Is Rebalancing Bad for Your Wealth?

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Abstract

The conventional wisdom suggests that retail investors should rebalance their portfolios back to a constant weight stock-bond mix. Although this sounds reasonable, empirical tests based on block bootstrap resampling of historical data do not support this claim. At best, the evidence for rebalancing the stock-bond split (as compared with buy and hold) for time horizons of less than ten years, is weak.

Only in the case where the investor wishes to avoid high instantaneous volatility (even though the CDF of the terminal wealth is arguably superior for buy and hold) will rebalancing be the better choice.

Keywords: Rebalancing, buy and hold, volatility pumping

1 Introduction

It is considered axiomatic in wealth management that investors should periodically rebalance their portfolios back to a target asset mix. The usual rationale is that this keeps the portfolio consistent with the investor's risk preferences. Rebalancing is fundamentally a contrarian strategy, selling winners and buying losers. This also allows the investor to *buy low and sell high.* There is a plethora of academic literature which seems to support this idea.

However, a classic example of a *buy and hold* investment is a capitalization weighted stock index exchange traded fund (ETF). The weights of each stock in the index are not constant, but drift in proportion to their market capitalization. After the initial purchase of the capitalization weighted ETF, the investor is basically following a buy and hold strategy. Consequently, anyone who holds a capitalization weighted index is actually following a buy and hold strategy for a significant proportion of their portfolio.

A sharp eyed reader will perhaps object to my characterization of a capitalization weighted index ETF as a pure buy and hold. Some rebalancing does take place. This is due to stocks being dropped from the index (as a result of not meeting market capitalization constraints). In addition, the implementation of dividend payments may also result in a small amount of rebalancing. For example, dividend payments may be distributed to ETF holders, or reinvested in new units of the total index. This can be viewed as a partial rebalancing. Nevertheless, a capitalization weighted index, at least to a first order approximation, can be considered to be a buy and hold strategy.

Consider a basket of stocks which have (i) low pair-wise correlation and (ii) high volatility. In this case, both theoretical and empirical analysis suggests that rebalancing to a constant weight in each

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underlying stock in the basket will produce superior investment results (Farago and Hjalmarsson, 2023; Forsyth, 2024), compared to a buy and hold (capitalization weighted) basket.

However, for most retail investors, the main investment decision is the split between bonds and stocks, usually implemented in terms of ETFs. Bond ETFs have low volatility (at least based on historical averages), hence the results concerning baskets of high volatility stocks may not apply to this case.

Traditional measures of risk and reward (e.g. Sharpe ratios) seem to favour rebalancing. However, many authors have criticized Sharpe ratios as being too simplistic (variance penalizes upside as well as downside). Some empirical studies even seem to favour buy and hold. For more discussion, we refer the reader to (Perold and Sharpe, 1988; Wise, 1996; Dayanandan and Lam, 2015; Dichtl et al., 2016; Edesess, 2017; Hilliard and Hilliard, 2018; Horn and Oehler, 2020; Bertrand and Prigent, 2022) and the references cited therein.

Can it be the case that one of buy and hold or rebalancing is the superior strategy? Consider the following example: an investment portfolio which consists of two assets: a stock index ETF and a risk free, inflation protected bond index. To illustrate the nuances involved, we examine two cases, where the investor has \$10,000 of initial wealth.

Buy and hold: The investor invests \$5,000 in the stock index, and \$5,000 in the inflation protected bond index, and never rebalances.

Rebalance yearly: The initial allocation is the same as the buy and hold case. However, the portfolio is rebalanced back to 50% in bonds and 50% in stocks annually.

Suppose the investment horizon is very long. Since stocks can be assumed to return more than bonds over the very long term, eventually, the buy and hold portfolio will be almost all stocks (by value). This will be undesirable for most investors, in terms of risk and reward.

On the other hand, the rebalanced portfolio has a positive probability of ending up (after a finite time) with a value less than \$5,000 (this is the *catch a falling knife* scenario).[1] By assumption (the bond fund is a government guaranteed, inflation protected index), the buy and hold portfolio has zero probability of ever being less than \$5,000.

These two extreme cases illustrate that it is not possible, *a priori* to conclude that rebalancing is superior to buy and hold. In the above example, we can see that we have to specify the time frame of the investment, and the desire to avoid left tail risk, in order to make a recommendation.

A more mathematical statement of this example is that neither strategy stochastically dominates the other [2]

In this paper, we consider an investor who holds only two assets: a stock index fund and a bond index fund.[3] Initially, in order to gain intuition, we will make simplifying assumptions (the stock index follows geometric Brownian motion, the bond index has zero volatility, and rebalancing is continuous). We will then go on to more realistic assumptions: a jump diffusion for the stock index and discrete rebalancing. We will examine the entire cumulative distribution function (CDF) of the final wealth distribution, as well as the wealth percentiles during the investment horizon (based on simulations) for both rebalancing and buy and hold.

Finally, we will again examine the wealth distributions for both strategies, but based on bootstrap resampling of historical data (Politis and Romano, 1994; Politis and White, 2004; Dichtl et al.,

[1]Suppose stocks trend down over a long period. Rebalancing will keep buying losers all the way down.

[2]First order stochastic dominance can be identified from the CDFs. If the CDF of strategy A plots below the CDF for strategy B, than A stochastically dominates (to first order) B. Any investor who prefers more rather than less will prefer strategy A. Stochastic dominance is rare. However, partial stochastic dominance can sometimes be observed (van Staden et al., 2021).

[3]This would be representative of the fundamental rebalancing issue for a retail investor.

2016; Anarkulova et al., 2022). For these simulations, we make no assumptions about the underlying stochastic processes of the stock and bond indexes. These results are purely data driven.

Based on historical US stock and bond data, and considering a ten year investment horizon, our main conclusion is that it is difficult to make the case that rebalancing is superior to buy and hold. The caveat here is that the buy and hold portfolio will show higher instantaneous volatility compared to the rebalanced portfolio, but only near the end of investment horizon, in cases where the buy and hold portfolio has large gains. Hence, this may be acceptable to many investors.

