Optimal Asset Allocation
For Outperforming A Stochastic Benchmark Target

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Abstract
We propose a data-driven Neural Network (NN) optimization framework to determine the optimal multi-period dynamic asset allocation strategy for outperforming a general stochastic target. We formulate the problem as an optimal stochastic control with an asymmetric, distribution shaping, objective function. The proposed framework is illustrated with the asset allocation problem in the accumulation phase of a defined contribution pension plan, with the goal of achieving a higher terminal wealth than a stochastic benchmark. We demonstrate that the data-driven approach is capable of learning an adaptive asset allocation strategy directly from historical market returns, without assuming any parametric model of the financial market dynamics. Following the optimal adaptive strategy, investors can make allocation decisions simply depending on the current state of the portfolio. The optimal adaptive strategy outperforms the benchmark constant proportion strategy, achieving a higher terminal wealth with a 90% probability, a 46% higher median terminal wealth, and a significantly more right-skewed terminal wealth distribution. We further demonstrate the robustness of the optimal adaptive strategy by testing the performance of the strategy on bootstrap resampled market data, which has different distributions compared to the training data.

1 Introduction
We propose a data-driven framework to compute optimal multi-period dynamic strategies for outperforming a general stochastic benchmark target, which is an important portfolio management problem with immediate practical applications. There is a large extant literature on techniques for constructing portfolios which outperform a stochastic benchmark, e.g., (Browne 1999, 2000; Tepla 2001; Basak et al. 2006; Davis and Lleo 2008; Lim and Wong 2010; Oderda 2015; Alekseev and Sokolov 2016; Samo and Vervuurt 2016; Al-Aradi and Jaimungal 2018).

Typically, outperforming a multi-period investment benchmark is formulated as an optimal stochastic control problem under an assumed model for trading asset price dynamics, e.g., (Oderda 2015; Al-Aradi and Jaimungal 2018). In Oderda (2015), under the assumption that stocks follow a geometric Brownian motion and no investing constraints (i.e. infinite leverage, trading continues if insolvent, and shorting is allowed), the authors find that a portfolio which outperforms (under certain criteria) the benchmark market capitalization index can be constructed by a combination of (i) the benchmark portfolio and (ii) rule-based
portfolios such as equal weight and minimum variance portfolios. The determination of the optimal weightsor these portfolios is independent of estimates of the expected returns of individual stocks. Hence this
outperformance portfolio is robust to uncertainty in the expected return parameters. A natural conjecture
is that determining asset allocation strategies that outperform a benchmark may be robust in general.

In [Al-Aradi and Jaimungal (2018)], optimal stochastic control techniques are also used in this context.
Based on several assumptions, [Al-Aradi and Jaimungal (2018)] formulate the control problem as a Hamilton-
Jacobi-Bellman (HJB) Partial Differential Equation (PDE), and are able to obtain a closed-form solution.
However, of necessity, this approach requires (i) the assumption of a parametric model for the Stochastic
Differential Equations (SDEs) governing the asset price processes and (ii) no constraints on the portfolio
(i.e. infinite leverage is allowed). It is possible, in some cases, to solve the HJB PDE numerically, and thus
include more realistic constraints.

There are two main challenges in the aforementioned methods for the stochastic optimal control outper-
forming benchmark problem. Firstly, unless the benchmark is specifically restricted, it can add additional
stochastic state variables in the optimal control problem ([Al-Aradi and Jaimungal 2018]). This makes solv-
ing the PDE formulated control problem numerically challenging, due to the curse of dimensionality. Hence,
this technique is limited to a small number of stochastic factors (i.e. less than four). Secondly, a parametric
model of the asset returns needs to be postulated, which adds challenges as the parameters can be difficult
to estimate accurately.

To overcome the aforementioned challenges, in this work we use market asset return data directly to
solve a scenario-based stochastic optimal control formulation, corresponding to the original stochastic control
problem. This avoids the need to make model assumptions and parameter estimations. In addition, we solve
the stochastic optimal control problem directly, without invoking dynamic programming to transform it into
a PDE problem (thus avoiding the curse of dimensionality). The optimal control is represented as a neural
network (NN) which is learned through training. The features for the NN can include any state variable that
influences the optimal strategy, including the state variables associated with a stochastic target. We design
a specific objective function to create a desirable terminal wealth distribution. This is done by measuring
the relative performance of the strategy against an elevated final wealth of the stochastic target strategy to
penalize extreme losses and limit unlikely extreme gains.

We formulate a general optimal control problem for the multi-period asset allocation portfolio which
outperforms a benchmark as an optimal stochastic control problem. We propose a benchmark target-based
objective function which measures the difference between the terminal wealth of the adaptive strategy and a
path-dependent elevated target (which is the terminal wealth of the constant proportion strategy multiplied
by a pre-defined growth factor). The objective function is designed as a double-sided penalty function to
force the terminal wealth of the adaptive strategy to be close to the elevated target. The NN model takes
three features as inputs: the current wealth of the adaptive portfolio, the current wealth of the constant
proportion portfolio, and the time remaining. In the case that the underlying assets follow simple stochastic
processes, it can be shown that the control is only a function of these variables.

Instead of formulating the problem as an HJB equation derived from dynamic programming, we solve the
single original optimal control problem directly as in [Li and Forsyth (2019)]. We define an objective function
in terms of the terminal wealth, and then solve for the control directly, using a data-driven approach. The
proposed data-driven approach does not require an estimation of the parameters of an assumed parametric
model for traded assets. We represent the control using a shallow neural network (NN). We remark that
shallow learning is found to outperform deep learning for asset pricing in [Gu et al. (2018)]. We also note that
good results are obtained in [Hejazi and Jackson (2016)] with an NN containing only one hidden layer (shallow
learning), in which the shallow neural network learns a good choice of distance function for efficiently and
accurately interpolating the Greeks for the input portfolio of Variable Annuity contracts.

To illustrate the proposed framework, we consider a practically relevant and important problem: optimal
multi-period asset allocation during the accumulation phase of a DC pension plan. A defined contribution
(DC) plan is a retirement plan in which the employer, employee, or both make contributions regularly with
no guarantee on the accumulated amount in the plan at the retirement date. In contrast, another type of
retirement plan is the defined benefit (DB) plan, which promises to pay a set income when the employee
In a DC plan, the employee (investor) is often presented with a list of eligible stock and bond funds, and then needs to specify how the DC account is to be allocated to each fund. Typically the employee has the opportunity to change the asset allocation at least yearly. Normally, the DC plan is tax-advantaged, so that there are no tax consequences triggered on rebalancing. A typical DC plan accumulation phase would occur over 30 years, assuming a 30-year-lifetime employment period. The choice of the asset allocation strategy is crucial to the terminal wealth in the DC fund.

A popular asset allocation strategy for retirement plans is the constant proportion strategy, in which the employee invests fixed proportions of the wealth into several assets. This idea can be traced back to [Graham (2003)]. Among the constant proportion strategies, a very popular one is the 50/50 strategy, in which 50% of the wealth is allocated to stocks and 50% of the wealth is allocated to bonds. It is conventional wisdom that a 50/50 portfolio is an appropriate tradeoff between risk and reward for those saving for retirement. Although there has been a popular shift to a 60/40 portfolio (60% in stocks) in recent years, for illustration, we will focus on the 50/50 portfolio in this article. This would be a typical average allocation to equities over the full accumulation phase of a lifecycle fund. Note that, in [Forsyth and Vetzal (2019)], it is shown that the final wealth distributions of a constant weight allocation, and any glide path strategy having the same average allocation as the constant weight strategy, are essentially the same. Hence there is little to be gained by using a (deterministic) glide path compared to a constant weight strategy. Using the proposed framework to determine the optimal multi-period dynamic asset allocation strategy for outperforming a stochastic target, we address a natural and interesting question of whether it is possible to develop a dynamic allocation strategy that outperforms the constant proportion strategy.

It is common practice in the financial industry to train and test strategy performance by splitting the historical market data path into two segments - one for training and the other for testing. We take a different approach. We aim to determine an investment strategy that would perform well statistically on a large set of data paths created through bootstrap resampling, rather than on a single historical data path. To achieve this, we generate additional data paths from the historical market data path by block bootstrap resampling of the historical data (see, e.g., [Politis and Romano (1994); Politis and White (2004); Patton et al. (2009)]). Once we have a large set of price paths from bootstrap resampling, we split them into training data set and testing data set.

To demonstrate the robustness of our approach, we test the optimal adaptive strategy on market data with different distributions from the training data. We first test the optimal adaptive strategy, learned from bootstrap resampled data with a given expected blocksize, on bootstrap resampled data with different expected blocksizes (thus different distributions, as noted by [Politis and Romano (1994)]). We then test the adaptive strategy learned from synthetic data generated from a parametric jump-diffusion stochastic process (estimated from the same single historic path) on bootstrap resampled data. Finally, we test the strategy learned on bootstrap resampling data from a segment of the historical market data path on bootstrap resampling data generated from another non-overlapping segment of the historical data path.

To the best of our knowledge, the closest work related to the research in this paper is [Samo and Vervuurt (2016)], in which the authors also use a data-driven machine learning approach for constructing a dynamic strategy which outperforms a benchmark. [Samo and Vervuurt (2016)] approximate the control by a Gaussian process and solve the optimal hyperparameters using Bayesian inference. However, they do not assess the distributional properties of the investment strategy, but rather evaluate the performance on a single historical path. In addition, they only validate the performance of the strategy for a relatively short period from 1992-2014. In contrast to our focus in this work, they consider the case of daily rebalancing with a large number of stocks which would not be typical of a defined contribution pension plan.

In this paper, we consider investment portfolios which are combinations of a stock index and a bond

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1A lifecycle fund is based on the intuitive concept of allocating a high equity weight during the early employment years, and then moving to bonds as retirement nears. However, as shown in [Graf (2017)], this strategy does not outperform a constant weight strategy.
index. This type of portfolio would be typical of a defined contribution (DC) pension plan. Our objective is to achieve a good balance of risk and return. Specifically, we aim to shape the cumulative distribution function of the terminal wealth so that it has desirable risk-return characteristics. We use yearly rebalancing so that trading costs are minimal. Our data is based on historical monthly index returns dating back to 1926, which provides a comprehensive coverage of historical financial market cycles. To augment the data for training and testing purposes, we use the stationary block bootstrap technique (Politis and Romano, 1994).

Furthermore, our approach differs from (Samo and Vervuurt, 2016) in the learning methodology, both with respect to learning algorithms and data utilization. Our approach can be applied to a general multi-period asset allocation problem with few assumptions. In addition, it can readily be scaled up to high dimensional problems (i.e. more assets and features). A shallow network is sufficient here, leading to a relatively small number of parameters and computationally efficient training. In contrast to Samo and Vervuurt (2016), we use a small number of feature variables that only depend on the state of the adaptive portfolio and the benchmark portfolio, rather than market-related signals. As a result, the trading strategy is easy to interpret, practical to implement and the model is less prone to overfitting. Furthermore, our computational results demonstrate that the optimal adaptive strategy has a higher expected terminal wealth as well as a more favorable terminal wealth distribution than the constant proportion benchmark strategy.

In summary, we make the following contributions in this research:

- We propose a data-driven solution to a general optimal dynamic asset allocation for outperforming a stochastic benchmark, which is formulated as a stochastic control problem. The data-driven learning bypasses the need for a robust estimation of parameters of an assumed parametric model. In addition, closed-form solutions are only available assuming simple parametric price processes and no portfolio constraints (i.e. infinite leverage is allowed) (see, for example, Oderda (2015); Al-Aradi and Jaimungal (2018)). Existing solution techniques, which require dynamic programming, are computationally infeasible due to the high dimensionality. In this work, we formulate the controls as the outputs of a neural network function and avoid the curse of dimensionality of a PDE approach. We use a gradient-based optimization method to solve for the controls. This approach naturally extends the method in (Li and Forsyth, 2019) to the problem of outperforming a stochastic benchmark. Our philosophy here is similar to that in Samo and Vervuurt (2016), although our method of implementation and scope are significantly different.

