Pricing Hydroelectric Power Plants with/without Operational Restrictions: a Stochastic Control Approach

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Abstract. In this paper, we value hydroelectric power plant cash flows under a stochastic control framework, taking into consideration the implication of operational constraints such as ramping and minimum flow rate constraints for the purpose of environmental protection. The power plant valuation problem under a ramping constraint is characterized as a bounded stochastic control problem, resulting in a Hamilton-Jacobi-Bellman (HJB) partial integrodifferential equation (PIDE). The valuation problem without the ramping restriction is characterized as an unbounded stochastic control problem; we propose an impulse control formulation, resulting in an HJB variational inequality, for the valuation problem under this scenario. We develop a consistent numerical scheme for solving both the HJB PIDE for the bounded control problem and the HJB variational inequality for the unbounded control problem. We prove the convergence of the numerical scheme to the viscosity solution of each pricing equation, provided a strong comparison result holds. Numerical results indicate that failing to consider operational constraints may considerably overestimate the value of hydroelectric power plant cashflows.

1 Introduction

A hydroelectric power plant generates electricity by releasing water from a reservoir through the turbine located downstream of the dam. The electricity power produced relies on the rate of water flowing through the turbine, which is controlled by the ramping rate (the rate of change of flow through the turbine) chosen by the operator. In order to gain profits, the operator of a hydroelectric power plant must determine the appropriate ramping rate in response to stochastic electricity prices. Furthermore, the operator is required to conform to various operational restrictions, such as ramping and minimum flow rate constraints, imposed by the government for the purpose of environmental protection [14]. The valuation of hydroelectric power plants is characterized as a stochastic control problem with the ramping rate (which can be either bounded or unbounded) as the control variable.

In this paper, we focus on valuing hydroelectric power plant cash flows by solving the corresponding stochastic control problem, and studying the implication of the operational constraints on the value of power plants.

Following [7, 8, 24], we can formulate the hydroelectric power plant pricing problem (i.e., the stochastic control problem) as a Hamilton-Jacobi-Bellman (HJB) partial integrodifferential equation (PIDE) and an HJB variational inequality respectively for the cases with and without a ramping constraint. Then we can use PDE based approaches to solve the HJB equation/variational inequality numerically.

In general, the solution to an HJB equation/variational inequality may not be unique. According to [15, 23], the viscosity solution to the HJB equation/variational inequality is normally identical to the value of the corresponding stochastic control problem. Therefore, as noted in [2, 16], it is important to ensure that a numerical scheme converges to the viscosity solution of the equation, which is the appropriate solution of the corresponding stochastic control problem.

Based on the work in [7, 8], we develop a consistent numerical scheme to solve both the HJB equation and the HJB variational inequality. In both cases, the numerical technique can be shown to converge to the viscosity solution of the pricing equation. Our main results are the following:

- We propose an HJB PIDE for the valuation of a hydroelectric power plant under a ramping constraint. We use a one-factor model for the electricity spot price that is able to capture major features of electricity prices such as mean-reverting dynamics, daily price trends, and price spikes.

- We present a semi-Lagrangian scheme, based on the scheme proposed in [8], to solve the HJB PIDE. The timestepping scheme is an implicit-explicit scheme with the integral terms treated explicitly. As discussed in [12], this scheme is unconditionally stable. The two integral terms in the PIDE are evaluated using a Fast Fourier Transform (FFT). Provided a strong comparison result holds, we prove the scheme converges to the unique viscosity solution of the pricing PIDE.

- We also consider a scenario with no ramping constraint. Formally, this case can be considered to be the limit obtained by allowing the ramping rate to become unbounded. We propose an impulse control formulation for this situation, resulting in an HJB variational inequality.

- We derive a simple extension of the the discretization scheme for the HJB PIDE under ramping constraints to solve the HJB variational inequality with unbounded ramping rates. Provided a strong comparison result holds, we prove the scheme converges to the unique viscosity solution of the HJB variational inequality.

- We study the implication of operational restrictions on the optimal control strategies and the value of the power plant. Through an example, our numerical results indicate that the operational constraints can reduce the value of the power plant cash flows by more than 37% compared to the case with no ramping constraints and no minimum flow requirements. Therefore, it is important to take these constraints into consideration in order to accurately value the plant revenues.