Only in the case where the investor wishes to avoid high instantaneous volatility (even though the CDF of the terminal wealth is arguably superior for buy and hold) will rebalancing be the better choice.

2 Intuition: geometric Brownian motion (GBM)

Let the value of the stock index ETF be denoted by $S(t)$ and the value of a bond index be denoted by $B(t)$. The stochastic process of the underlying stock index ETF with value S is assumed to follow geometric Brownian motion (GBM)

$$\begin{aligned} \frac{dS}{S} &= \mu \; dt + \sigma \; dZ \\ &\mu = \text{ arithmetic return} \\ &\sigma = \text{ volatility} \\ &dZ = \text{ increment of a Wiener process } . \end{aligned} \tag{2.1}$$

The bond index is considered to be risk-free and non-volatile

$$dB = rB \; dt \; . \tag{2.2}$$

The value of a portfolio $W(t)$, which is continuously rebalanced to a weight of β in the stock index and $(1-\beta)$ in the bond index then follows the process

$$\begin{aligned} \frac{dW}{W} &= \beta\left(\frac{dS}{S}\right) + (1-\beta)\left(\frac{dB}{B}\right) \\ &= \Big((1-\beta)r + \beta\mu\Big) \; dt + \beta\sigma \; dZ \; . \end{aligned} \tag{2.3}$$

The exact solution to equation (2.3) is

$$\begin{aligned} \frac{W(t)}{W(0)} &= e^{(1-\beta)rt - \beta^2\sigma^2 t/2} e^{\beta\mu t + \beta\sigma(Z(t)-Z(0))} \\ &\qquad (Z(t)-Z(0)) \simeq \mathcal{N}(0,t) \; , \end{aligned} \tag{2.4}$$

where $\mathcal{N}(0,t)$ is a draw from a normal distribution with mean zero and variance t.

The exact solution to equation (2.1) is

$$\frac{S(t)}{S(0)} = e^{\mu t - \sigma^2 t/2} e^{\sigma(Z(t)-Z(0))} \tag{2.5}$$

or

$$\left(\frac{S(t)}{S(0)}\right)^{\beta} e^{\beta\sigma^2 t/2} = e^{\beta\mu t + \beta\sigma(Z(t)-Z(0))} \tag{2.6}$$

Substitute equation (2.6) into equation (2.4) to obtain

$$\frac{W(t)}{W(0)} = e^{(1-\beta)rt+\beta(1-\beta)\sigma^2 t/2}\left(\frac{S(t)}{S(0)}\right)^{\beta} \tag{2.7}$$

Assume that $W(0) = S(0)$, so that this becomes

$$\text{Rebalanced portfolio} = \frac{W^{rebal}(t)}{S(0)} = e^{(1-\beta)rt+\beta(1-\beta)\sigma^2 t/2}\left(\frac{S(t)}{S(0)}\right)^{\beta} \tag{2.8}$$

Now consider a buy and hold portfolio with $\beta W^{bh}(0)$ in the stock index, $(1-\beta)W^{bh}(0)$ in the bond index at $t = 0$, never rebalanced, and liquidated at time t. Assume $W^{bh}(0) = S(0)$. Then

$$\text{Buy and hold} = \frac{W^{bh}(t)}{S(0)} = \frac{\beta S(t) + (1-\beta)e^{rt}S(0)}{S(0)} . \tag{2.9}$$

Finally, the value of the pure stock ETF, denoted by W^s, at time t is simply (initial value $S(0)$)

$$\text{All Stock ETF} = \frac{W^s(t)}{S(0)} = \frac{S(t)}{S(0)} . \tag{2.10}$$

3 Derivative contracts

If $\beta < 1$ in equation (2.8) we can see that rebalancing results in a nonlinear, option-like payoff. For example if $\beta = 0.5$, then rebalancing produces a square root payoff. We can think of W^{rebal}, W^{bh}, W^s as derivative contracts, each costing $S(0)$ to purchase, with payoffs given by equations (2.8-2.10).

We consider the stock index to be the Center for Research in Securities Prices (CRSP) capitalization weighted index. We take the bond index to be based on the return of 30-day T-bills. The GBM stock index parameters are obtained by fitting to the inflation adjusted CRSP capitalization weighted index, 1926:1-2023:12 (see Appendix A) using maximum likelihood. The interest rate r is based on the average (real) return of 30-day T-bills from CRSP as well.

Note that equations (2.8-2.10) are independent of the stock drift rate μ (see equation (2.1)). Figure 3.1(a) compares the payoffs, as a function of $(S(t)/S(0)))$ for all three contracts. Figure 3.1(b) plots

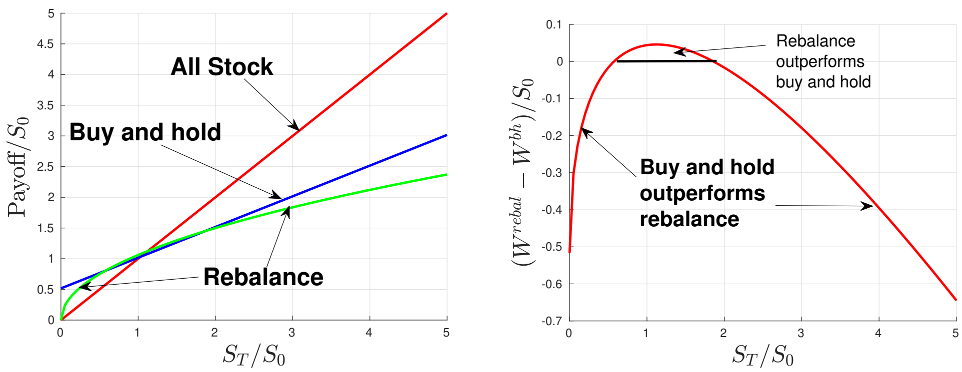
$$\frac{W^{rebal} - W^{bh}}{S(0)} . \tag{3.1}$$

Observe from Figure 3.1(b), that for values of $(S_T/S_0) \in (.5,1.75)$, the payoff of the rebalanced portfolio is above the buy and hold strategy, but by most $\simeq 6\%$. Outside these ranges for (S_T/S_0), buy and hold has a superior payoff, sometimes by very large amounts. This payoff diagram confirms our intuition. If stocks trend up for long periods, then rebalancing gives up some of these large stock gains. If stocks trend down for long periods, then rebalancing will erode the bond protection, since bonds are sold and losing stocks purchased. If stocks trade in a limited range, then rebalancing does generate a *volatility pumping* return. But, for reasonable market parameters, this effect is not large.

