Hedging strategy	95% VaR ($)	95% CTE ($)	99% VaR ($)	99% CTE ($)
No hedge	13.90	21.55	28.87	36.15
PDE hedge	5.76	7.12	7.80	9.69

TABLE 6: *Comparison of VaR measures and conditional tail expectations (CTE) for our prototypical segregated fund contract when $r_e = 1\%$ when no hedging is performed and when using a dynamic delta hedging strategy which is updated twenty times per year.*

5.3 Effect of Hedging on Reserve Requirements

As mentioned in the introduction, new capital requirements have recently been imposed for providers of segregated fund guarantees. By implementing a hedging strategy, it may be possible for these regulatory requirements to be reduced significantly.

One standard measure which can be used to determine an appropriate reserve of capital is the standard value at risk (VaR) measure. Briefly, a 95% VaR measure is the amount of capital that is required to ensure that there are sufficient resources to cover all but the worst 5% of the possible outcomes.[9]

We can use the simulation techniques described in this paper to find the 95^{th} percentile for our prototypical segregated fund contract. In Table 6 we see that when a proportional fee of $r_e = 1\%$ is charged to cover the cost of providing the guarantee we find that a reserve of $13.90 is required if the contract is unhedged. By using a dynamic delta hedging strategy which is updated twenty times per year, the required reserve can be reduced to $5.76. These amounts represent the capital that is required per $100 notional value to ensure that the insurer has sufficient resources to back up the guarantee 95% of the time. For a 99% VaR, the unhedged strategy requires a reserve more than twice as high (compared to the 95% VaR). By contrast, the increase for the hedged strategy from the 95% to the 99% VaR is about 35%.

Recently, Wirch and Hardy (1999) have demonstrated that additional measures should be considered when managing certain types of risk. This is particularly relevant to the case of segregated funds, since much of the risk exposure comes from large losses which occur with small probability. As one alternative, Wirch and Hardy (1999) propose the conditional tail expectation (CTE). The 95% CTE is simply the mean value of the outcomes conditional on being in the worst 5% from the viewpoint of the insurer. The CTE helps to quantify the remaining risk which has not been considered during the VaR calculation. For the segregated fund guarantee studied in this paper, when a proportional fee of $r_e = 1\%$ is charged, the 95% CTE is $21.55 if the position is unhedged, but is $7.12 when the dynamic hedging strategy is implemented (and updated twenty times per year). The differences are similar for the 99% CTE. These results indicate that the dynamic hedge performs very well at reducing downside risk for the providers of these contracts.

[9]See Duffie and Pan (1997) for a comprehensive review. In addition, a vast amount of information is available at `http://www.gloriamundi.org`.

6 Conclusions and Future Work

In this paper we describe techniques which allow us to perform stochastic simulations on problems in finance which involve optimal decisions by the investor. During the Monte Carlo simulation, we use a previously computed solution to determine the timing of optimal decisions.

As a case study, we demonstrate these techniques by simulating the performance of a hedging strategy for segregated fund guarantees. In particular, we look at how frequently the hedges for these contracts should be updated and find that when the hedge is updated twenty times per year the standard deviation of the outcomes is approximately reduced by a factor of six when compared with an unhedged contract.

Another important aspect of this work is that it allows us to study the effect of non-optimal investor behaviour on the cost of providing the guarantee. In this work we study a heuristic rule where the investor resets the guarantee if the asset level is larger than the current guarantee level by a factor of 1.15. It is possible that for sufficiently sub-optimal investor behaviour, the downside risk of these contracts can be hedged with the current fees being charged.

Recently, new capital requirements have been imposed on companies providing these contracts. By implementing a hedging strategy, it may be possible for these companies to reduce their capital requirements substantially. In this paper we find that the 95% VaR requirements are reduced from $13.90 to $5.76 per $100 notional value for the prototypical contract studied. The 95% CTE was also substantially improved by implementing the hedging strategy. Similar results were obtained for the 99% VaR and CTE cases.

There are numerous avenues for future research in this area. Since these contracts are very long term, the assumption of constant volatility and interest rates is questionable. It would be informative to study the effects of stochastic volatility and interest rates on the hedging strategies produced. Moreover, we have ignored transactions costs. It would be interesting to investigate the tradeoff between these costs and the required frequency of rehedging. Additionally, it would be desirable to investigate alternative dynamic hedging plans. One example would be a "move-based" strategy, in which the hedge position is rebalanced if the value of the underlying instrument has changed by a specified amount.[10] It would also be interesting to explore delta-gamma strategies, rather than just delta-hedging.

Another important financial modelling issue is that as mentioned, it is often not possible for the insurer to establish a hedge by trading directly in the underlying asset. In this paper we assume that we are insuring a segregated fund guarantee which is offered on an index tracking fund. If we do not have a perfectly correlated asset with which to establish a hedge, it will be necessary to set up a minimum variance cross hedge.

Moreover, if investors act sub-optimally, it may be possible to establish a hedge which removes downside risk while still having positive expected present value even though the correct no-arbitrage fee is not being charged. In this paper we study one simple heuristic

[10]Note that Boyle and Hardy (1997) investigate these types of issues (i.e. transactions costs, alternative hedging strategies) in the much simpler context of a maturity guarantee with no embedded optionality. They find that the move-based strategy outperforms the type of strategy we have considered here, where the hedge is rebalanced at fixed points in time.

rule to model investor behaviour. It would be interesting to collect data on the behaviour of actual investors in order to better model their actual behaviour.

Finally, it may be worthwhile to investigate the impact of alternative simulation methods, especially with regard to variance reduction. This would appear to require separate simulations to assess expected values and various risk measures, rather than the all-in-one approach taken here. In particular, recent developments in low discrepancy sequences and quasi-Monte Carlo simulation (see e.g. Joy et al. (1996)) are a promising avenue for future research.

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