- Unlike the commonly used one-sided quadratic shortfall objective function, we propose an asymmetric distribution shaping objective function for the optimal asset allocation problem. The proposed objective function aims to produce an optimal dynamic and adaptive strategy which can yield significantly higher median terminal wealth than the stochastic benchmark, with only a small probability (and magnitude) of underperformance.

- Recognizing financial data scarcity, we use block bootstrap resampling to generate both training data and testing data. We observe that the block bootstrap resampling data sets generated using different expected block sizes lead to performance testing against different distributions. We mathematically establish upper bounds on the probability of a training path being equivalent to a testing path to justify the soundness of the proposed stationary block bootstrap method even when the same expected block sizes are used to generate training and testing data sets.

- We apply the proposed data-driven framework to the allocation of the DC pension plan. In this context, the constant proportion strategy is a popular asset allocation strategy because of its simplicity in execution and its capability of diversifying market risks effectively. However, constant proportion strategies are not able to adapt to different market scenarios because of the predefined fixed allocations. It is a popular active research problem within the financial industry to devise schemes that consistently outperform the constant proportion strategy.

\(^2\)Since we consider benchmark and optimal portfolios as having two assets each, this would result in a four-dimensional PDE problem, assuming discrete rebalancing.
• Our work has significant empirical importance and implications. The optimal adaptive asset allocation strategy learned from the data-driven framework has a more favorable terminal wealth distribution than the constant proportion strategy with a higher expected terminal wealth and significantly less downside risk. In addition, the optimal adaptive strategy has consistently higher expected wealth compared to the constant proportion strategy over the entire investment period. Finally, the optimal adaptive strategy is robust in the sense that it performs well on bootstrapped market data with different distributions.

2 Formulation for Outperforming a Stochastic Benchmark

Let the initial time \( t_0 = 0 \) and consider a set \( \mathcal{T} \) of rebalancing times

\[
\mathcal{T} \equiv \{ t_0 = 0 < t_1 < \ldots < t_N = T \}. \tag{2.1}
\]

The fraction of total wealth allocated to each asset is adjusted at times \( t_n, \ n = 0, \ldots, N - 1 \), with the investment horizon \( t_N = T \). Consider an investment problem in \( M \) risky and riskless assets.

Assume that, at time \( t \), a fund holds wealth of amount \( W_m(t) \) in asset \( m, m = 1, \ldots, M \). The total value of the portfolio at \( t \) is then

\[
W(t) = \sum_{m=1}^{M} W_m(t). \tag{2.2}
\]

For any given time \( t \) and arbitrary function \( f(t) \), define \( f(t^+) = \lim_{\epsilon \to 0^+} f(t + \epsilon) \), and \( f(t^-) = \lim_{\epsilon \to 0^+} f(t - \epsilon) \).

Assume that \( W(t_0^-) = 0 \), i.e., the initial value of the portfolio before any cash injection is zero, and let \( q(t_n) \) represent an a priori specified cash injection schedule.

We denote the allocation at stage \( n \) by an allocation vector \( p_n, n = 0, \ldots, N - 1 \). Given the allocation control vectors \( p_0, p_1, \ldots, p_{N-1} \), the statistical properties of the terminal wealth of the adaptive portfolio \( W(T) \) can be determined. Similarly, given a benchmark allocation vector \( \tilde{p}_n \), the final wealth of the benchmark portfolio \( W_b(T) \) can also be determined. The time evolution of \( W(t) \) and \( W_b(t) \) is given by

\[
\begin{align*}
\text{for } n = 0, 1, ..., N - 1, \\
W(t_n^+) &= W(t_n^-) + q(t_n) \\
W_b(t_n^+) &= W_b(t_n^-) + q(t_n) \\
W(t_{n+1}^-) &= p_n^T R(t_n) W(t_n^+) \\
W_b(t_{n+1}^-) &= \tilde{p}_n^T R(t_n) W_b(t_n^+)
\end{align*}
\]

end,

where \( R(t_n) \) is the vector of returns on assets in \( (t_n^-, t_n^+) \).

Our first goal is to minimize some measure of underperformance against the benchmark. A natural choice is to quadratically penalize the underperformance of the terminal wealth of the adaptive strategy compared to a benchmark of the terminal wealth of the constant proportion strategy, as in [Li and Forsyth (2019)]. Note, however, that in our case, the benchmark is stochastic. This leads to the following optimization problem

\[
\min_{p_0, p_1, \ldots, p_{N-1}} \mathbb{E} \left[ \min \left( W(T) - W_b(T), 0 \right)^2 \right]. \tag{2.3}
\]

Unfortunately, the optimal solution to (2.3) is trivially the benchmark strategy \( p_n = \tilde{p}_n, \forall n \), which indicates the formulation (2.3) does not sufficiently capture properties of the desired solution.
We propose to generate a more ambitious strategy by using an elevated target $e^{sT} \cdot W_b(T)$ in the objective function, i.e.,

$$\min_{p_0,p_1,\ldots,p_{N-1}} \mathbb{E} \left[ \min (W(T) - e^{sT} \cdot W_b(T), 0)^2 \right], \quad (2.4)$$

where $s$ is the yearly pre-determined target outperformance spread. Consequently, in an ideal case, the adaptive strategy will have a terminal wealth of $e^{sT} \cdot W_b(T)$ which indicates that the adaptive strategy achieves an annual outperformance spread of return $s$ compared to the benchmark strategy.

We note, however, that if the outperformance spread $s$ is large, (2.4) will tend to generate a strategy that concentrates on the asset with the highest rate of returns, which can potentially lead to an unacceptable probability of underperformance. We can see that outperforming a target is a complex distribution shaping problem with multiple criteria which is difficult to formulate and to compute. Recognizing that low probability underperformance scenarios often come with low probability high outperformance scenarios, we choose an asymmetric objective function which controls the loss-side tail by penalizing the underperformance quadratically, while at the same time penalizing the outperformance linearly.

Our asymmetric distribution shaping benchmark outperforming formulation becomes

$$\min_{p_0,p_1,\ldots,p_{N-1}} \mathbb{E} \left[ \min (W(T) - e^{sT} \cdot W_b(T), 0)^2 + \max (W(T) - e^{sT} \cdot W_b(T), 0) \right]. \quad (2.5)$$

Figure 2.1 illustrates this asymmetric distribution shaping objective function.

![Asymmetric distribution shaping objective function with elevated target $e^{sT} \cdot W_b(T)$](image-url)

Figure 2.1: Asymmetric distribution shaping objective function with elevated target $e^{sT} \cdot W_b(T)$.

We remark that distribution shaping objectives can be problem dependent and we choose the objective function (2.5) for the pension investment problem. Furthermore, our proposed framework does not depend on any specific form of the objective function.

If we postulate parametric stochastic processes for prices of the traded assets, mathematically, the controls $p_0,\ldots,p_{N-1}$ can be determined using dynamic programming. This will result in a nonlinear HJB PDE (see (Al-Aradi and Jaimungal, 2018) for example). In the absence of any closed-form solution, computing a solution of this problem numerically would be costly, particularly when the problem has a high dimension. Consider the simplest allocation problem, for which the portfolio consists of a stock index and a bond index. In the case of discrete rebalancing, the state variables would be the dollar amounts in the bond and stock indices, for both the adaptive and target portfolios (Dang and Forsyth, 2014). Consequently, even for this comparatively simple case, this would result in a four-dimensional HJB PDE.
Assume that samples of asset returns are available. These samples can come directly from market observations or from simulations of postulated parametric models. Instead of solving \( p_0, \ldots, p_{N-1} \) using dynamic programming, we propose a data driven approach as follows. We represent the optimal control as a function of several features \( F(t) \), i.e., at \( t_n, n = 0, 1, \ldots, N - 1 \),

\[
p_n = p(F(t_n))
\]

Example 1 (Two Asset Problem with Benchmark \( W_{50/50} \)). In our numerical examples, we will focus on portfolios consisting of two assets: a stock index and a bond index. The benchmark portfolio in this case will be a constant proportion strategy, with 50% stocks and 50% bonds. We will denote the wealth of the benchmark strategy in this case as \( W_{50/50}(t) \). For this example, for the stochastic target pension allocation problem, we use three features for \( F(t) \): (i) \( W(t_n) \), the wealth of the adaptive portfolio at \( t_n \), (ii) \( W_{50/50}(t_n) \), the wealth of the constant proportion portfolio at \( t_n \), (iii) \( T-t \), time remaining in the investment period. In the case that simple stochastic processes are assumed, then it can be shown (in the absence of transaction costs) that the controls are only a function of these features [Dang and Forsyth 2014].

We remark that our feature set \( F(t) \) for Example 1 is different from the features in Samo and Vervuurt (2016) which explicitly use security prices. Instead, at time \( t \) our feature set consists of the accumulated wealth at \( t \) from allocation strategy and benchmark strategy, which depend on the returns of traded assets from prior periods. Traded asset prices are not directly used as features for the neural network model. This is essentially because, at each rebalancing time, we search for the optimal adaptive strategy amongst all strategies with the current level of wealth. In addition, since we evaluate the performance of a trading strategy based on the terminal wealth \( W(T) \) only, the trading decision at time \( t \) depends on the current accumulated wealth and return distribution of future trading periods. Unless the asset price has predictability in its future return, including the prices as features is redundant in this context and will likely lead to overfitting of the model.

We use a 2-layer neural network as the functional form for the optimal control. As a result, the goal of the optimization problem is to find the optimal parameters of the neural network.

![Figure 2.2: A 2-Layer NN representing the control functions](image)

Assume that \( h \in \mathbb{R}^l \) is the output of the hidden layer. Let the matrix \( z \in \mathbb{R}^{dl} \) be the weights from the input features \( F(t_n) \in \mathbb{R}^d \) to the hidden nodes \( h \). We use the sigmoid activation function,

\[
\sigma(u) = \frac{1}{1 + e^u},
\]

and have

\[
h_j(F(t_n)) = \sigma(F_i(t_n)z_{ij}).
\]
Here we use double summation convention, i.e.
\[
F_i(t_n)z_{ij} \equiv \sum_{i=1}^d F_i(t_n)z_{ij}, \ j = 1, \ldots, l.
\]

At the output layer, we use the logistic sigmoid function as the activation function. Let the matrix \( x \in R^{LM} \) be the weights for output layer. For the \( m \)th asset, the asset allocation on this asset is given by:
\[
(p(F(t_n)))_m = \frac{e^{x_{km}h_k(F(t_n))}}{\sum_i e^{x_{ki}h_k(F(t_n))}}, \ 1 \leq m \leq M.
\]

Note that with the logistic sigmoid activation function, the following constraint is automatically satisfied
\[
0 \leq p(F(t_n)) \leq 1, \quad 1^T p(F(t_n)) = 1.
\]

This enforces the constraints of no-shorting and no leverage. In addition, insolvency cannot occur.

The dynamics of the terminal wealth of the adaptive portfolio then becomes
\[
\text{for } n = 0, 1, \ldots, N - 1

W(t_n^+) = W(t_n^-) + q(t_n)

W(t_{n+1}^-) = p(F(t_n))TR(t_n)W(t_n^+)
\]

end .
\[ (2.6) \]

We approximate the expectation in equation (2.5) by a finite number of wealth samples of \( W(T) \), computed from return samples of \( R(t_n) \) obtained by bootstrapping the historical data. Let \( W^\ell(T), W^\ell_b(T) \) be the final wealth samples for the adaptive and benchmark strategies, obtained using equation (2.6), along the \( \ell \)th return sample path \( R(t_n)^\ell, \ n = 0, 1, \ldots, N - 1 \).

Denote
\[
g(x) \equiv \min(x, 0)^2 + \max(x, 0).
\]

The expectation in equation (2.5) is approximated by
\[
E\left[g(W(T) - e^{sT} \cdot W_b(T))\right] \simeq \frac{1}{L} \sum_{\ell=1}^L g(W^\ell(T) - e^{sT} \cdot W^\ell_b(T))
\]

Since the approximate function on the right hand side of (2.8) is a nonconvex, continuous but piecewise differentiable function of the NN weights, solving the optimization problem is challenging.