In Bertrand and Prigent (2022), it is demonstrated that for a wide range of μ, σ, T (T up to 30 years), the probability of falling inside the region where rebalancing has a higher payoff than buy and hold is $< 70\%$. In summary, we can say that, in general, the payoff of rebalancing is slightly better than buy and hold at most 70% of the time. On the other hand, the payoff of buy and hold can be much larger than rebalancing $\simeq 30\%$ of the time.

The CRSP index, fit to GBM and adjusted for inflation, gives an arithmetic return of $\mu = .0818$, with a median value of $(S_T/S(0))$ at $T = 10$ of about 1.91. From Figure 3.1(b) this suggests that for this data, at least 50% of the time, buy and hold is superior to rebalancing, sometimes by a large amount.

At this point, we really cannot say much more here unless we know the probabilities of being in the various regions of the payoff diagrams.



(a) $W^{bh}/S(0), W^{rebal}/S(0), W^s d/S(0)$

(b) $(W^{rebal} - W^{bh})/S(0)$ *from Figure 3.1(a)*

FIGURE 3.1: *Payoff diagrams comparing* $W^{bh}/S(0), W^{rebal}/S(0), W^s/S(0)$. $T = 10.0$ *years, from equations (2.8-2.10). Fraction in equities* $\beta = 0.5$. *Stock data fit of equation (2.1) to inflation adjusted CRSP index, 1926:1-2023:12. Interest rate r from T-bills, inflation adjusted, 1926:1-2023:12.* $\sigma = .1849, r = .0032$.

4 Jump diffusion, discrete rebalancing

Clearly, it is simplistic to assume that (i) stocks follow GBM and (ii) rebalancing is continuous. In this Section, we remove these two assumptions. Assume that the stock index follows a jump diffusion process, which allows for non-normal returns. If a jump occurs $S(t) = \xi S(t^-)$, and

$$\begin{aligned}
\frac{dS}{S(t^-)} &= (\mu - \lambda\kappa)\ dt + \sigma\ dZ + (\xi - 1)d\mathbb{Q} \\
& d\mathbb{Q} = \begin{cases} 0 & ;\ \text{probability } (1 - \lambda\ dt) \\ 1 & ;\ \text{probability } \lambda\ dt \end{cases} \\
& \kappa = E[\xi - 1] \\
& \lambda = \text{ intensity of the Poisson process }. \qquad (4.1)
\end{aligned}$$

Assume that $y = \log \xi$ follows a double exponential process(Kou, 2002), with density $g(y)$ given by

$$g(y) = p_{up}\eta_1 e^{-\eta_1 y}\mathbf{1}_{y\geq 0} + (1 - p_{up})\eta_2 e^{\eta_2 y}\mathbf{1}_{y<0}. \qquad (4.2)$$

where p_{up} is the probability of an upward jump. Note as well that

$$E[\xi] = \tfrac{p_{up}\eta_1}{\eta_1 - 1} + \tfrac{(1-p_{up})\eta_2}{\eta_2 + 1}\ . \qquad (4.3)$$

Assume that

$$dB = rB\ dt \tag{4.4}$$

and so the SDE for total wealth, assuming continuous rebalancing, is

$$\begin{aligned} \frac{dW}{W} &= \beta\left(\frac{dS}{S}\right) + (1-\beta)\left(\frac{dB}{B}\right) \\ &= \Big((1-\beta)r + \beta(\mu - \lambda\kappa)\Big)\ dt + \beta\sigma\ dZ + \beta(\xi - 1)\ d\mathbb{Q}\ . \end{aligned} \tag{4.5}$$

We can proceed as for the GBM case, and attempt to determine the payoff function for equation (4.5). However, the payoff is not a deterministic function of $S(t)$ anymore, so this is not so useful. Instead, we will simply carry out Monte Carlo simulations. For the interested reader, we give the payoff type function for jump-diffusion in Appendix B.

As before, we assume that the bond index is non-volatile, and follows equation (2.2). We use a filtering method (Cont and Mancini, 2011; Dang and Forsyth, 2016) to estimate the jump diffusion parameters, based on the CRSP data 1926:1-2023:12 (see Appendix A). The parameters are listed in Appendix C. Our basic scenario is given in Table 4.1. We will use Monte Carlo simulation to determine the CDFs for these strategies.

T	10 years
Initial Investment	1000
Rebalancing frequency	1 month
T-bill return r	0.0031
Jump diffusion parameters	Table C.1

TABLE 4.1: *Data for example payoffs.*

Table 4.2 shows the summary statistics for these simulations. ES(5%) is the expected shortfall at the five per cent level, i.e. the mean of the worst 5% of the outcomes. The Omega ratio (Keating and Shadwick, 2002) at level L is defined as

$$\begin{aligned} \text{Omega(L)} &= \frac{E[\max(W_T - L, 0)]}{E[\max(L - W_T, 0)]} \\ &= 1 + \frac{E[W_T - L]}{E[\max(L - W_T, 0)]}\ . \end{aligned} \tag{4.6}$$

The Omega ratio is a measure of upside versus downside, with respect to the level L. Since both rebalancing and buy and hold have similar median values, Table 4.2 shows the Omega ratio at level $L = 1481$, the median of the rebalanced portfolio.