1.1 Previous work

Pricing and scheduling hydroelectric and thermal power generation assets is a popular research topic (see, e.g., [14, 18, 19, 21, 24] for the work on hydroelectric power generators and [10, 24, 25, 28] for the research on thermal power generators).

The authors of [14, 18, 19] study the short-term scheduling of hydroelectric power generation systems with various operational constraints including a ramping constraint. Their research focuses on scheduling a number of power systems under the assumption of known electricity prices with the main effort placed on solving nonlinear optimization problems, while in this paper we consider the operation of a single power plant under electricity price uncertainty.

The authors of [21] consider the long and medium term production planning of a hydropower system under price and inflow uncertainty. However, they assume a parametric form for the control strategies in order to solve the valuation problem. As such, the resulting control strategies may not be optimal, and hence the method tends to undervalue the power system. Furthermore, no operational constraints are considered in [21].

The research in [24] focuses on the value of a pump-storage facility in the PDE framework. The pump storage facility allows water to be pumped to the reservoir when the electricity price is low and to be released from the storage to generate electricity when the electricity price is high, and thus resulting in profit. Their work, however, does not consider operational constraints. Moreover, the pricing equation is solved numerically using an explicit finite differencing method, which is known to suffer from timestep restrictions due to stability considerations.

2 Hydroelectric Power Plant Valuation Problem

This section defines the short-term hydroelectric power plant valuation problem and proposes the pricing equations for the value of a hydroelectric power plant. The section is arranged as follows: first, we define some notation for the problem; we then present a one-factor model for electricity spot prices that is able to capture main features of the market electricity prices. Based on the spot price model, we propose two pricing equations for the problem: one incorporates the regulatory operational constraints, the other does not. We also provide the boundary conditions for the pricing equations to completely specify the hydroelectric power plant valuation problem.

2.1 Problem notation

We use the following notation for the hydroelectric power plant valuation problem:

- t: time in hours.
- P: current electricity spot price.

- h: head of the water above the turbine (i.e., the difference in elevation between water levels upstream and downstream of a dam). We assume that h can be any value lying within the domain $[h_{\min}, h_{\max}]$, where $h_{\min}, h_{\max}$ are positive constants specified in the management plan.

- c: rate of the water flowing through the turbine (outflow rate). We assume c can be any value residing within the domain $[c_{\min}, c_{\max}]$, where $c_{\min} \geq 0$ and $c_{\max}$ is determined by the turbine's head due to Bernoulli's law [24]. For a hydroelectric power plant, the change on the head is usually much smaller (e.g., within meters) compared with the head itself (e.g., a hundred meters). As a result, instead of treating $c_{\max}$ as a function of the head h, in this paper, we regard $c_{\max}$ as a positive constant.

- T: Cash flow horizon. We will typically choose $T = 1$ week. This will determine the risk neutral revenues obtained through optimal operation of the plant in a typical weekly cycle.

- $V(P, c, h, \tau)$: value of the hydroelectric power plant as a function of electricity price P, outflow rate c, head h and current time to the cash flow horizon $\tau = T - t$, where t denotes the current time.

- z: control variable that represents the ramping rate imposed by the power plant. That is, we have
$$\frac{dc}{dt} = z. \tag{2.1}$$
According to equation (2.1), $z > 0$ represents ramp-up (i.e., increasing the outflow rate c), $z < 0$ represents ramp-down (i.e., decreasing the outflow rate c), while $z = 0$ represents keeping the current outflow rate unchanged.

- f: rate of the water flowing into the dam (inflow rate). In this paper, we assume that f is a positive constant. It is straightforward to extend f to be a deterministic function of t or to follow a stochastic process [24].

- a: surface area of the hydroelectric reservoir. We assume that the volume of the reservoir is $a \cdot \bar{h}$, where $\bar{h}$ is the depth of the reservoir.