We recognize however that \( E\left[g(W(T) - e^{sT} \cdot W_b(T))\right] \) is a continuously differentiable function of the NN weights assuming that the continuous return distribution is continuous. This motivates us to use the smoothing technique from Alexander et al. (2006). In equation (2.8), we replace \( g(x) \) by the smoothed approximation \( \bar{g}(x) \), to obtain a continuously differentiable approximation,
\[
\bar{g}(x) = \begin{cases} 
  x, & \text{if } x > \epsilon, \\
  \frac{x^2}{4} + \frac{1}{2}x + \frac{1}{4} \epsilon, & \text{if } -\epsilon \leq x \leq \epsilon, \\
  (x + \epsilon)^2, & \text{if } x < -\epsilon, 
\end{cases}
\]

where \( \epsilon \) is a predetermined small number. Since we are essentially optimizing the parameters \( x \) and \( z \), we write the final problem as
\[
\min_{x, z} \frac{1}{L} \sum_{\ell=1}^L \bar{g}(W^\ell(T) - e^{sT} \cdot W^\ell_b(T)).
\]

(2.10)
Similar to Li and Forsyth (2019), we use the same trust region optimization method Coleman and Li (1996) to solve the resulting optimization problem. More specifically, the optimization method requires the evaluation of the objective function, its derivative with respect to the weight parameters $x$ and $z$, and the Hessian matrix. The gradients can be explicitly evaluated via the chain rule, and the Hessian matrix can be numerically computed via the finite-difference of the gradients. The detailed gradient computation can be found in Li and Forsyth (2019).

### 3 Testing on Different Distributions with Bootstrap Resampling

Success in data-driven learning critically depends on the efficient use of data. Standard machine learning measures success based on testing the model performance on unseen data which are assumed to have the same distribution as the training data. In other words, test results are typically computed based on test samples from the same distributions as training samples.

For training of the optimization problem (2.10), we only have access to a single path of historical returns. This lack of data presents a unique challenge in data-driven financial model learning.

For financial model learning and testing, it is a common practice to train and test strategy performance by splitting the historical market data path into two segments - one for training and the other for testing. A critical problem in this approach is insufficient data for robust learning and testing. This is especially problematic in the context of pension planning due to the long-term investment horizon.

Li and Forsyth (2019) uses block bootstrap resampling to generate training and testing data in data-driven financial decision learning. Standard block bootstrap resampling is done by dividing the historical market sequential data into blocks with fixed block sizes and randomly choosing blocks to construct the bootstrap resampled data series. To reduce the impact of a fixed block size and to mitigate the edge effects at each block end, the stationary block bootstrap (Patton et al., 2009; Politis and White, 2004) can be used. A single bootstrap resampled path is constructed as follows.

- First, randomly select a block of the historical market data time series. The block size is randomly sampled from a shifted geometric distribution with an expected block size $\hat{b}$. The optimal choice for $\hat{b}$ is determined using the algorithm described in (Patton et al., 2009).
- Repeat the previous step and concatenate the new block after the existing data series until the new resampled path has reached the desired length.
- If the selected block exceeds the range of historical data, wrap around the historical data as in the circular bootstrap method (Politis and White, 2004; Patton et al., 2009).

Algorithm 1 presents pseudocode for the stationary block bootstrap.

In Li and Forsyth (2019), the training dataset is generated using stationary block resampling with one expected block size and the testing dataset is generated with a different expected block size. As Politis and Romano (1994) points out, changing the expected block sizes for block bootstrap resampling essentially changes the distribution of the bootstrap resampled data paths. Consequently, such training and testing assessments actually perform out-of-distribution tests.

Intuitively, using the block bootstrap resampling time-series financial market data seems natural. We have trained a model, considering all permutations of the financial market data with respect to different and random concatenations of time horizons. In addition, testing has been performed on a different distribution of the financial market random horizon concatenations, since the testing data uses a different expected block size from that of the training data. Indeed, evaluating testing performance in this fashion seems to uphold a more stringent standard in comparison to the standard machine learning which evaluates testing performance assuming (unseen) testing samples are from the same distribution of the training data.

Still, one may have concerns that when the training data and testing data are block bootstrap resampled from the same underlying historical market data sequence, one path may appear in both training and testing datasets so that the learning algorithm may benefit from such an unfair edge. To address such concerns, we
Algorithm 1: Pseudocode for stationary block bootstrap

/* initialization */
bootstrap_samples = [ ];
/* loop until the total number of required samples are reached */
while True do
    /* choose random starting index in [1,...,N], N is the index of the last historical sample */
    index = UniformRandom( 1, N );
    /* actual blocksize follows a shifted geometric distribution with expected value of \( \exp \text{block size} \) */
    blocksize = GeometricRandom( \( \frac{1}{\exp \text{block size}} \) );
    for ( i = 0; i < blocksize; i = i + 1 ) {
        /* if the chosen block exceeds the range of the historical data array, do a circular bootstrap */
        if index + i > N then
            bootstrap_samples.append( historical_data[ index + i - N ] );
        else
            bootstrap_samples.append( historical_data[ index + i ] );
        end
        if bootstrap_samples.len() == number_required then
            return bootstrap_samples;
        end
    }
end

establish a theoretical bound on the probability of training and testing sample sequences being exactly the same.

**Theorem 1.** Consider generating a sequence of \( N \) data points using fixed block resampling from a sequence of \( N_{\text{tot}} \) distinct observations. Let path \( P_1 \) be a bootstrap resampled with a fixed blocksize of \( b_1 \) and path \( P_2 \) be a bootstrap resampled with a fixed blocksize of \( b_2 \). Then the probability of \( P_1 \) and \( P_2 \) being identical is

\[
\left( \frac{1}{N_{\text{tot}}} \right)^{\text{lcm}(b_1, b_2)},
\]

where \( \text{lcm}(a, b) \) is the least common multiple of integer \( a, b \).

The proof of Theorem 1 is in Appendix A.1. To put this into perspective, assume a fixed blocksize for the training paths of 6 months, and a fixed blocksize for the testing path of 24 months (or 2 years). Consider a 30-year investment horizon of monthly return paths randomly generated from historical monthly data over 90 years, i.e. \( N = 30 \times 12 = 360 \) and \( N_{\text{tot}} = 90 \times 12 = 1080 \). Then the probability of a training path being identical to a testing path is \( \left( \frac{1}{1080} \right)^{\text{lcm}(6, 24)} = \left( \frac{1}{1080} \right)^{60} < 10^{-180} \). Assume that we use a total of 100,000 training paths in the training data and 10,000 testing paths in the testing data. By the union bound, the probability of the existence of a pair of identical training and testing paths is bounded by \( 100,000 \times 10,000 \times 10^{-180} = 10^{-171} \).

Next, we consider the stationary block bootstrap case, in which the block sizes are randomly generated from a shifted geometric distribution. We are able to establish the following theorem about the probability of two paths generated with stationary block bootstrap being identical.

**Theorem 2.** Consider generating a sequence of \( N \) data points using stationary block resampling from a sequence of \( N_{\text{tot}} \) distinct observations. Let \( P_1 \) and \( P_2 \) be two paths generated from the stationary block bootstrap resampling from this observation sequence with the expected block sizes of \( b_1 \) and \( b_2 \) respectively, and both have a length of \( N \). The probability of \( P_1 \) and \( P_2 \) being identical is
The proof of Theorem 2 is also in Appendix A.1. Consider the following example. If the training paths are bootstrap resampled with an expected blocksize of 6 months (0.5 years) and the testing paths with an expected blocksize of 24 (2 years), for $N = 30 \times 12 = 360$ (30-year horizon) and $N_{tot} = 90 \times 12 = 1080$. Then the probability of a training path being identical to a testing path is $8.737 \times 10^{-39}$.

If training data set consists of a total of 100,000 training paths and testing data set consists of 10,000 testing paths, by union bound, the probability of existing a pair of training and testing path being identical is bounded by $100,000 \times 10,000 \times 8.737 \times 10^{-39} < 10^{-29}$. Therefore, even when the training set and testing set are generated from the same data sequence, the probability of observing the same path in the training and testing dataset is near zero. This suggests that using the block bootstrap resampling to generate training and testing data sets is a robust method for enhancing data for the learning framework.

**Remark 1.** Under stationary block bootstrap, a path is likely to have large actual block sizes even if the expected block size is relatively small, which can result in a higher probability of observing two identical paths than under fixed block bootstrap. For example, a path with expected block size of 10 years has a 5% probability of only containing one block of 30 years, which increases the probability of one path being identical to another path, according to Theorem 4.

## 4 Performance Assessment and Comparison

We evaluate and report the performance of the proposed data-driven approach for outperforming a stochastic target in the context of a 30 year DC pension plan. In our numerical tests, we focus on portfolios with only two assets: a stock index and a bond index, as described in Example 1. The benchmark portfolio is a constant weight strategy, which is rebalanced to 50% bonds and 50% stocks annually. We denote the wealth of the benchmark strategy at time $t$ by $W_{50/50}(t)$.

### 4.1 Original Data and Its Augmentation

#### 4.1.1 Historical Data

Our main objective here is to consider the core allocation problem between a risky and a defensive asset. To that end, we use monthly historical data from the Center for Research in Security Prices (CRSP) from January 1, 1926 to December 31, 2015. Specifically, we use the CRSP 3-month Treasury bill (T-bill) index and the CRSP cap-weighted total return index. The latter index includes all distributions for all domestic stocks trading on major U.S. exchanges. Since both indexes are in nominal terms, we adjust them for inflation using the U.S. CPI index, also supplied by CRSP. We use real indexes since investors saving for retirement should be focused on real (not nominal) wealth goals. Note that in (Li and Forsyth, 2019), in the context of a fixed (non-stochastic) target based objective function, we have also tested the use of the CRSP equal weighted index (for the risky asset) and the ten year treasury index (for the defensive asset). The control strategies are qualitatively similar for either choice of risky and defensive asset. For simplicity here, we will focus on the CRSP index and the 3-month T-bill case.

For illustration, we consider here a two-asset allocation in which the wealth of the portfolio is allocated to the two indexes. We subsequently refer to the two assets simply as the stock and the bond.

For the stock index and bond index, Table 4.1 shows the optimal expected block size for each index estimated from the historical data. When using the resampling method in the proposed data-driven NN
approach, we simultaneously sample the same block across all asset data sets (i.e. the stock index and bond index). Since the optimal blocksize varies with the index, it is not clear which blocksize to use since we need to simultaneously resample both indices. Consequently, we will carry out tests with a variety of blocksizes, in the ranges reported in Table 4.1.

<table>
<thead>
<tr>
<th>Data Series</th>
<th>Optimal expected block size $\hat{b}$ (months)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Real 3-month T-bill index</td>
<td>50.1</td>
</tr>
<tr>
<td>Real CRSP cap-weighted index</td>
<td>1.8</td>
</tr>
</tbody>
</table>

Table 4.1: Optimal expected blocksize $\hat{b} = 1/v$ when the blocksize follows a geometric distribution $Pr(b = k) = (1 - v)^{k-1}v$. The algorithm in Patton et al. (2009) is used to determine $\hat{b}$.

4.2 Experiment Setting

The parameters used in training and testing the proposed data-driven approach are as below:

- $L$: a total of $L = 100,000$ bootstrap paths are used for training;
- $L_{test}$: a total of $L_{test} = 10,000$ paths are bootstrap resampled from a different expected blocksize than the training data for testing the strategy performance;
- $W(0)$: initial wealth is $W(0) = 0$;
- $T$: the entire investment period is $T = 30$ years;
- $N$: the entire period is divided into $N = 30$ periods. At the beginning of each period rebalancing occurs, i.e., annual rebalancing;
- $q$: annual cash injection is $q = 10$;
- $s$: the annual target outperformance rate $s = 1\%$ for calculating the elevated target $e^{sT}W_{50/50}(T)$, where $W_{50/50}(T)$ is the terminal wealth of the constant proportion portfolio;

- 3 features:
  - $T - t$: time remaining in the investment period,
  - $W(t)$: wealth of the adaptive portfolio at time $t$,
  - $W_{50/50}(t)$: wealth of the constant proportion portfolio at time $t$.