We can see that the buy and hold portfolio has a higher expected terminal wealth W_T compared to the rebalanced portfolio. The median values of W_T are essentially the same, for both strategies. However, the 5th percentile is smaller (worse) than the rebalanced portfolio by about 6%, but the expected shortfalls (the tail risk measure) are essentially the same. The standard deviation of the buy and hold portfolio is much larger (947 versus 574) compared to the rebalanced portfolio. However, the Omega ratio indicates that this is primarily due to more upside variation, i.e. more extreme values above the median. Hence, we can see here that the standard deviation is not a good measure of risk.

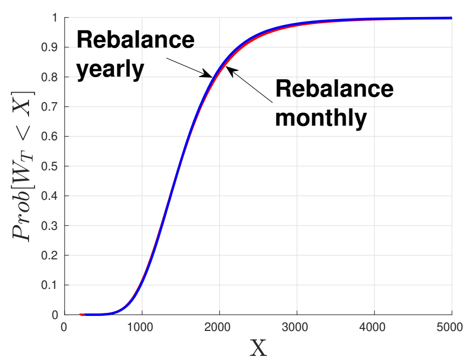
	$E[W_T]$	Median$[W_T]$	$W_T : 5^{th}$ percentile	$W_T : 95^{th}$ percentile	ES(5%)	std$[W_T]$	Omega(1481)
Rebalance	1572 (2.2)	1481	870	2563	754	574	1.58
Buy and Hold	1710 (3.7)	1484	827	3302	746	947	2.29
All stocks	2392 (7.3)	1937	624	5599	462	1894	n/a

TABLE 4.2: *Jump diffusion model for stocks. Statistics for: rebalanced monthly, $\beta = 0.5$, buy and hold (initial stock fraction $\beta = 0.5$), and an all stock portfolio (i.e. buy and hold with $\beta = 1.0$).* 2.56×10^5 *Monte Carlo simulations. Numbers in brackets are the standard error estimate at the 95% confidence level. Stocks follow the jump diffusion model 4.1-4.2. Parameters fit to value-weighted CRSP index deflated by the CPI. Sample period 1926:1 to 2023:12, see Table C.1. The average real return of a 30 day T-bill in the same period was $r = .0.0031$. ES(5%) is the mean of the worst 5% of the outcomes.*

Figure 4.1(a) compares the CDFs of W_T for rebalancing at monthly intervals, buy and hold, and the 100% stock portfolio. We can see from Figure 4.1(a) that buy and hold underperforms compared to rebalancing, below the median, and outperforms rebalancing above the median. However, the extreme left tail performance for both strategies is about the same (from Table 4.2).

Figure 4.1(b) compares yearly and monthly rebalancing strategies. The CDF curves are virtually indistinguishable, indicating that frequent rebalancing appears unnecessary.

(a) *Jump diffusion CDFs*



(b) *Monthly vs yearly rebalancing*

FIGURE 4.1: *Jump diffusion model (4.1). Data in Appendix C. Scenario in Table 4.1. Rebalancing fraction in equities $\beta = 0.5$. Stock data fit of equation (4.1) to inflation adjusted CRSP index, 1926:1-2023:12. Interest rate r from T-bills, inflation adjusted, 1926:1-2023:12.*

5 Bootstrap resampling

Our last test consists of examining the performance of rebalancing and buy and hold using a pure data driven approach. We will use bootstrap resampling of the inflation adjusted CRSP capitalization weighted index, and the inflation adjusted 30-day T-bill index (see Appendix A). The data set covers the historical range 1926:1-2023:12.

A ten year investment scenario consists of 120 consecutive one month returns. A single scenario is constructed as follows. We select a month at random from the historical data, and use this as our first month's return. Then, we select another month at random (with replacement) which is the second month's return in our ten year scenario. We keep doing this until we have a set of 120 returns (one thirty-year path). We then repeat this procedure many times, to produce many 30-year return paths.

However, this bootstrapping approach does not take into account possible serial correlation in the returns. This is just another way of saying that next month's returns may be affected by the returns of the past few months or years.

To take this into account, we select an initial month at random, but use b consecutive monthly returns (starting at the initial random month). We repeat this (120/b) times to generate a single 10 year path. We call b the blocksize.

But we are not done yet. It turns out that a better approach is to not use a fixed blocksize, but to specify an average blocksize b, and randomly vary the blocksize within each ten year path. This is called the stationary block bootstrap method.

For more details about this method, see (Politis and Romano, 1994; Politis and White, 2004; Patton et al., 2009; Dichtl et al., 2016; Forsyth and Vetzal, 2019; Anarkulova et al., 2022). Detailed pseudo-code for block bootstrap resampling is given in Ni et al. (2022).

We will block bootstrap the returns for both the CRSP capitalization index, and for the CRSP 30 day T-bill index (both inflation adjusted), based on the historical data over the period 1926:1-2023:12. We will simultaneously draw returns from both the stock index and the bond index (preserving any possible correlations. We use an expected blocksize of one year. Experiments with expected blocksizes ranging from 3 months to two years do not change the results significantly. The basic scenario is shown in Table 5.1.

For the buy and hold case, we initially invest $0.5W(0)$ in the stock index and $0.5W(0)$ in the bond index, and never rebalance. In the rebalancing case, we start off with the same initial investment as buy and hold, but then rebalance to a weight of 0.50 in stocks annually.

T	10 years
Initial Investment	$W_0 = 1000$
Rebalancing frequency	1 year
T-bill returns	CRSP data
Stock returns	CRSP data
Rebalancing fraction	$\beta = 0.5$

TABLE 5.1: *Data for the bootstrap simulations.*

Table 5.2 shows summary statistics for the various strategies. Note that for the rebalancing case, the summary statistics for rebalancing monthly are quite similar to the statistics for annual rebalancing. This is not unexpected, in view of Figure 4.1(b).

From now on, we will focus exclusively on annual rebalancing. Annual rebalancing will be straightforward to implement for a retail investor.