- $H(c, h)$: amount of power produced by the power generator as a function of outflow rate c and head h. According to [24], in theory, the power generated by a hydroelectric power plant is given by
$$H_m(c, h) = g\rho c h, \tag{2.2}$$
where g is the acceleration of gravity, ρ is the density of water. Let $\eta(c, h)$ be a nonlinear function of c and h representing the efficiency of the turbine. Then the actual power generated by the turbine is represented by
$$H(c, h) = H_m(c, h)\eta(c, h). \tag{2.3}$$

Following [24], we use the following quadratic function to represent $\eta(c,h)$:

$$\eta(c,h) = -k_1 \left(\frac{H_m(c,h)}{k_2} - 1 \right)^2 + k_1, \tag{2.4}$$

where $0 < k_1 < 1$ and $k_2 > H_m(c_{\max}, h_{\max})/2$ so that $0 < \eta(c,h) < 1$. The above equation has the physical meaning that the turbine achieves the maximum efficiency k_1 when it generates a power of k_2. According to (2.2), $H_m(c,h)$ will achieve the maximum at $(c,h) = (c_{\max}, h_{\max})$. Note that the maximum efficiency η in (2.4) may not correspond to $(c_{\max}, h_{\max})$. In other words, if $H_m(c_{\max}, h_{\max}) \neq k_2$ then $\eta(c_{\max}, h_{\max}) < k_1$.

Given the above notation, when h is away from the extreme values $h_{\min}$ or $h_{\max}$, the rate of change of the head satisfies

$$\frac{dh}{dt} = \frac{f-c}{a}. \tag{2.5}$$

Equation (2.5) implies that if the outflow rate c is larger than the inflow rate f, then the head decreases. Similarly, the head increases if the outflow rate is smaller than the inflow rate. Nevertheless, equation (2.5) cannot be generalized to the case when either when $h \to h_{\min}$ and $c > f$, or when $h \to h_{\max}$ and $c < f$. As $h \to h_{\min}$ and $c > f$, the head h can fall below $h_{\min}$ according to equation (2.5), which is not allowed. As $h \to h_{\max}$ and $c < f$, equation (2.5) implies that the head can increase above $h_{\max}$, the highest head permitted. To handle these cases, we generalize equation (2.5) to

$$\frac{dh}{dt} = G(c,h) \cdot \frac{f-c}{a} \tag{2.6}$$

for any $h \in [h_{\min}, h_{\max}]$ and any $c \in [c_{\min}, c_{\max}]$, where G is a smooth function of c and h satisfying

$$\begin{array}{ll} G(c,h) \to 0 & \text{if } c > f \text{ and } h \to h_{\min}, \\ G(c,h) \to 0 & \text{if } c < f \text{ and } h \to h_{\max}, \\ G(c,h) = 1 & \text{otherwise.} \end{array} \tag{2.7}$$

Consequently, equations (2.6-2.7) ensure that $h \in [h_{\min}, h_{\max}]$. For future reference, note that equations (2.6-2.7) imply that

$$G(c,h) \cdot \frac{f-c}{a} \geq 0 \;\Leftrightarrow\; \frac{dh}{dt} \geq 0 \;\; ; \;\; \forall c \in [c_{\min}, c_{\max}] \;, \;\; h = h_{\min} \;, \tag{2.8}$$

$$G(c,h) \cdot \frac{f-c}{a} \leq 0 \;\Leftrightarrow\; \frac{dh}{dt} \leq 0 \;\; ; \;\; \forall c \in [c_{\min}, c_{\max}] \;, \; h = h_{\max} \;. \tag{2.9}$$

In order to satisfy (2.7), we assume that function $G(c,h)$ has the following property

$$G(c,h) = \begin{cases} O\left((h - h_{\min})^\nu\right) & \text{if } c > f \text{ and } h \to h_{\min}, \\ O\left((h_{\max} - h)^\nu\right) & \text{if } c < f \text{ and } h \to h_{\max}, \\ 1 & \text{otherwise,} \end{cases} \tag{2.10}$$

where ν is any small positive constant.