4.3 Performance

We now evaluate the performance of the optimal adaptive strategy trained on bootstrap resampled data. First, we show the performance of the optimal adaptive strategy trained on the bootstrap resampled data with the expected blocksize $\hat{b} = 0.5$ years, and tested on bootstrap resampled data with expected blocksize of $\hat{b} = 2$, which is the average optimal blocksize. When discussing robustness in Section 5.1, we show that the strategy performance using alternative training-testing expected blocksize pairs is qualitatively similar.
Table 4.2: Terminal wealth statistics of the optimal adaptive strategy, trained on bootstrap resampled data with blocksize $b = 0.5$ years and tested on bootstrap resampled data with blocksize $b = 2$ years.

Table 4.2 reports performance statistics and the probability of the terminal wealth less than the median of the terminal wealth of both strategies. From Table 4.2, we observe that

- The median and mean of the optimal adaptive strategy is significantly higher than the constant proportion strategy.
- The optimal adaptive strategy has only 26% probability of achieving a lower terminal wealth than the median terminal wealth of the constant proportion strategy ($\text{median}(W_{CP}^T)$), while the constant proportion strategy has an 84% probability of achieving a lower terminal wealth than the median terminal wealth of the NN adaptive strategy ($\text{median}(W_{NN}^T)$).

It is also worth noting that the standard deviation of the terminal wealth of the optimal adaptive strategy is higher than the standard deviation of the terminal wealth of the constant proportion strategy. In the context of dynamic trading, a higher standard deviation does not imply that the performance of the strategy is poor. In fact, we can observe from Figure 4.1a that the distribution of the terminal wealth of the optimal adaptive strategy is significantly more right-skewed. A higher standard deviation of terminal wealth is desirable in the right-skewed situation (van Staden et al. 2019). This illustrates why standard deviation and Sharpe Ratio are poor measures of risk for inherently non-linear strategies (Lhabitant, 2000). In fact, the optimal adaptive dynamic strategy has properties in common with option-based strategies. We also plot the CDF plot for the optimal adaptive strategy and the constant proportion strategy in Figure 4.1b.

We should point out that the terminal wealth distribution of the optimal adaptive strategy has a slightly worse left tail than the constant proportion strategy. The 90% VaR of terminal wealth is 340 for the
optimal adaptive strategy and 394 for the constant proportion strategy. These tail events occur when the bootstrapped path corresponds to consistently bearish market periods when stocks underperform bonds for a long period of time. We remark that the investor can include risk measures such as VaR and CVaR in the objective function if reducing the tail risk is of a higher priority in the investment plan. This, of course, will produce a lower probability of outperformance.

Figure 4.1c shows the cumulative distribution function (CDF) of the wealth difference $W(T) - W_{50/50}(T)$ to give a more direct comparison between the optimal adaptive strategy and the constant proportion strategy along the same paths. From Figure 4.1c we can see that the probability of the optimal adaptive strategy underperforming the constant proportion strategy is less than 10%. When underperformance occurs, the magnitude of underperformance is small compared to the magnitude of outperformance.

We have analyzed and compared the overall performance based on the terminal wealth adaptive strategy. Next, we provide more detailed comparisons of the various characteristics of the strategies.

### 4.3.1 Strategy Performance Over Time

Since the objective function for the optimal control (2.5) is defined from the terminal wealth, we examine how the optimal adaptive strategy performs over the entire period of investment.

![Wealth Difference Over Time](image)

(a) Percentiles of wealth difference $W(t) - W_{50/50}(t)$ over time

![Relative Wealth Difference Over Time](image)

(b) Percentiles of relative wealth difference $\frac{W(t) - W_{50/50}(t)}{W_{50/50}(t)}$ over time

Figure 4.2: Wealth difference and relative wealth difference over time: $W(t)$ denotes the optimal adaptive is wealth and $W_{50/50}(t)$ denotes the benchmark

Figure 4.2 graphs the average and various percentiles of the wealth difference $W(t) - W_{50/50}(t)$ in the investment time horizon. From Figure 4.2, we observe that

- With a high probability, the optimal adaptive strategy achieves higher wealth than the constant proportion strategy over time.
- The outperformance of the optimal adaptive strategy in terms of the relative wealth difference is not as significant as the wealth difference in dollar values.

The observations indicate that larger outperformance of the optimal adaptive strategy often occurs when the constant proportion strategy performs well. Nevertheless, the outperformance of the optimal adaptive strategy...
strategy in terms of the relative wealth difference is still very impressive with a median value of almost 40% at the terminal stage. Of course, if we are primarily interested in relative outperformance, it is a simple matter to alter our objective function to focus on achieving this goal.

Figure 4.2 shows that, even though the objective function only targets the wealth difference of the portfolios at the terminal time, without having any direct restrictions on the wealth of the optimal adaptive strategy in the interim period, the adaptive strategy still manages to have a statistically higher wealth throughout the entire investment period.

4.3.2 Strategy Characteristics

We further examine the characteristics of the optimal adaptive strategy. Figure 4.3a shows different percentiles of the stock allocation of the optimal adaptive strategy over time. We observe that

- In general, the stock allocation (fraction of wealth invested in stocks) decreases when approaching the end of the investment horizon.
- The stock allocation almost always stays above the benchmark allocation of 50%.

![Stock Allocation Over Time](image1.png)

(a) Percentiles of the fraction invested in stocks over time for the adaptive strategy

![Stock Allocation Heatmap](image2.png)

(b) Heatmap, fraction invested in stocks for the adaptive strategy

Figure 4.3: Fraction invested in stocks over time for the optimal adaptive strategy: percentiles and the heatmap

With a red-blue color scheme, Figure 4.3b shows the heatmap of the stock allocation with respect to time \( t \) and the wealth difference \( W(t) - W_{50/50}(t) \). Darker shades of the red color indicate more allocation in stocks and darker shades of the blue color indicate more allocation in bonds.

From Figure 4.3b, we observe that when \( W(t) - W_{50/50}(t) \) is positive and large (optimal adaptive strategy outperforming), the allocation of wealth to the stock becomes small. The intuitive explanation is that the optimal adaptive strategy tends to decrease the wealth allocation to stocks once it has established an advantage over the benchmark constant proportion strategy. This also explains why the stock allocation almost always stays above 50%. In most cases where the optimal adaptive strategy has established an advantage over the constant proportion strategy (as we have seen in Figure 4.2), decreasing the stock allocation to 50% to maintain the same allocation strategy as the 50/50 constant proportion strategy locks in the outperformance.

On the other hand, when \( W(t) - W_{50/50}(t) < 0 \) (i.e. the adaptive strategy underperforms), the optimal policy allocates more wealth to stocks. This is because the stock index has a higher expected return than
the bond index. To eventually outperform the constant proportion strategy, the adaptive strategy invests more wealth in stocks, in an attempt to make up for the lost ground.

In fact, the optimal adaptive strategy appears to be a contrarian strategy, following which an investor buys and sells in opposition to the prevailing sentiment at the time.

### 4.3.3 Historical Backtest Performance

As a special out-of-sample test, we consider the actual historical path from 1985 to 2015 to backtest the performance of the optimal adaptive strategy. We note that the historical path is not a path in the training data set.

From Figure 4.4, we see that the optimal adaptive portfolio always maintains a higher wealth than the constant proportion strategy over the entire investment period. While optimizing the performance of the adaptive strategy on a specific path is not the goal of our study, it is still quite interesting to see that historically the optimal adaptive strategy does better than the constant proportion strategy.

Note that the adaptive strategy does show a large drawdown in 2002 and 2008. However, our objective function is posed in terms of outperformance of the terminal wealth. We see that the adaptive strategy outperforms, in the sense that its wealth is always above the benchmark wealth, even in 2002 and 2008. It is, of course, possible to add penalties on drawdowns in the objective function. However, this would result in less favorable terminal statistics.

The solid line without markers in Figure 4.4 illustrates the time evolution of the stock allocation on the historical path. When the adaptive strategy performs poorly, such as in 2002 and 2008, the strategy allocates more wealth to stocks. When the adaptive strategy performs well, the strategy decreases allocation to stocks and invests more in bonds.

![Wealth Growth From 1985 to 2015 (Year End)](image)

Figure 4.4: Backtest of strategy performance over the historical period from 1985-2015 (single path)

### 4.4 Comparison with the 80/20 Constant Proportion Strategy

While the average stock allocation from the optimal adaptive strategy varies over time, its average over time is about 80%. A natural question is how the optimal adaptive strategy compares with the 80/20 constant proportion strategy which invests 80% of the wealth in the stocks and 20% in the bonds.

Here we compare the optimal adaptive strategy with the 80/20 constant proportion strategy. Recall that in Section 4.3, the optimal adaptive strategy is trained on bootstrap resampled data with the expected blocksize of 0.5 years and the test dataset is bootstrap resampled data with the expected blocksize of 2 years.

We compare the optimal adaptive strategy and 80/20 strategy on the same test dataset.
In Figure 4.5, we plot CDFs of $W_{NN}(T) - W_{50/50}(T)$ and $W_{80/20}(T) - W_{50/50}(T)$, i.e., the wealth difference of the optimal adaptive strategy and the 80/20 strategy from the 50/50 strategy respectively.

We observe that the optimal adaptive strategy controls tail risk better than the 80/20 strategy. Specifically, the probability of the optimal adaptive strategy underperforming the 50/50 strategy is lower than the 80/20 strategy. When underperformance against the 50/50 strategy occurs, the magnitude of underperformance for the optimal adaptive strategy is less than the magnitude of underperformance for the 80/20 strategy, as in Figure 4.5.

It is worth noting that the 80/20 strategy has more upside than the optimal adaptive strategy. However, we should remind the readers that less upside is a natural result of our choice of the double-sided penalty objective function. As reflected in the asymmetric objective function, our goal is not to achieve extremely large outperformance over the 50/50 strategy, but to reach the elevated target with high probability and to control the downside risk. The optimal adaptive strategy achieves those goals better than the 80/20 strategy. To better demonstrate this, we plot the following CDF of outperformance of both strategies over the elevated target $e^{ST} \cdot W_{50/50}(T)$, in Figure 4.6b.

We also observe that the optimal adaptive strategy has a smaller probability of underperforming the elevated target (37.3%) than the 80/20 strategy (46.8%). This means the optimal adaptive strategy is more likely to reach the elevated target and thus achieve the pre-determined annual outperformance spread. Moreover, we observe from the enlarged CDF plot in Figure 4.6b that the optimal adaptive strategy consistently controls underperformance better than the 80/20 strategy, in the sense that the optimal adaptive strategy underperforms less than the 80/20 strategy when the elevated target is not met.

5 Robustness Assessment

To further evaluate the robustness of the optimal adaptive strategy, we assess optimal control models from the following three perspectives:

- We test the strategy learned from the bootstrap data with a given expected blocksize on bootstrap data with multiple different expected blocksizes.
We train the model on a dataset simulated from a synthetic parametric model and test it on the bootstrap resampled dataset.

We train the strategy learned on bootstrap data from one segment of the historical data and test the strategy on bootstrap data from another segment of the historical data.

We generate the bootstrap resampled data by sampling directly from the specified historical data sequence for training the optimal control model.

### 5.1 Testing Using Different Blocksizes

We test the adaptive strategy learned on bootstrap resampled data with a given blocksize on bootstrap resampled data with different blocksizes.

For illustration, here we only show the testing results of the strategy learned on bootstrap resampled data with expected blocksize of 0.5 years, where test data sets are bootstrap resampled data with blocksizes ranging from 1-10 years. We note that training on data sets using a different blocksize, and testing on other blocksizes produces qualitatively similar results.