From Table 5.2 and Figure 5.1(a) we can see that

- Rebalancing outperforms buy and hold, below the median of W_T, by a small amount, however the extreme left tail statistic $ES(5\%)$ is similar for both strategies

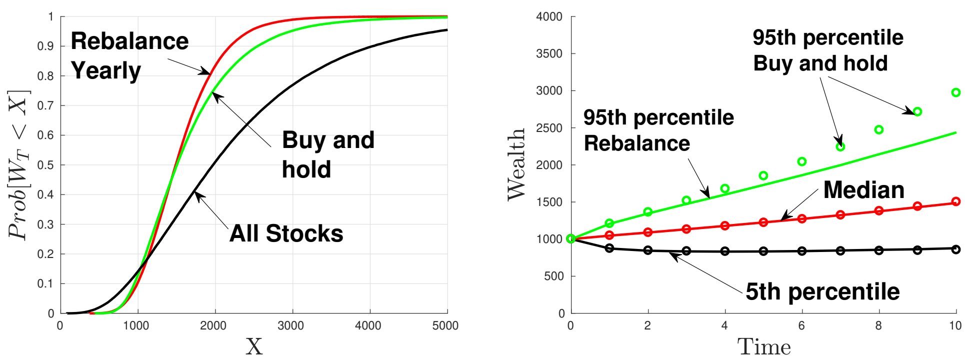
- Buy and hold outperforms rebalancing, sometimes by a large amount, above the median of W_T.

Figure 5.1(b) shows the wealth percentiles for rebalancing and buy and hold, at each rebalancing time. We can observe that the $5th$ percentile and median values of wealth for both strategies is

	$E[W_T]$	Median$[W_T]$	$W_T : 5^{th}$ percentile	$W_T : 95^{th}$ percentile	ES(5%)	std$[W_T]$
			Rebalance monthly			
Rebalance	1534	1477	870	2386	758	476
			Rebalance yearly			
Rebalance	1552	1487	878	2438	768	491
			Never rebalance			
Buy and Hold	1656	1501	854	2970	764	706
All stocks	2272	1969	671	4869	510	1400

TABLE 5.2: *Bootstrap simulations, rebalanced yearly (monthly) $\beta = 0.5$, buy and hold (initial stock fraction $\beta = 0.5$), and an all stock portfolio (i.e. buy and hold with $\beta = 1.0$). 10^5 block bootstrap simulations, expected blocksize one year. CRSP data, 1926:1-2023:12. Stock index: CRSP capitalization weighted index. Bond index: 30 day US T-bills. Indexes inflation adjusted. See Table 5.1. ES(5%) is the mean of the worst 5% of the outcomes.*

almost identical. On the other hand, the $95th$ percentile wealth for buy and hold is significantly larger than for the rebalanced portfolio, as $t \to T$.



(a) *Bootstrap CDF of W_T.*

(b) *Percentiles wealth. Solid lines: rebalance. Symbols: buy and hold.*

FIGURE 5.1: *Bootstrap simulations: rebalanced yearly $\beta = 0.5$, buy and hold (initial stock fraction $\beta = 0.5$), and an all stock portfolio (i.e. buy and hold with $\beta = 1.0$). 10^5 block bootstrap simulations, expected blocksize one year. CRSP data, 1926:1-2023:12. Stock index: CRSP capitalization weighted index. Bond index: 30 day US T-bills. See Table 5.1.*

6 Pathwise comparison

A more rigorous comparison of buy and hold and rebalancing can be determined by examining the pathwise comparison of rebalancing and buy and hold. To this end, we will consider the statistics of (W^{bh}/W^{rebal}) along each path. Buy and hold outperformance, along each path, is indicated if $(W^{bh}/W^{rebal}) > 1.0$

Table 6.1 shows the statistics for the ratio (W^{bh}/W^{rebal}) at $t = T$. The Omega ratio for this pathwise test is defined as

$$\begin{aligned} \text{Omega(L)} &= \frac{E[\max(R_T - L, 0)]}{E[\max(L - R_T, 0)]} \\ R_T &= \frac{W_T^{bh}}{W_T^{rebal}} \ . \end{aligned} \tag{6.1}$$

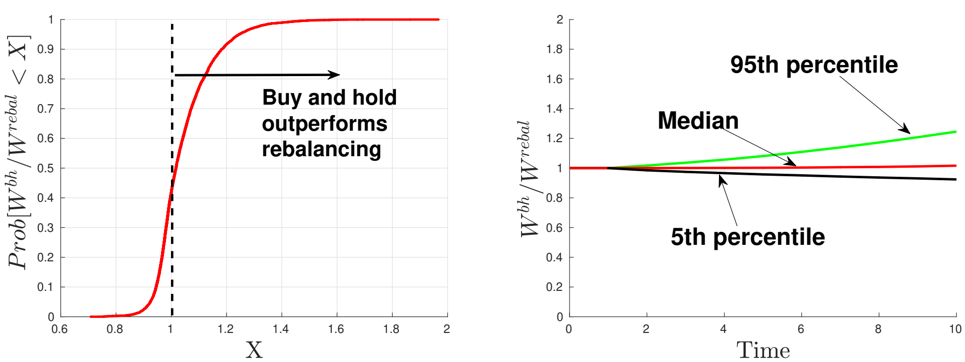
We will examine the Omega ratio for $L = 1$, i.e. we consider the expected value of buy and hold outperforming rebalancing, compared to underperforming.

We can see from Table 6.1 that, along any path, the 5^{th} percentile of the wealth of the buy and hold portfolio is about 92% of the rebalanced portfolio, while at the 95^{th} percentile, (W_T^{bh}/W_T^{rebal}) is 125%. This indicates a better upside, compared to the downside, for buy and hold. This is also reflected in the Omega ratio.