As outflow rate $c \to c_{\min}$, condition

$$\frac{dc}{dt} \geq 0 \tag{2.11}$$

needs to be satisfied so that the outflow rate does not decrease below the minimum flow rate $c_{\min}$. Similarly, condition

$$\frac{dc}{dt} \leq 0 \tag{2.12}$$

must be satisfied when $c \to c_{\max}$ so that the outflow rate does not increase above the maximum flow rate $c_{\max}$. From equations (2.1) and (2.11-2.12), the above analysis results in the following conditions on the ramping rate z:

$$z \geq 0\,, \quad c = c_{\min}, \tag{2.13}$$

$$z \leq 0\,, \quad c = c_{\max}. \tag{2.14}$$

In order to satisfy conditions (2.13-2.14), we will further assume that

$$\min\{z\} = O\Big((c - c_{\min})^{\theta}\Big)\,, \quad \text{if } c \to c_{\min}, \tag{2.15}$$

$$\max\{z\} = O\Big((c_{\max} - c)^{\theta}\Big)\,, \quad \text{if } c \to c_{\max}, \tag{2.16}$$

where θ is any positive constant. Condition (2.15) has the physical meaning that the operator of the power plant has to gradually decrease the maximum ramp-down rate $\min(z)$ towards zero when c is close to $c_{\min}$, instead of suddenly switching it to zero at $c = c_{\min}$. A similar explanation applies to the condition (2.16).

In the following, we only require that $\theta, \nu > 0$. If $\theta \ll 1$, then according to (2.15-2.16), for all practical purposes, the operator can switch ramping rates virtually instantaneously to zero at $c_{\min}$ and $c_{\max}$. If $\nu \ll 1$, then (2.7) implies that the change of head can be reduced to zero virtually instantaneously at $h_{\min}$ and $h_{\max}$. We require $\theta, \nu > 0$ to simplify some of the proofs in subsequent sections.

At a point (P, c, h), the rate of cash flow (instantaneous revenue) received by the operator of the power plant is $H(c, h) \cdot P$, obtained from selling the generated electricity to the spot market at price P. Similar to the above analysis, we need to pay attention to two cases when $h \to h_{\min}$, $c > f$ and $h \to h_{\max}$, $c < f$. Since these two cases can result in potential decrease/increase of the head below/above the lower/upper bound, we will impose a penalty on the revenue function in these cases to keep the head and the outflow rate from reaching these states (since there are penalties imposed, it would not be optimal to reach these states). Specifically, we will set the rate of cash flow as $G(c, h)H(c, h)P$, where the function G is given in (2.7), so that the revenue rate decreases to zero as $h \to h_{\min}$, $c > f$ or $h \to h_{\max}$, $c < f$, and otherwise remains at $H(c, h)P$.

2.2 Electricity spot price model

Electricity spot prices mainly exhibit three features: mean-reverting dynamics, seasonality trends on daily, weekly and annual time scales, and occasional price spikes caused by short-term disparities between supply and demand. A price spike consists of a cluster of up-jumps of relatively large size with respect to normal fluctuations, shortly followed by a return to normal price levels [9, 17]. Since we are interested in considering short-term (e.g., one week) valuation and optimal operation of a hydroelectric generation asset, we will model hourly electricity spot price and ignore the weekly and annual price trends.

In this section, we specify a one-factor hourly electricity spot price model similar to the model given in [24] which is capable of qualitatively capturing the realistic electricity spot price dynamics. We will consider directly the risk adjusted (or risk neutral) price process with parameters given under the $\mathbb{Q}$ measure since we will value the cash flows under the risk neutral measure.

We assume that the risk adjusted hourly spot price is modeled by the a stochastic differential equation (SDE) given by

$$\begin{aligned} dP &= [\alpha(K(t) - P) - \lambda_1 \kappa_1 P - \lambda_2(P)\kappa_2 P]dt + \sigma P dZ \qquad (2.17) \\ &\quad + (J_1 - 1)P dq_1 + (J_2 - 1)P dq_2 \, , \\ K(t) &= K_0 + \beta \sin\left(\frac{2\pi(t - t_0)}{24}\right) , \qquad (2.18) \end{aligned}$$

where

- $\alpha > 0$ is the mean-reversion rate,
- $K(t) \geq 0$ is the long-term equilibrium price that incorporates the daily price cycle,
- σ is the volatility,
- dZ is an increment of the standard Gauss-Wiener process,
- $K_0 \geq 0$ is the equilibrium price without incorporating a daily price fluctuation,
- β is the parameter for the daily price trend,
- t_0 is the centering parameter, representing the time of daily peak of the equilibrium price,
- dq_1 is the independent Poisson process representing the up-jump of a price spike event, that is, $dq_1 = \begin{cases} 0 & \text{with probability } 1 - \lambda_1 dt \, , \\ 1 & \text{with probability } \lambda_1 dt \, . \end{cases}$
- λ_1 is the constant jump intensity representing the mean arrival time of the up-jump event,