We can observe from Table 5.1 that

- The mean and the median terminal wealth of the adaptive strategy remain similar across different blocksizes.
- The adaptive strategy has a more favorable terminal wealth distribution as it is more likely to achieve the terminal wealth higher than the median terminal wealth of the constant proportion strategy.

Table 5.1 demonstrate that the outperformance of the adaptive strategy over the benchmark strategy is robust across different expected blocksizes. We include more testing results from strategies trained with other expected blocksizes in the Appendix.
For future reference, note that \( \lambda \) is a Poisson process with positive intensity parameter \( \xi \) distribution (5.1). Moreover, \( y \) for the probability of an upward jump, while 1 follows a double exponential distribution ((Kou, 2002); (Kou and Wang, 2004)). If a jump occurs, \( \xi \) explores the effects of severe market crashes on the stock holding, and nonnormal returns. We assume that \( \mu \) where \( S \) represent a jump multiplier. When a jump occurs, \( \mathbf{t} \) denote the wealth invested in the stocks and bonds at time \( t \).

The synthetic data is generated based on a jump-diffusion stochastic process. Let \( S(t) \) and \( B(t) \) respectively denote the wealth invested in the stocks and bonds at time \( t \), \( t \in [0, T] \). Specifically, we will assume that \( S(t) \) represents the unit amount invested in a broad stock market index (CRSP cap-weighted index), while \( B(t) \) is the unit amount invested in short term default-free government bonds (our case, the 3-month T-bill).

Recall that \( t^- = t - \epsilon, \epsilon \to 0^+ \), i.e. \( t^- \) is the instant of time before \( t \), and let \( \psi \) be a random number representing a jump multiplier. When a jump occurs, \( S(t^-) = \xi S(t^-) \). Allowing discontinuous jumps lets us explore the effects of severe market crashes on the stock holding, and nonnormal returns. We assume that \( \xi \) follows a double exponential distribution ((Kou, 2002); (Kou and Wang, 2004)). If a jump occurs, \( \eta \) is the probability of an upward jump, while \( 1 - \eta \) is the chance of a downward jump. The density function for \( y = \log \xi \) is

\[
\hat{f}(y) = p_{up} \eta_1 e^{-\eta_1 y} 1_{y \geq 0} + (1 - p_{up}) \eta_2 e^{\eta_2 y} 1_{y \leq 0}.
\]

For future reference, note that

\[
E[y = \log \xi] = \frac{p_{up}}{\eta_1} \frac{(1 - p_{up})}{\eta_2}, \quad E[y = \xi] = \frac{p_{up} \eta_1}{\eta_1 - 1} + \frac{(1 - p_{up}) \eta_2}{\eta_2 - 1}
\]

We assume that \( S(t) \) evolves according to

\[
dS(t) = (\mu - \lambda E[\xi - 1])dt + \sigma dZ + d(\sum_{i=1}^{\pi_1}(\xi_i - 1)),
\]

where \( \mu \) is the (uncompensated) drift rate, \( \sigma \) is the volatility, \( dZ \) is the increment of a Wiener process, \( \pi \) is a Poisson process with positive intensity parameter \( \lambda \), and \( \xi_i \) are i.i.d. positive random variables having distribution (5.1). Moreover, \( \xi_i, \pi_i \), and \( dZ \) are assumed to all be mutually independent.

### Table 5.1: Terminal wealth statistics of the adaptive strategy trained on bootstrap resampled data with expected blocksize \( \hat{b} = 0.5 \) years. Tested on bootstrap resampled data with blocksizes from 1 to 10 years.

<table>
<thead>
<tr>
<th>Strategy</th>
<th>( E(W_T) )</th>
<th>( std(W_T) )</th>
<th>median ( W_T )</th>
<th>( Pr(W_T &lt; median(W_T^{1..})) )</th>
<th>( Pr(W_T &lt; median(W_T^{2..})) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>constant proportion ((p = 0.5)) adaptive</td>
<td>678</td>
<td>270</td>
<td>624</td>
<td>0.50</td>
<td>0.26</td>
</tr>
<tr>
<td>NN adaptive</td>
<td>963</td>
<td>474</td>
<td>913</td>
<td>0.50</td>
<td>0.50</td>
</tr>
<tr>
<td>Expected Blocksize ( b = 1 ) years</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>constant proportion ((p = 0.5)) adaptive</td>
<td>674</td>
<td>273</td>
<td>624</td>
<td>0.50</td>
<td>0.50</td>
</tr>
<tr>
<td>NN adaptive</td>
<td>955</td>
<td>466</td>
<td>909</td>
<td>0.27</td>
<td>0.50</td>
</tr>
<tr>
<td>Expected Blocksize ( b = 2 ) years</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>constant proportion ((p = 0.5)) adaptive</td>
<td>676</td>
<td>263</td>
<td>631</td>
<td>0.50</td>
<td>0.50</td>
</tr>
<tr>
<td>NN adaptive</td>
<td>958</td>
<td>445</td>
<td>917</td>
<td>0.26</td>
<td>0.50</td>
</tr>
<tr>
<td>Expected Blocksize ( b = 5 ) years</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>constant proportion ((p = 0.5)) adaptive</td>
<td>669</td>
<td>244</td>
<td>626</td>
<td>0.50</td>
<td>0.50</td>
</tr>
<tr>
<td>NN adaptive</td>
<td>953</td>
<td>409</td>
<td>915</td>
<td>0.24</td>
<td>0.50</td>
</tr>
<tr>
<td>Expected Blocksize ( b = 8 ) years</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>constant proportion ((p = 0.5)) adaptive</td>
<td>669</td>
<td>233</td>
<td>632</td>
<td>0.50</td>
<td>0.50</td>
</tr>
<tr>
<td>NN adaptive</td>
<td>960</td>
<td>393</td>
<td>928</td>
<td>0.23</td>
<td>0.50</td>
</tr>
<tr>
<td>Expected Blocksize ( b = 10 ) years</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>constant proportion ((p = 0.5)) adaptive</td>
<td>667</td>
<td>223</td>
<td>635</td>
<td>0.50</td>
<td>0.50</td>
</tr>
<tr>
<td>NN adaptive</td>
<td>961</td>
<td>383</td>
<td>928</td>
<td>0.22</td>
<td>0.50</td>
</tr>
</tbody>
</table>

549 5.2 Strategy Trained on Synthetic Data
550
551 In this section, we generate synthetic data from a parametric model calibrated to historical data. We then test the strategy on bootstrap resampled data. Clearly, the synthetic data from the parametric model will have a different distribution compared to the resampled data.

553 5.2.1 Synthetic Data Generation
554
555 The synthetic data is generated based on a jump-diffusion stochastic process. Let \( S(t) \) and \( B(t) \) respectively denote the wealth invested in the stocks and bonds at time \( t \), \( t \in [0, T] \). Specifically, we will assume that \( S(t) \) represents the unit amount invested in a broad stock market index (CRSP cap-weighted index), while \( B(t) \) is the unit amount invested in short term default-free government bonds (our case, the 3-month T-bill).

558 Recall that \( t^- = t - \epsilon, \epsilon \to 0^+ \), i.e. \( t^- \) is the instant of time before \( t \), and let \( \psi \) be a random number representing a jump multiplier. When a jump occurs, \( S(t^-) = \xi S(t^-) \). Allowing discontinuous jumps lets us explore the effects of severe market crashes on the stock holding, and nonnormal returns. We assume that \( \xi \) follows a double exponential distribution ((Kou, 2002); (Kou and Wang, 2004)). If a jump occurs, \( \eta \) is the probability of an upward jump, while \( 1 - \eta \) is the chance of a downward jump. The density function for \( y = \log \xi \) is

\[
f(y) = p_{up} \eta_1 e^{-\eta_1 y} 1_{y \geq 0} + (1 - p_{up}) \eta_2 e^{\eta_2 y} 1_{y \leq 0}.
\]

563 For future reference, note that

\[
E[y = \log \xi] = \frac{p_{up}}{\eta_1} \frac{(1 - p_{up})}{\eta_2}, \quad E[y = \xi] = \frac{p_{up} \eta_1}{\eta_1 - 1} + \frac{(1 - p_{up}) \eta_2}{\eta_2 - 1}
\]

565 We assume that \( S(t) \) evolves according to

\[
dS(t) = (\mu - \lambda E[\xi - 1])dt + \sigma dZ + d(\sum_{i=1}^{\pi_1}(\xi_i - 1)),
\]

566 where \( \mu \) is the (uncompensated) drift rate, \( \sigma \) is the volatility, \( dZ \) is the increment of a Wiener process, \( \pi \) is a Poisson process with positive intensity parameter \( \lambda \), and \( \xi_i \) are i.i.d. positive random variables having distribution (5.1). Moreover, \( \xi_i, \pi_i \), and \( dZ \) are assumed to all be mutually independent.
We assume that the dynamics of the amount $B(t)$ invested in the risk-free asset are

$$dB(t) = rB(t)dt,$$  \hspace{1cm} (5.4)

where $r$ is the (constant) risk-free rate. This is obviously a simplification of the real bond market. We remind the reader that, ultimately, our NN method is entirely data-driven, and will be based on bootstrapped stock and bond indexes.

Based on (5.3) and (5.4), we use the methods in (Dang and Forsyth, 2016) to calibrate the process parameters. We use a threshold technique (Cont et al., 2011) to identify jump frequency and distribution, and the methods in (Dang and Forsyth, 2016) to determine the remaining parameters. Annualized estimated parameters for the cap-weighted stock index is provided in Table 5.2.

<table>
<thead>
<tr>
<th>$\mu$</th>
<th>$\sigma$</th>
<th>$\lambda$</th>
<th>$p_{up}$</th>
<th>$\eta_1$</th>
<th>$\eta_2$</th>
<th>$r$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.08889</td>
<td>0.14771</td>
<td>0.32222</td>
<td>0.27586</td>
<td>4.4273</td>
<td>5.2613</td>
<td>0.00827</td>
</tr>
</tbody>
</table>

Table 5.2: Estimated annualized parameters for double exponential jump diffusion model. Cap-weighted index, deflated by the CPI. Sample period 1926:1 to 2015:12.

We then generate the synthetic data based on the parametric model with the calibrated parameters through Monte Carlo simulations.

### 5.2.2 Strategy Performance

We test the performance of the strategy trained on synthetic data on bootstrap data with expected blocksize $\hat{b} = 2$ years. Note that the testing performance with other expected blocksizes is very similar to each other so we only show results for $\hat{b} = 2$ years.

<table>
<thead>
<tr>
<th>Strategy</th>
<th>$E(W_T)$</th>
<th>std($W_T$)</th>
<th>median($W_T$)</th>
<th>$Pr(W_T &lt; median(W_{T-N}^N))$</th>
<th>$Pr(W_T &lt; median(W_{T-N}^N))$</th>
</tr>
</thead>
<tbody>
<tr>
<td>constant proportion ($p = 0.5$)</td>
<td>714</td>
<td>383</td>
<td>630</td>
<td>0.50</td>
<td>0.82</td>
</tr>
<tr>
<td>adaptive</td>
<td>1019</td>
<td>651</td>
<td>930</td>
<td>0.29</td>
<td>0.50</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Strategy</th>
<th>$E(W_T)$</th>
<th>std($W_T$)</th>
<th>median($W_T$)</th>
<th>$Pr(W_T &lt; median(W_{T-N}^N))$</th>
<th>$Pr(W_T &lt; median(W_{T-N}^N))$</th>
</tr>
</thead>
<tbody>
<tr>
<td>constant proportion ($p = 0.6$)</td>
<td>679</td>
<td>267</td>
<td>630</td>
<td>0.50</td>
<td>0.84</td>
</tr>
<tr>
<td>adaptive</td>
<td>944</td>
<td>431</td>
<td>912</td>
<td>0.26</td>
<td>0.50</td>
</tr>
</tbody>
</table>

Table 5.3: Terminal wealth statistics the adaptive strategy trained on synthetic data and tested on bootstrap resampled data with expected blocksize $\hat{b} = 2$ years.