$E[(W_T^{bh}/W_T^{rebal})]$	(W_T^{bh}/W_T^{rebal}) Median	(W_T^{bh}/W_T^{rebal}) 5^{th} percentile	(W_T^{bh}/W_T^{rebal}) 95^{th} percentile	ES(5%)	Omega (L=1)
1.04	1.02	.924	1.25	.889	3.53

TABLE 6.1: *Statistics for (W^{bh}/W^{rebal}) at $t = T$. W^{bh} : buy and hold (initial stock fraction $\beta = 0.5$). W^{rebal} : rebalance yearly ($\beta = 0.5$). 10^5 block bootstrap simulations, expected blocksize one year. CRSP data, 1926:1-2023:12. Stock index: CRSP capitalization weighted index. Bond index: 30 day US T-bills. Both indexes inflation adjusted. See Table 5.1. ES(5%) is the mean of the worst 5% of the outcomes of (W^{bh}/W^{rebal}). The Omega ratio for (W^{bh}/W^{rebal}) is defined in equation (6.1).*

Figure 6.1(a) shows the CDF of (W^{bh}/W^{rebal}) at $t = T$. We can see that the probability that buy and hold will outperform rebalancing is about 55%, with greater upside compared to downside, consistent with the Omega ratio in Table 6.1. Figure 6.1(b) shows that the 5^{th} and 95^{th} percentiles of (W^{bh}/W^{rebal}) show a larger deviation from the median as time goes on. However, at each time in $[0,T]$, the upside is larger than the downside.



(a) *CDF (W^{bh}/W^{rebal}) at $t = T$.* (b) *Percentiles (W^{bh}/W^{rebal}).*

FIGURE 6.1: *Ratio (W^{bh}/W^{rebal}). Bootstrap simulations: rebalanced yearly $\beta = 0.5$, buy and hold (initial stock fraction $\beta = 0.5$), and an all stock portfolio (i.e. buy and hold with $\beta = 1.0$). 10^5 block bootstrap simulations, expected blocksize one year. CRSP data, 1926:1-2023:12. Stock index: CRSP capitalization weighted index. Bond index: 30 day US T-bills. Both indexes inflation adjusted. See Table 5.1.*

7 Instantaneous volatility: buy and hold

As a first approximation, we can ignore the volatility of the bond index. Consequently, the instantaneous volatility of the buy and hold portfolio will be proportional to fraction in equities, compared to the rebalanced portfolio which maintains the fraction 0.5 in stocks.

Figure 7.1 shows the percentiles of the fraction in stocks, through time, for the buy and hold policy. We can observe that the buy and hold strategy has a median fraction in stocks which increases steadily as time goes on, ending up with about 66% stocks at $t = T$. The terminal fraction in stocks, at the 95^{th} percentile is about 0.80, indicating that at this percentile, the volatility of buy and hold is $(.8/.5) \simeq 1.60$ times larger than the rebalanced portfolio. However, this large volatility will only occur at large values of wealth. On the other hand, at the 5^{th} percentile, the terminal fraction in stocks is about 0.4, indicating a smaller volatility compared to rebalancing, along paths with poor stock performance.

In summary, it is clear that when stocks perform well, the fraction in stocks for buy and hold will increase, and the volatility of the portfolio will be larger compared to rebalancing. We can also expect larger drawdowns. However, from Table 6.1 and Figure 6.1(b), the pathwise worst case 5^{th} percentile of (W^{bh}/W^{rebal}) is about 92%. This pathwise criteria is very strict. Looking at the 5^{th} percentiles of wealth for rebalancing and buy and hold (not pathwise), Figure 5.1(b) indicates that the 5^{th} percentiles for the wealth are almost the same for both strategies.

It would appear then that there is at least a behavioral argument in favour of rebalancing, since it is probable that the instantaneous volatility of the buy and hold portfolio will be larger than the rebalanced portfolio. This smaller volatility, of the rebalanced portfolio, comes at the cost of giving up potential upside. However, note that if stocks do poorly, then the buy and hold portfolio will have less volatility than the rebalanced portfolio.

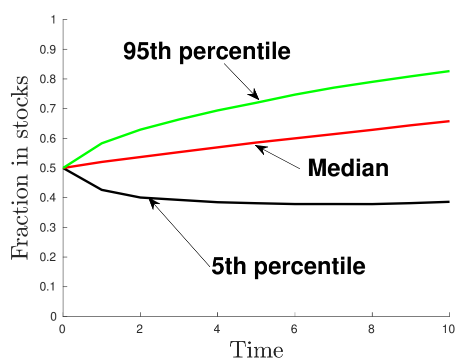


FIGURE 7.1: *Fraction of wealth in stocks, buy and hold.* 10^5 *block bootstrap simulations, expected blocksize one year. CRSP data, 1926:1-2023:12. Stock index: CRSP capitalization weighted index. Bond index: 30 day US T-bills. See Table 5.1.*

8 Alternate assets

In order to verify that there is nothing special about our choice of assets, we carry out block bootstrap resampling using the CRSP equal weighted stock index, and for the bond index, we use the 10 year US Treasury index. Both indexes are constructed for the period 1926:1-2023:12, and deflated using the CPI. The results are reported in Appendix D, and are qualitatively similar to those with carried out the capitalization weighted CRSP index, and the 30-day T-bill index.

9 Other strategies

A popular alternative to constant weight rebalancing is a deterministic glide path. A simple example of a glide path is

$$\text{fraction in equities} \quad = \quad \frac{100 - \text{ your age}}{100} \ . \tag{9.1}$$

However, we know that for any glide path, there is a constant weight strategy which has almost the same CDF of the final wealth (Forsyth and Vetzal, 2019; Ni et al., 2022). This implies that, in terms of the final wealth CDF, if we replaced constant weight rebalancing by (any) glide path, our conclusions would be similar.

On the other hand, use of dynamic, adaptive strategies would generally be superior to buy and hold, in terms of meeting the specified objective function (van Staden et al., 2021; Forsyth, 2022; Forsyth and Vetzal, 2022).

In other words, compared to buy and hold, there is little to be gained by using a constant weight strategy, or a deterministic glide path. Significent improvements can only be obtained using dynamic, adaptive strategies.

10 Summary

We have carried out a detailed analysis of the CDFs of the final wealth for buy and hold compared to annually rebalancing over a ten year period. Consistent results are obtained using various levels of modelling: (i) GBM models of stock index returns (ii) jump diffusion models of stocks (iii) bootstrap resampling of historical data.