- J_1, $J_1 \geq 1$, is a random variable representing the up-jump size of electricity price. When $dq_1 = 1$, price jumps from P up to PJ_1. We assume that J_1 follows a probability density function $g_1(J_1)$.
- dq_2 is the independent Poisson process representing the down-jump of a price spike event, that is, $dq_2 = \begin{cases} 0 & \text{with probability } 1 - \lambda_2(P)dt \ , \\ 1 & \text{with probability } \lambda_2(P)dt \ . \end{cases}$
- $\lambda_2(P)$ is the jump intensity as a function of P, representing the mean arrival time of the down-jump event,
- J_2, $0 < J_2 \leq 1$, is a random variable representing the down-jump size of electricity price. When $dq_2 = 1$, price jumps from P down to PJ_2. We assume that J_2 follows a probability density function $g_2(J_2)$.
- κ_1 (or κ_2) is $E[J_1 - 1]$ (or $E[J_2 - 1]$), where $E[\,\cdot\,]$ is the expectation operator.

We assume that the two Poisson processes are uncorrelated, i.e., $Cov(dq_1, dq_2) = 0$.

Following [24], we use parameters λ_1 and J_1 to simulate the upward jumps, and use $\lambda_2(P)$ and J_2 to simulate the downward jumps. Let $P_0 > 0$ be a constant representing the threshold value for the spike state. In other words, with high probability, we regard any spot price $P > P_0$ as a price resulting after up-jumps, but before down-jumps, during a price spike event. According to [24], we set

$$\lambda_2(P) = \begin{cases} \bar{\lambda}_2 & \text{if } P \geq P_0 \ , \\ 0 & \text{if } P < P_0 \ , \end{cases} \tag{2.19}$$

where $\bar{\lambda}_2$ is a positive constant. Consequently, if the price resides in the spike state after upward jumps (i.e., $P \geq P_0$), then with high probability a down-jump will occur, bringing the price back to the normal state; conversely, if the price lies in the normal state (i.e., $P < P_0$), the down-jump never occurs. Note that in the spike state another up-jump of the spot price can still occur before dropping back to the normal state, which will form a cluster of up-jumps, as frequently observed in the electricity spot market [9, 17].

2.3 Pricing equations for hydroelectric power plant valuation

The valuation of power generation assets is characterized as a stochastic control problem resulting in Hamilton-Jacobi-Bellman (HJB) equations. In [24], a pricing PIDE is proposed for the pump-storage facility valuation. The pricing equation in [24] does not consider environmental and regulatory operational constraints for hydroelectric power generators such as ramping and minimum flow rate constraints [14, 18, 19]. As we demonstrate in a later section, ignoring these constraints will lead to over-valuing the assets.

In this section, we first give the pricing equation, an HJB PIDE, with a ramping constraint imposed. We then present the pricing equation, an HJB variational inequality, for the case without imposing a ramping constraint.

Incorporating the ramping constraint

Under a ramping constraint, the ramping rate z must satisfy

$$z_{\min} \leq z \leq z_{\max} \,, \tag{2.20}$$

where $z_{\min} \leq 0$, $z_{\max} \geq 0$ are bounded constants. We denote by $Z(c)$ the set of admissible controls that satisfy constraint (2.20) as well as conditions (2.15-2.16). As a result, we have $Z(c) \subseteq [z_{\min}, z_{\max}]$.

Based on the standard hedging arguments in the financial valuation literature [24], the value of a hydroelectric power plant $V(P, c, h, \tau)$, assuming that the risk adjusted electricity spot price follows the stochastic process defined in (2.17-2.18), is given by the following HJB PIDE:

$$V_\tau = \mathcal{C}V + \mathcal{B}V + \frac{1}{a}(f-c)G(c,h)V_h + \sup_{z \in Z(c)} \big(zV_c\big) + G(c,h)H(c,h)P, \tag{2.21}$$

where the operators $\mathcal{C}$ and $\mathcal{B}$ are

$$\begin{aligned} \mathcal{C}V &= \frac{1}{2}\sigma^2 P^2 V_{PP} + \big[\alpha(K(t)-P) - \lambda_1\kappa_1 P - \lambda_2(P)\kappa_2 P\big]V_P \\ &\quad -\big[r + \lambda_1 + \lambda_2(P)\big]V \qquad (2.22)\\ \mathcal{B}V &= \lambda_1 \int_{-\infty}^{\infty} V(J_1P)g_1(J_1)dJ_1 + \lambda_2(P)\int_{-\infty}^{\infty} V(J_2P)g_2(J_2)dJ_2. \qquad (2.23) \end{aligned}$$

Here r is the riskless interest rate.