Table 5.3 shows that the adaptive strategy learned from synthetic data performs well on the test set of bootstrap resampled data. The adaptive strategy have significantly higher median and mean terminal wealth than the constant proportion strategy in both training and testing.

We do notice that in the testing results, the adaptive strategy has slightly lower mean and median terminal wealth, as well as a lower standard deviation than in training results. This is hardly surprising since the training and test data have different distributions. However, overall, the strategy appears to be quite robust. Further distribution comparisons can be found in Appendix A.3.

### 5.3 Robustness Test With Training/Testing Split

In §5.1 and §5.2, both training and testing datasets are generated from either a parametric model or bootstrap resampled data from a single historical return path from 1926-2015. A possible criticism of such an approach is that both the training data and testing data share the same information source. In particular, is it possible for the training data to have a forward-looking bias?
We argue that there is no forward-looking bias in the described training and testing data generation process. Recall that in the experiments, training data and testing data have different expected block sizes, and thus different distributions. Specifically, when bootstrap resampling randomly with different expected block sizes, the ordering of blocks of data points is randomly shuffled and any sequential ordering information is destroyed. Further, Theorem 1 and 2 show that the probability of an entire path in the training dataset reappearing in the testing dataset is vanishingly small. This is due to the random block resampling nature of the bootstrap algorithm.

Nonetheless, to provide additional evidence of robustness, we compare the following two different cases:

**Case #1:** We train the adaptive strategy on bootstrap resampled data from the entire historical path from 1926 to 2015. We test the strategy on bootstrap resampled data from the last 30 years of the historical path from 1986-2015. There is an overlap between the underlying historical path for training and testing (1986-2015). We show that such overlap does not introduce an advantage in terms of the strategy performance by comparing it with case #2 - the non-overlap case.

**Case #2:** We train the adaptive strategy on bootstrap resampled data from the first 60 years of the historical path from 1926 to 1985. We test the strategy on the same bootstrap resampled data generated from the last 30 years of the historical path from 1986-2015 as in case #1. Consequently, there is no overlap between the underlying historical paths we use for generating training data and testing data at all.

Figure 5.1 and Figure 5.2 show these two cases schematically. Case #2 is the more stringent test case as there are zero overlaps between the underlying historical data for the generation of the training set and the testing set.

![Figure 5.1: Case #1: use historical data from 1926-2015 for generating training data, and 1986-2015 for testing. There is an overlap between the underlying historical paths for training and testing.](image1)

![Figure 5.2: Case #2: “non-overlap” case where underlying market data for training and testing data has no overlaps. Case #2 uses the same testing dataset as case #1.](image2)

Note that, for Case #1 and #2, the underlying historical data for testing data has only a 30-year window. Recall that our investment horizon in our previous experiments was $T = 30$ years. In order to obtain more meaningful block bootstrap resampling results for the non-overlap window, we will reduce the investment horizon to $T = 15$ years, for both cases in this section.

We first compare the CDF of the terminal wealth of the two cases. From Figure 5.3 we can observe that Case #1 and Case #2 have almost identical CDF curves. The almost identical CDF curves for Case #1 and Case #2 (the non-overlap case) - supports our argument that forward-looking bias is not a concern in our
Figure 5.3: CDF of wealth difference $W(T) - W_{50/50}(T)$ for the two cases: case #1: train: 1926-2015, test: 1986-2015; case #2: train: 1926-1985, test: 1986-2015.

approach. Despite using the entire historical period as the underlying data for training, case #1 does not have a superior CDF than Case #2, in which the underlying market data for training data and testing data have no overlaps.

In Figure 5.4, we show the different percentiles of wealth difference between the adaptive strategy and the constant proportion strategy for both cases. Again, Case #1 and Case #2 have almost identical performances, except that Case #2 has slightly better tail risk control than Case #1 (5th percentile). This further proves that the overlap does not introduce performance advantage as the non-overlap case actually has less tail risk.

In Figure 5.5, we compare the actual strategies, i.e., stock allocations of both cases. This time we can observe some differences between Case #1 and Case #2. From the median and mean plot, we can observe that Case #2 tends to derisk (decrease allocation in the stocks) more aggressively over time than Case #1. We believe the difference comes from the difference in the distributions between the different segments of the underlying historical market returns. However, the difference between allocation strategies is not significant. In fact, the average stock holding over time are quite similar for both cases. In addition, we have already observed similar strategy performances in terms of terminal wealth distributions from figure 5.3 and figure 5.4.

In conclusion, the results further illustrate the robustness of our approach and show that forward-looking bias is not a concern in our work.

6 Conclusions

In this article, we propose a data-driven framework for computing the optimal asset allocation for outperforming a stochastic benchmark target based on market asset return observations. The scenario-based dynamic asset allocation problem is solved directly assuming a neural network representation for the optimal control, without using dynamic programming. This leads to a method that avoids the curse of dimensionality.
Figure 5.4: Percentiles of wealth difference $W(T) - W_{50/50}(T)$ for the two cases

Figure 5.5: Stock allocation for the two cases
which is a critical issue in dynamic allocation for outperforming a stochastic benchmark.

In addition, we design an asymmetric distribution shaping objective function which is capable of producing an optimal strategy which can yield significantly larger median terminal wealth than the target, with only a small probability (and magnitude) of underperformance. We emphasize that our methodology can encompass a wide class of objective functions, which can be tailored to the risk preferences of individual investors.

We use block bootstrap resampling to augment historical financial market data. The training data is generated by block bootstrap resampling from market asset returns. This leads to a data-driven approach for determining the optimal dynamic asset allocation, avoiding the need to make a parametric asset price model as well as model parameter estimations. We further provide mathematical justifications for using block bootstrap resampling to generate both training and testing datasets.

The proposed method is illustrated in the DC pension allocation problem, which is a practically relevant and important problem on its own. We evaluate and analyze the performance of the optimal NN adaptive strategy based on CRSP 3-month Treasury bill (T-bill) index for the risk-free asset and the CRSP cap-weighted total return index for the risky asset from 1926:1-2015:12.

We illustrate the robustness of our approach from three different perspectives.

• We show that the adaptive strategy trained on bootstrap resampled data with a given expected blocksize performs consistently well on bootstrap resampled data with different expected blocksizes (thus different distributions).

• We show that the adaptive strategy learned on synthetic data performs well on bootstrap resampled data, despite the fact that the methodology for generating the datasets are quite different.

• We compare the performance of our strategy with the strategy learned in an non-overlap setting where the underlying market data for the training dataset and testing dataset has no overlap. We show that the non-overlap case has a comparable performance which supports our argument that forward-looking bias should not be a concern in our approach.

Basing our optimal control on a shallow Neural Network representation using only a small number of financially relevant feature variables results in a strategy that is financially intuitive and implementable.

7 Acknowledgements

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8 Conflicts of interest

The authors have no conflicts of interest to report.

A Appendix

A.1 Proofs for Theorem 1 and 2

We mathematically establish Theorem 1 and 2.
For a path $\mathcal{P}$, we use the following notations:

\[
\begin{align*}
\hat{b} & = \text{expected blocksize in stationary block bootstrap} \\
N & = \text{number of total datapoints in the path} \\
N_{\text{tot}} & = \text{number of total datapoints to bootstrap from} \\
\mathcal{P}[i] & = \text{the } i\text{th data point in path } \mathcal{P}
\end{align*}
\] (A.1)

We also make the following definitions.

**Definition 1.** Assume that a path $\mathcal{P}$ of length $N$, which contains blocks $[B_1, \ldots, B_k]$, is resampled from the original data path of length $N_{\text{tot}}$. The **decision index list** $[I_1, \ldots, I_k]$ of the path $\mathcal{P}$ is defined as the list of starting indices of every block in the resampled path with $I_1 = 1$, $I_i = 1 + \sum_{j=1}^{i-1} |B_j|$, $i = 2, \ldots, k$, where $|B_j|$ denotes the number of points in the block $B_j$. If $I_k$ is the starting index of the last block in the path, then, for index completeness, we define $I_{k+1} \equiv N + 1$.

**Remark 2** (Decision Index List Example). Given a decision index list $[I_1, \ldots, I_k]$, associated with a path $\mathcal{P}$, then the data point of the path, which starts at decision index $I_i$, is $\mathcal{P}[I_i]$.

**Definition 2.** For any two paths $\mathcal{P}_1$ and $\mathcal{P}_2$, the **combined decision index list** of $\mathcal{P}_1$ and $\mathcal{P}_2$ is the merged index list (with only a single copy of each index) of the decision index lists of $\mathcal{P}_1$ and $\mathcal{P}_2$. The merged list $[I_1, \ldots, I_p]$ retains the order properties of the original lists, i.e. $I_{i+1} > I_i$ and $I_{p+1} = N + 1$.

**Definition 3.** For any two paths $\mathcal{P}_1$ and $\mathcal{P}_2$, we define $N_{\text{cdi}}(\mathcal{P}_1, \mathcal{P}_2)$ as the length of the combined decision index list of $\mathcal{P}_1$ and $\mathcal{P}_2$.

**Lemma 1.** Consider either the fixed block resampling or stationary resampling from a sequence of $N_{\text{tot}}$ distinct observations. Two paths $\mathcal{P}_1$ and $\mathcal{P}_2$ with $[I_1, I_2, \ldots, I_{\text{cdi}}]$ as the combined decision index list are identical if and only if $\mathcal{P}_1[I_j] = \mathcal{P}_2[I_j]$ at any $I_j$, $j = 1, \ldots, N_{\text{cdi}}$.

**Proof.** First, $\mathcal{P}_1$ equals to $\mathcal{P}_2$ clearly implies that $\mathcal{P}_1[I_j] = \mathcal{P}_2[I_j]$ at any $I_j$, $j = 1, \ldots, N_{\text{cdi}}$. Conversely, assume that $\mathcal{P}_1[I_j] = \mathcal{P}_2[I_j], j = 1, \ldots, N_{\text{cdi}}$. For any $j$, $j = 1, \ldots, N_{\text{cdi}}$, the entire segment $\mathcal{P}_1[I_j], \ldots, \mathcal{P}_1[I_{j+1} - 1]$ is from the same resampled subblock of the original data. Similarly, the entire segment $\mathcal{P}_2[I_j], \ldots, \mathcal{P}_2[I_{j+1} - 1]$ is from the same resampled subblock of the original data. Since $\mathcal{P}_1[I_j] = \mathcal{P}_2[I_j]$, then $\mathcal{P}_1[I_j], \ldots, \mathcal{P}_1[I_{j+1} - 1]$ and $\mathcal{P}_2[I_j], \ldots, \mathcal{P}_2[I_{j+1} - 1]$ are identical. Thus, the entire paths $\mathcal{P}_1$ and $\mathcal{P}_2$ are identical.

**Theorem 1.** Consider fixed block resampling sequences of $N$ points from a sequence of $N_{\text{tot}}$ distinct observations. Let path $\mathcal{P}_1$ be a bootstrap resampled path with a fixed blocksize of $b_1$ and path $\mathcal{P}_2$ be a bootstrap resampled path with a fixed blocksize of $b_2$. Then the probability of $\mathcal{P}_1$ and $\mathcal{P}_2$ being identical is $\left(\frac{1}{N_{\text{tot}}}\right)^{\text{lcm}(N_{\text{cdi}}^1, N_{\text{cdi}}^2)}$, where lcm$(a, b)$ is the least common multiple of integer $a, b$.

**Proof.** Let $I$ denote the combined decision index list of $\mathcal{P}_1$ and $\mathcal{P}_2$, with $N_{\text{cdi}}$ the total number of combined decision points and $I_j$ denoting the $j$th index within $I$.

From Lemma 1, two paths are identical if and only if $\mathcal{P}_1[I_j] = \mathcal{P}_2[I_j]$ at any $I_j$, $j = 1, \ldots, N_{\text{cdi}}$. 