Generally, rebalancing outperforms buy and hold by a small amount, below the median of the final wealth, but underperforms buy and hold by larger amounts above the median. In addition, the extreme left tail, as measured by the average of the worst 5% of the outcomes, is similar for both buy and hold and rebalancing. In other words, buy and hold has more upside than downside, compared to rebalancing, with similar worst case performance.

We also note that, considering a ten year investment horizon for rebalancing investors, it is unnecessary to rebalance more frequently than annually.

The bootstrap results were qualitatively similar for the cases (i) stock index: CRSP capitalization weighted index; bond index: 30-day T-bills and (ii) stock index: CRSP equal weight index; bond index: 10 year US treasuries.

The negative aspect of buy and hold is that the instantaneous volatility of this strategy will be generally larger than the rebalanced portfolio. This will be particularly pronounced when buy and hold has large returns in stocks, meaning that the wealth of the buy and hold portfolio will be large. It is possible that this may be a behavioral reason to recommend rebalancing to some investors.

11 Conclusions

Based on the CDFs of the terminal wealth, the percentiles of wealth through time, and considering the extreme left tail of the final wealth, it is difficult to recommend that investors should rebalance a stock-bond portfolio, over ten year horizons. Buy and hold has more upside than rebalancing, with slightly worse downside, but similar risk in the extreme left tail. Buy and hold is also generally more volatile than rebalancing, but this effect is large in cases where the buy and hold wealth is also large. This may not be a problem for many investors.

We have not considered taxes or transaction costs in this paper. Transaction costs for infrequently traded index ETFs are negligible. However, in a taxable account, rebalancing can trigger capital gains taxes (winners are sold). Buy and hold, of course, defers taxes. Consequently, our results would favour buy and hold in a taxable account.

Appendices

A Data

We use data from the Center for Research in Security Prices (CRSP) on a monthly basis over the 1926:1-2023:12 period.[4] Our base case tests use the CRSP US 30 day T-bill for the bond asset and the CRSP value-weighted total return index for the stock asset. This latter index includes all distributions for all domestic stocks trading on major U.S. exchanges. All of these various indexes are in nominal terms, so we adjust them for inflation by using the U.S. CPI index, also supplied by CRSP. We use real indexes since investors should be focused on real (not nominal) wealth goals. We also include examples using the CRSP equal weighted index, and the CRSP 10-year U.S. Treasury index.[5]

B Jump diffusion payoff function

Recall equations (4.1) and (4.5)

$$
\begin{aligned}
\frac{dS}{S(t^-)} &= (\mu - \lambda\kappa)\ dt + \sigma\ dZ + (\xi - 1) d\mathbb{Q} && \text{(B.1)}\\
\frac{dW}{W} &= \Big((1-\beta) r + \beta(\mu - \lambda\kappa) \Big)\ dt + \beta\sigma\ dZ + \beta(\xi - 1)\ d\mathbb{Q}\ . && \text{(B.2)}
\end{aligned}
$$

Equation (B.1) implies that

$$
\begin{aligned}
\frac{S(t)}{S(0)} &= e^{(\mu-\lambda\kappa)t - \sigma^2 t/2} e^{\sigma(Z(t)-Z(0)) + \sum_{i=0}^{\pi(t)} \log \xi_i} \\
&= e^{(\mu-\lambda\kappa)t + \sigma(Z(t)-Z(0))} e^{-\sigma^2 t/2 + \sum_{i=0}^{\pi(t)} \log \xi_i}\ , && \text{(B.3)}
\end{aligned}
$$

where $\pi(t)$ counts the number of Poisson jumps with intensity λ in $(0,t)$. Rearrange equation (B.3) to obtain

$$
\left(\frac{S(t)}{S(0)}\right)^{\beta} e^{\beta\sigma^2 t/2 - \beta \sum_{i=0}^{\pi(t)} \log \xi_i} = \left(e^{(\mu-\lambda\kappa)t + \sigma(Z(t)-Z(0))} \right)^{\beta} . \qquad \text{(B.4)}
$$

Equation (B.2) implies that

$$
\begin{aligned}
\frac{W(t)}{W(0)} &= e^{(1-\beta)rt + \beta(\mu-\lambda\kappa)t - \beta^2\sigma^2 t/2} e^{\beta\sigma(Z(t)-Z(0)) + \sum_{i=0}^{\pi(t)} \log(1+\beta(\xi_i - 1))} \\
&= e^{(1-\beta)rt - \beta^2\sigma^2 t/2} \left(e^{(\mu-\lambda\kappa)t + \sigma(Z(t)-Z(0))} \right)^{\beta} e^{\sum_{i=0}^{\pi(t)} \log(1+\beta(\xi_i - 1))}\ . && \text{(B.5)}
\end{aligned}
$$

[4]More specifically, results presented here were calculated based on data from Historical Indexes, ©2023 Center for Research in Security Prices (CRSP), The University of Chicago Booth School of Business. Wharton Research Data Services (WRDS) was used in preparing this article. This service and the data available thereon constitute valuable intellectual property and trade secrets of WRDS and/or its third-party suppliers.

[5]The 10-year Treasury index was calculated using monthly returns from CRSP dating back to 1941. The data for 1926-1941 were interpolated from annual returns in Homer and Sylla (2005). The bond index is constructed by (i) purchasing a 10-year Treasury at the start of each month, (ii) collecting interest during the month and (iii) selling the Treasury at the end of the month.