For any \( j = 1, \ldots, N_{cdi} \), since each starting point of either \( \mathcal{P}_1 \) or \( \mathcal{P}_2 \) is chosen independently with equal probability \( \mathbb{P}(\mathcal{P}_1[I_j] = \mathcal{P}_2[I_j]) = \frac{1}{N_{tot}} \). In addition
\[
\mathbb{P}(\mathcal{P}_1[I_j] = \mathcal{P}_2[I_j], j = 1, \ldots, N_{cdi}(\mathcal{P}_1, \mathcal{P}_2)) = \prod_{j=1}^{N_{cdi}(\mathcal{P}_1, \mathcal{P}_2)} \mathbb{P}(\mathcal{P}_1[I_j] = \mathcal{P}_2[I_j]) = \left( \frac{1}{N_{tot}} \right)^{N_{cdi}(\mathcal{P}_1, \mathcal{P}_2)}
\]
\[
\text{Since } N_{cdi}(\mathcal{P}_1, \mathcal{P}_2) = \text{lcm}(\frac{N}{b_1}, \frac{N}{b_2}), \text{ the probability of } \mathcal{P}_1 \text{ and } \mathcal{P}_2 \text{ being identical is } \left( \frac{1}{N_{tot}} \right)^{\text{lcm}(\frac{N}{b_1}, \frac{N}{b_2})}. \]

Next, we consider the stationary block bootstrap resampling, in which the blocksizes are randomly generated from a shifted geometric distribution.

**Properties 1 (Properties of a Geometric Distribution).** Suppose the integer \( m > 0 \) is drawn from a shifted geometric distribution, with \( \mathbb{E}[m] = 1/p \), then
\[
\begin{align*}
\mathbb{P}[m = k] &= (1 - p)^{k-1}p \\
\mathbb{P}[m \geq k] &= (1 - p)^{k-1}.
\end{align*}
\]

We rewrite equation (A.2) in a form amenable to manipulation. Let
\[
(1 - p) = e^{-\lambda},
\]
so that equation (A.2) becomes
\[
\begin{align*}
\mathbb{P}[m = k] &= e^{-\lambda k}(e^\lambda - 1) \\
\mathbb{P}[m \geq k] &= e^{-\lambda(k-1)} \\
\lambda &= -\log[1 - p].
\end{align*}
\]

Denote the expected blocksize by \( \hat{b} \), then in our case, \( p = 1/\hat{b} \), and consequently
\[
\lambda = -\log \left[ 1 - \frac{1}{\hat{b}} \right].
\]

**Lemma 2.** Suppose \([I_1, \ldots, I_k]\) be the decision index list of a block resampled path of length \( N \) with the expected blocksize of \( \hat{b} \). Then the probability of the decision index list \([I_1, \ldots, I_k]\) occurring is \( e^{-\lambda(N-1)}(e^\lambda - 1)^{k-1} \), with \( \lambda = -\log[1 - \frac{1}{\hat{b}}] \).

**Proof.** By definition, \( I_{j+1} > I_j \) for any \( j = 1, \ldots, k-1 \), and \( I_{k+1} = N + 1 \). The probability of path \( \mathcal{P} \) having \([I_1, \ldots, I_k]\) as the decision index list is equal to the probability of path \( \mathcal{P} \) having the first block with blocksize of \( I_2-I_1, \ldots, \) the \( k \)th block with blocksize of \( I_{k+1}-I_k \). Denote the blocks of path \( \mathcal{P} \) as \( B_1, \ldots, B_k \).

According to Properties 1
\[
\mathbb{P}(\text{blocksize}(B_j) = I_{j+1}-I_j) = \begin{cases} 
  e^{-\lambda(I_{j+1}-I_j)}(e^\lambda - 1), & \text{if } j < k \\
  e^{-\lambda(I_{k+1}-I_k-1)}, & \text{if } j = k 
\end{cases}
\]

The probability of path \( \mathcal{P} \) having \([I_1, \ldots, I_k]\) as the decision index list is
\[
\prod_{j=1}^{k} \mathbb{P}(\text{blocksize}(B_j) = I_{j+1}-I_j) = e^{-\lambda(I_{k+1}-I_1-1)}(e^\lambda - 1)^{k-1} = e^{-\lambda(N-1)}(e^\lambda - 1)^{k-1}.
\]
Lemma 2 shows that the probability of a stationary block resampled path \( P \) with an expected blocksize of \( \hat{b} \) having a decision index list is uniquely determined by the expected blocksize \( \hat{b} \), the path length \( N \), and the length \( k \) of the decision index list.

**Lemma 3.** Suppose two paths \( P_1 \) and \( P_2 \) of the length \( N \) are generated by stationary block bootstrap resampling with the expected blocksize of \( \hat{b}_1 \) and \( \hat{b}_2 \) respectively. Then

\[
P(N_{cdi}(P_1, P_2) = k) = \binom{N-1}{k-1} \left( e^{-(\lambda_1+\lambda_2)(N-1)}(e^{\lambda_1+\lambda_2} - 1)^k \right)
\]

\[
\lambda_1 = -\log \left[ 1 - \frac{1}{\hat{b}_1} \right]; \quad \lambda_2 = -\log \left[ 1 - \frac{1}{\hat{b}_2} \right].
\]

**Proof.** Let \( f(\hat{b}, n) \) denote the occurrence probability of a stationary block resampled path of length \( N \) with the expected blocksize of \( \hat{b} \) and a decision index list of length \( n \) (this is given by Lemma 2).

Suppose \([I_1, \ldots, I_k]\) is a combined index list of any two paths \( P_1 \) and \( P_2 \). Let \( v \) be the number of overlapped indices and \( i \) be the number of non-overlapped indices for \( P_1 \) respectively, corresponding to \([I_1, \ldots, I_k]\).

Enumerating the possible values for \( v \), the number of overlapped indices and values for \( i \), the number non-overlapped indices in \( P_1 \), the probability of a combined decision index list \([I_1, \ldots, I_k]\) occurring equals

\[
\sum_{v=1}^{k} \left( \binom{k-1}{v-1} \sum_{i=0}^{k-v} \binom{k-v}{i} f(\hat{b}_1, v+i)f(\hat{b}_2, k-i) \right).
\]

Note that

\[
\sum_{v=1}^{k} \left( \binom{k-1}{v-1} \sum_{i=0}^{k-v} \binom{k-v}{i} f(\hat{b}_1, v+i)f(\hat{b}_2, k-i) \right)
\]

\[
= \sum_{v=1}^{k} \left( \binom{k-1}{v-1} \sum_{i=0}^{k-v} \binom{k-v}{i} e^{-(\lambda_1+\lambda_2)(N-1)}(e^{\lambda_1} - 1)^{v+i-1}e^{-\lambda_2(N-1)}(e^{\lambda_2} - 1)^{k-i-1} \right)
\]

\[
= e^{-(\lambda_1+\lambda_2)(N-1)} \sum_{v=1}^{k} \left( \binom{k-1}{v-1} \left( e^{\lambda_1+\lambda_2} - e^{\lambda_1} - e^{\lambda_2} + 1 \right)^{v-1} \sum_{i=0}^{k-v} \binom{k-v}{i} (e^{\lambda_1} - 1)^i(e^{\lambda_2} - 1)^{k-v-i} \right)
\]

\[
= e^{-(\lambda_1+\lambda_2)(N-1)} \sum_{v=1}^{k} \left( \binom{k-1}{v-1} \left( e^{\lambda_1+\lambda_2} - e^{\lambda_1} - e^{\lambda_2} + 1 \right)^{v-1} \left( e^{\lambda_1} + e^{\lambda_2} - 2 \right)^{k-v} \right)
\]

\[
= e^{-(\lambda_1+\lambda_2)(N-1)}(e^{\lambda_1+\lambda_2} - 1)^k
\]

Since there are \( \binom{N-1}{k-1} \) combinations of the decision index list of length \( k \), we conclude

\[
P(N_{cdi}(P_1, P_2) = k) = \binom{N-1}{k-1} e^{-(\lambda_1+\lambda_2)(N-1)}(e^{\lambda_1+\lambda_2} - 1)^k.
\]

Using Lemma 1 and Lemma 3 we establish the probability of two paths generated with stationary block bootstrap resampling being identical.

**THEOREM 2.** Let \( P_1 \) and \( P_2 \) be two paths of the length \( N \) generated from the stationary block bootstrap resampling from a sequence of \( N_{tot} \) distinct observations with the expected blocksizes of \( \hat{b}_1 \) and \( \hat{b}_2 \) respectively. The probability of \( P_1 \) and \( P_2 \) being identical is
\[
\frac{1}{N_{tot}} \left( (1 - \frac{1}{b_1})(1 - \frac{1}{b_2}) + \frac{1}{b_1} + \frac{1}{b_2} - \frac{1}{b_1b_2} \right)^{N-1}.
\]

Proof. Using Lemma 1, \( P_1 = P_2 \) if and only if the observations from \( P_1 \) and \( P_2 \) are equal at each of the index in the combined decision index list. Thus
\[
P(\mathcal{P}_1 = \mathcal{P}_2|\mathcal{N}_{cdi}(\mathcal{P}_1, \mathcal{P}_2) = k) = \left( \frac{1}{N_{tot}} \right)^k.
\]

Additionally, following Lemma 3 we have
\[
P(\mathcal{P}_1 = \mathcal{P}_2) = \sum_{k=1}^{N} P(\mathcal{N}_{cdi}(\mathcal{P}_1, \mathcal{P}_2) = k) \cdot P(\mathcal{P}_1 = \mathcal{P}_2|\mathcal{N}_{cdi}(\mathcal{P}_1, \mathcal{P}_2) = k)
\]
\[
= \sum_{k=1}^{N} \left( \frac{N-1}{k-1} \right) e^{-(\lambda_1+\lambda_2)(N-1)}(e^{\lambda_1+\lambda_2} - 1)^{k-1}\left( \frac{1}{N_{tot}} \right)^k
\]
\[
= e^{-(\lambda_1+\lambda_2)(N-1)} \sum_{k=1}^{N} \left( \frac{N-1}{k-1} \right) \left( \frac{e^{\lambda_1+\lambda_2} - 1}{N_{tot}} \right)^{k-1}
\]
\[
= \frac{e^{-(\lambda_1+\lambda_2)(N-1)}}{N_{tot}} \left( 1 + \frac{e^{\lambda_1+\lambda_2} - 1}{N_{tot}} \right)^{N-1}
\]
\[
= \frac{1}{N_{tot}} e^{-(\lambda_1+\lambda_2)} \left( 1 - e^{-(\lambda_1+\lambda_2)} \right)^{N-1}
\]
\[
= \frac{1}{N_{tot}} \left( (1 - \frac{1}{b_1})(1 - \frac{1}{b_2}) + \frac{1}{b_1} + \frac{1}{b_2} - \frac{1}{b_1b_2} \right)^{N-1}.
\]

A.2 Additional Robustness Testing Results

As mentioned in section 4.3 we only showed terminal wealth statistics for the strategy trained with bootstrap resampled with expected blocksize \( \hat{b} = 0.5 \) years. Here we show the testing performance of strategies trained on bootstrap data with different blocksizes on different testing sets (bootstrap resampled from different blocksizes). The results show that the adaptive strategy consistently outperforms the constant proportion strategy.
<table>
<thead>
<tr>
<th>Strategy</th>
<th>$E(W_T)$</th>
<th>$\text{std}(W_T)$</th>
<th>$\text{median}(W_T)$</th>
<th>$\text{Pr}(W_T &lt; \text{median}(W_T^{WT}))$</th>
<th>$\text{Pr}(W_T &lt; \text{median}(W_T^{NN}))$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Expected Blocksize $b = 0.5$ years</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>constant proportion ($p = .5$)</td>
<td>678.68</td>
<td>286.28</td>
<td>623.07</td>
<td>0.50</td>
<td>0.81</td>
</tr>
<tr>
<td>NN adaptive</td>
<td>949.49</td>
<td>478.84</td>
<td>874.84</td>
<td>0.27</td>
<td>0.50</td>
</tr>
<tr>
<td>Expected Blocksize $b = 1$ years</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>constant proportion ($p = .5$)</td>
<td>674.27</td>
<td>273.53</td>
<td>623.99</td>
<td>0.50</td>
<td>0.81</td>
</tr>
<tr>
<td>NN adaptive</td>
<td>942.49</td>
<td>459.60</td>
<td>878.60</td>
<td>0.27</td>
<td>0.50</td>
</tr>
<tr>
<td>Expected Blocksize $b = 2$ years</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>constant proportion ($p = .5$)</td>
<td>676.26</td>
<td>263.26</td>
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<tr>
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<td>438.74</td>
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<tr>
<td>constant proportion ($p = .5$)</td>
<td>669.24</td>
<td>244.24</td>
<td>626.11</td>
<td>0.50</td>
<td>0.84</td>
</tr>
<tr>
<td>NN adaptive</td>
<td>940.40</td>
<td>404.87</td>
<td>881.87</td>
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</tr>
<tr>
<td>Expected Blocksize $b = 8$ years</td>
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<tr>
<td>constant proportion ($p = .5$)</td>
<td>669.23</td>
<td>233.23</td>
<td>632.24</td>
<td>0.50</td>
<td>0.84</td>
</tr>
<tr>
<td>NN adaptive</td>
<td>945.38</td>
<td>470.84</td>
<td>892.84</td>
<td>0.22</td>
<td>0.50</td>
</tr>
<tr>
<td>Expected Blocksize $b = 10$ years</td>
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<td></td>
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<td></td>
</tr>
<tr>
<td>constant proportion ($p = .5$)</td>
<td>667.22</td>
<td>223.22</td>
<td>635.29</td>
<td>0.50</td>
<td>0.85</td>
</tr>
<tr>
<td>NN adaptive</td>
<td>942.37</td>
<td>373.95</td>
<td>895.88</td>
<td>0.22</td>
<td>0.50</td>
</tr>
</tbody>
</table>

Table A.1: Trained on bootstrap resampled data with $\hat{b} = 1$ years

<table>
<thead>
<tr>
<th>Strategy</th>
<th>$E(W_T)$</th>
<th>$\text{std}(W_T)$</th>
<th>$\text{median}(W_T)$</th>
<th>$\text{Pr}(W_T &lt; \text{median}(W_T^{WT}))$</th>
<th>$\text{Pr}(W_T &lt; \text{median}(W_T^{NN}))$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Expected Blocksize $b = 0.5$ years</td>
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<td></td>
</tr>
<tr>
<td>constant proportion ($p = .5$)</td>
<td>678.68</td>
<td>286.28</td>
<td>623.07</td>
<td>0.50</td>
<td>0.83</td>
</tr>
<tr>
<td>NN adaptive</td>
<td>962.96</td>
<td>491.03</td>
<td>903.07</td>
<td>0.27</td>
<td>0.50</td>
</tr>
<tr>
<td>Expected Blocksize $b = 1$ years</td>
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<tr>
<td>constant proportion ($p = .5$)</td>
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<td>273.53</td>
<td>623.99</td>
<td>0.50</td>
<td>0.83</td>
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<tr>
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<td>905.02</td>
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<tr>
<td>constant proportion ($p = .5$)</td>
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<td>263.26</td>
<td>631.06</td>
<td>0.50</td>
<td>0.84</td>
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<tr>
<td>NN adaptive</td>
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<tr>
<td>constant proportion ($p = .5$)</td>
<td>669.24</td>
<td>244.24</td>
<td>626.11</td>
<td>0.50</td>
<td>0.85</td>
</tr>
<tr>
<td>NN adaptive</td>
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<td>491.34</td>
<td>914.34</td>
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<td>0.50</td>
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<tr>
<td>Expected Blocksize $b = 8$ years</td>
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<td></td>
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<tr>
<td>constant proportion ($p = .5$)</td>
<td>669.23</td>
<td>233.23</td>
<td>632.24</td>
<td>0.50</td>
<td>0.87</td>
</tr>
<tr>
<td>NN adaptive</td>
<td>961.39</td>
<td>392.89</td>
<td>928.89</td>
<td>0.22</td>
<td>0.50</td>
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<tr>
<td>Expected Blocksize $b = 10$ years</td>
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<tr>
<td>constant proportion ($p = .5$)</td>
<td>667.22</td>
<td>223.22</td>
<td>635.29</td>
<td>0.50</td>
<td>0.88</td>
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<tr>
<td>NN adaptive</td>
<td>961.38</td>
<td>380.15</td>
<td>930.15</td>
<td>0.21</td>
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Table A.2: Trained on bootstrap resampled data with $\hat{b} = 2$ years
<table>
<thead>
<tr>
<th>Strategy</th>
<th>$E(W_T)$</th>
<th>$\text{std}(W_T)$</th>
<th>$\text{median}(W_T)$</th>
<th>$\Pr(W_T &lt; \text{median}(W_{T}^{CP}))$</th>
<th>$\Pr(W_T &lt; \text{median}(W_{T}^{NN}))$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Expected Blocksize $b = 0.5$ years</strong></td>
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<td></td>
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<td></td>
</tr>
<tr>
<td>constant proportion($p = .5$)</td>
<td>678</td>
<td>286</td>
<td>623.07</td>
<td>0.50</td>
<td>0.86</td>
</tr>
<tr>
<td>NN adaptive</td>
<td>995</td>
<td>495</td>
<td>963.03</td>
<td>0.26</td>
<td>0.50</td>
</tr>
<tr>
<td><strong>Expected Blocksize $b = 1$ years</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
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</tr>
<tr>
<td>constant proportion($p = .5$)</td>
<td>674</td>
<td>273</td>
<td>623.99</td>
<td>0.50</td>
<td>0.87</td>
</tr>
<tr>
<td>NN adaptive</td>
<td>988</td>
<td>478</td>
<td>963.28</td>
<td>0.25</td>
<td>0.50</td>
</tr>
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<td><strong>Expected Blocksize $b = 2$ years</strong></td>
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<td></td>
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</tr>
<tr>
<td>constant proportion($p = .5$)</td>
<td>676</td>
<td>263</td>
<td>631.06</td>
<td>0.50</td>
<td>0.88</td>
</tr>
<tr>
<td>NN adaptive</td>
<td>994</td>
<td>458</td>
<td>973.65</td>
<td>0.25</td>
<td>0.50</td>
</tr>
<tr>
<td><strong>Expected Blocksize $b = 5$ years</strong></td>
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<td></td>
<td></td>
<td></td>
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</tr>
<tr>
<td>constant proportion($p = .5$)</td>
<td>668</td>
<td>244</td>
<td>626.11</td>
<td>0.50</td>
<td>0.89</td>
</tr>
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<td>NN adaptive</td>
<td>997</td>
<td>427</td>
<td>976.51</td>
<td>0.22</td>
<td>0.50</td>
</tr>
<tr>
<td><strong>Expected Blocksize $b = 8$ years</strong></td>
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</tr>
<tr>
<td>constant proportion($p = .5$)</td>
<td>669</td>
<td>233</td>
<td>632.24</td>
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<tr>
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<td>993.88</td>
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<td>0.50</td>
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<td>constant proportion($p = .5$)</td>
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<td>223</td>
<td>635.29</td>
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<td>409</td>
<td>996.57</td>
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Table A.3: Trained on bootstrap resampled data with $\hat{b} = 5$ years

<table>
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<tr>
<th>Strategy</th>
<th>$E(W_T)$</th>
<th>$\text{std}(W_T)$</th>
<th>$\text{median}(W_T)$</th>
<th>$\Pr(W_T &lt; \text{median}(W_{T}^{CP}))$</th>
<th>$\Pr(W_T &lt; \text{median}(W_{T}^{NN}))$</th>
</tr>
</thead>
<tbody>
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</tr>
<tr>
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<td>678</td>
<td>286</td>
<td>623.07</td>
<td>0.50</td>
<td>0.86</td>
</tr>
<tr>
<td>NN adaptive</td>
<td>980</td>
<td>480</td>
<td>945.12</td>
<td>0.25</td>
<td>0.50</td>
</tr>
<tr>
<td><strong>Expected Blocksize $b = 1$ years</strong></td>
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<td></td>
<td></td>
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</tr>
<tr>
<td>constant proportion($p = .5$)</td>
<td>674</td>
<td>273</td>
<td>623.99</td>
<td>0.50</td>
<td>0.88</td>
</tr>
<tr>
<td>NN adaptive</td>
<td>973</td>
<td>464</td>
<td>947.99</td>
<td>0.25</td>
<td>0.50</td>
</tr>
<tr>
<td><strong>Expected Blocksize $b = 2$ years</strong></td>
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<td></td>
<td></td>
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</tr>
<tr>
<td>constant proportion($p = .5$)</td>
<td>676</td>
<td>263</td>
<td>631.06</td>
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<td>0.87</td>
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<tr>
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<td>979</td>
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<tr>
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</tr>
<tr>
<td>constant proportion($p = .5$)</td>
<td>669</td>
<td>244</td>
<td>626.11</td>
<td>0.50</td>
<td>0.88</td>
</tr>
<tr>
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<td>412</td>
<td>959.86</td>
<td>0.21</td>
<td>0.50</td>
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<tr>
<td><strong>Expected Blocksize $b = 8$ years</strong></td>
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</tr>
<tr>
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<td>669</td>
<td>233</td>
<td>632.24</td>
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<td>0.90</td>
</tr>
<tr>
<td>NN adaptive</td>
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<td>399</td>
<td>976.44</td>
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</tr>
<tr>
<td><strong>Expected Blocksize $b = 10$ years</strong></td>
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<td>635.29</td>
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<td>0.91</td>
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<tr>
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<td>980.07</td>
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</table>

Table A.4: Trained on bootstrap resampled data with $\hat{b} = 8$ years
Test Results: Market Cap Weighted

<table>
<thead>
<tr>
<th>Strategy</th>
<th>$E(W_T)$</th>
<th>$std(W_T)$</th>
<th>median($W_T$)</th>
<th>$Pr(W_T &lt; median(W_{N,T})$)</th>
<th>$Pr(W_T &lt; median(W_{N,T})$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>constant proportion($p = .5$)</td>
<td>678</td>
<td>286</td>
<td>623.07</td>
<td>0.50</td>
<td>0.84</td>
</tr>
<tr>
<td>NN adaptive</td>
<td>963</td>
<td>468</td>
<td>920.86</td>
<td>0.25</td>
<td>0.50</td>
</tr>
<tr>
<td>constant proportion($p = .5$)</td>
<td>674</td>
<td>273</td>
<td>623.99</td>
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<td>0.84</td>
</tr>
<tr>
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<td>957</td>
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<td>925.63</td>
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<td>0.50</td>
</tr>
<tr>
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<td>676</td>
<td>263</td>
<td>631.06</td>
<td>0.50</td>
<td>0.85</td>
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<td>626.11</td>
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<td>0.84</td>
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<tr>
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<td>954.63</td>
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</table>

Table A.5: Trained on bootstrap resampled data with $\hat{b} = 10$ years

A.3 Robustness: Distribution Comparison Based on Test Results From the Synthetic Model

We observe from Figure A.1 that the terminal wealth distributions of the adaptive strategy are consistently right-skewed and have similar shapes in training and testing, which indicates that the NN strategy similarly outperforms the constant proportion in both training and testing.

We also show the plot of the CDF of the wealth difference $W(T) - W_{50/50}(T)$ to give a more direct comparison between the adaptive strategy and constant proportion strategy on the same paths.

From Figure A.2 we can see that the probability of the adaptive strategy underperforming the constant proportion strategy is less than 10% for both training and testing. When underperformance occurs, the scale of underperformance is small compared to the scale of potential outperformance. Therefore, we conclude that the adaptive strategy controls tail risks consistently in both training and testing, despite the fact that...
the training dataset is synthetically generated and the testing dataset is bootstrap resampled data.

References