Substitute equation (B.4) into (B.5) to obtain

$$
\begin{aligned}
\frac{W(t)}{W(0)} &= e^{(1-\beta)rt+\beta(1-\beta)\sigma^2 t/2} \exp\left(\sum_i^{\pi(t)} \left\{ \log(1+\beta(\xi_i - 1)) - \beta \log(\xi_i) \right\} \right) \left(\frac{S(t)}{S(0)} \right)^{\beta} \\
&= e^{(1-\beta)rt+\beta(1-\beta)\sigma^2 t/2} \left(\frac{S(t)}{S(0)} \right)^{\beta} H(\beta, t) \ , \qquad \text{(B.6)}
\end{aligned}
$$

where

$$
\begin{aligned}
H(\beta, t) &= \exp\left(\sum_i^{\pi(t)} \left\{ \log(1+\beta(\xi_i - 1)) - \beta \log(\xi_i) \right\} \right) \\
&= \Pi_{i=0}^{\pi(t)} \left(\frac{1+\beta(\xi_i - 1)}{\xi_i^{\beta}} \right) \\
&= \Pi_{i=0}^{\pi(t)} F(\xi_i) \\
F(\xi_i) &= \left(\frac{1+\beta(\xi_i - 1)}{\xi_i^{\beta}} \right) \ . \qquad \text{(B.7)}
\end{aligned}
$$

Unfortunately, $H(\beta, t)$ is not deterministic, so the payoff function is not a deterministic function of $S(t)$, in contrast to the GBM case.

Note that $\pi(t) \to 0$ as $t \to 0$, so that equation (B.6) becomes

$$
\lim_{t \to 0} \frac{W(t)}{W(0)} = \left(\frac{S(t)}{S(0)} \right)^{\beta} \ . \qquad \text{(B.8)}
$$

So, even if there are jumps, the payoff of a rebalanced portfolio is a power law for small t.

It also interesting to examine the extra jump term $H(\beta, t)$ in equation (B.6). This extra term involves products of terms like

$$
F(\xi_i) = \frac{1+\beta(\xi_i - 1)}{\xi_i^{\beta}} \ . \qquad \text{(B.9)}
$$

Since $F(\xi = 1) = 1$, $\xi \in [0,\infty]$, and assuming $0 < \beta < 1$, then

$$
F(\xi) = \begin{cases} \infty & \xi \to 0 \\ \infty & \xi \to \infty \end{cases} \qquad \text{(B.10)}
$$

In addition,

$$
\frac{dF}{d\xi} = \frac{\beta(\beta - 1)(\xi^{\beta-1} - \xi^{\beta})}{\xi^{2\beta}} \qquad \text{(B.11)}
$$

which implies that

$$
\frac{dF}{d\xi} = \begin{cases} < 0 & \xi < 1 \\ > 0 & \xi > 1 \end{cases} \qquad \text{(B.12)}
$$

hence $F(\xi) \geq 1, \forall \xi$, so jumps always increase the value of rebalancing if $S(t) = S(0)$.

	μ	σ	λ	p_{up}	η_1	η_2
CRSP Index (real)	0.08732	0.1477	0.3163	0.2258	4.3591	5.5337

TABLE C.1: *Estimated annualized parameters for double exponential jump diffusion model. Value-weighted CRSP index deflated by the CPI. Sample period 1926:1 to 2023:12. The average real return of a 30 day T-bill in the same period was* $r = .0.0031$.

C Jump diffusion parameters

The parameters for equations (4.1) and (4.2) are fit to the CRSP data, with results in Table C.1.

D Alternate assets

In order to verify that our findings are robust to the choice of assets, we carry out block bootstrap resampling using the CRSP equal weighted stock index, and for the bond index, we use the 10 year US Treasury index. Both indexes are constructed for the period 1926:1-2023:12. As before, we deflate these indexes using the CPI.

Table D.1 shows that median returns for all methods are larger than for the case where the underlying assets are the capitalization weighted CRSP index, and the 30-day T-bill index. This is hardly unexpected. However, the qualitative results are similar to that reported in Section 4. In particular, compare Figure D.1 with Figure 5.1.

	$E[W_T]$	Median$[W_T]$	5^{th} percentile	95^{th} percentile	ES(5%)	std$[W_T]$
Rebalance	2026	1811	895	3900	758	1023
Buy and Hold	2229	1796	872	5047	756	1617
All stocks	3222	2340	613	8800	435	3205

TABLE D.1: *Statistics for bootstrap simulations: rebalanced yearly* $\beta = 0.5$*, buy and hold (initial stock fraction* $\beta = 0.5$*), and an all stock portfolio (i.e. buy and hold with* $\beta = 1.0$*).* 10^5 *block bootstrap simulations, expected blocksize one year. CRSP data, 1926:1-2023:12. Stock index: CRSP equal weighted index. Bond index: 10 year Treasuries. See Table 5.1. ES(5%) is the mean of the worst 5% of the outcomes.*

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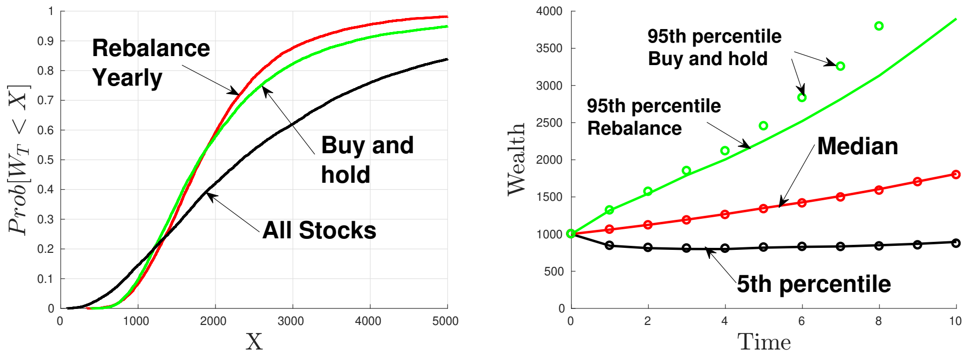

(a) *Bootstrap CDF of W_T.*

(b) *Percentiles wealth. Solid lines: rebalance. Symbols: buy and hold.*

FIGURE D.1: *Bootstrap simulations: rebalanced yearly $\beta = 0.5$, buy and hold (initial stock fraction $\beta = 0.5$), and an all stock portfolio (i.e. buy and hold with $\beta = 1.0$). 10^5 block bootstrap simulations, expected blocksize one year. CRSP data, 1926:1-2023:12. Stock index: CRSP equal weighted index. Bond index: 10 year US Treasuries.*

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