Remark 2.1. *In PIDE (2.21), if $V_c > 0$, then the supremum is achieved at $z = \max\{Z(c)\}$ (note that $Z(c)$ is a closed region). Similarly, if $V_c < 0$, then the supremum is achieved at $z = \min\{Z(c)\}$; if $V_c = 0$, then for any $z \in Z(c)$ the supremum is zero.*

Relaxing the ramping constraint

In the absence of a ramping constraint, the ramping rate z is unbounded, that is, $z \in [-\infty, \infty]$. According to equation (2.1), this means that the outflow rate c can instantaneously switch from one state to another. In this case, we follow the steps in [7] and formulate the valuation problem as an impulse control problem by introducing positive fixed costs d_- and d_+ for control values $z = -\infty$ and $z = +\infty$, respectively (d_- and d_+ can be infinitesimally small). The optimal ramping rate will either be $-\infty$, 0, or $+\infty$.

At a point (P, c, h, τ), one of the following three scenarios occurs:

- The optimal control satisfies $z = 0$. Substituting $z = 0$ into the PIDE (2.21) gives

$$V_\tau - \mathcal{C}V - \mathcal{B}V - \frac{1}{a}(f-c)G(c,h)V_h - G(c,h)H(c,h)P = 0 \tag{2.24}$$

- The optimal control satisfies $z = -\infty$, which corresponds to switching the outflow rate c to a smaller value instantaneously. In this case, the value of V satisfies the following no-arbitrage jump condition:

$$V - \sup_{\delta \in [c_{\min} - c, 0)} \big[V(P, c + \delta, h, \tau) - d_- \big] = 0. \tag{2.25}$$

- The optimal control satisfies $z = +\infty$, which corresponds to switching the outflow rate c to a higher value instantaneously. In this case, V satisfies the following jump condition:

$$V - \sup_{\delta \in (0, c_{\max} - c]} \big[V(P, c + \delta, h, \tau) - d_+ \big] = 0. \tag{2.26}$$

More generally, at each point (P, c, h, τ), we have

$$\begin{aligned} V_\tau - \frac{1}{a}(f - c)G(c, h)V_h - \mathcal{C}V - \mathcal{B}V - G(c, h)H(c, h)P &\geq 0 \\ V - \sup_{\delta \in [c_{\min} - c, 0)} \big[V(P, c + \delta, h, \tau) - d_- \big] &\geq 0 \\ V - \sup_{\delta \in (0, c_{\max} - c]} \big[V(P, c + \delta, h, \tau) - d_+ \big] &\geq 0 \end{aligned} \tag{2.27}$$

with at least one of these inequalities holding with equality. Therefore, the value function V satisfies the following HJB variational inequality

$$\begin{aligned} \min \Big\{ & V_\tau - \mathcal{C}V - \mathcal{B}V - \frac{1}{a}(f - c)G(c, h)V_h - G(c, h)H(c, h)P, \\ & V - \sup_{\delta \in [c_{\min} - c, 0)} \big[V(P, c + \delta, h, \tau) - d_- \big], \\ & V - \sup_{\delta \in (0, c_{\max} - c]} \big[V(P, c + \delta, h, \tau) - d_+ \big] \Big\} = 0. \end{aligned} \tag{2.28}$$

2.4 Boundary conditions for pricing equations

In order to completely specify the power plant valuation problem, we need to provide boundary conditions. As for the terminal boundary conditions, we use the following zero payoff as specified in [24]:

$$V(P, c, h, \tau = 0) = 0. \tag{2.29}$$

The domain for pricing equations (2.21) and (2.28) is $(P, c, h) \in [0, \infty] \times [c_{\min}, c_{\max}] \times [h_{\min}, h_{\max}]$. For computational purposes, we need to solve the equations in a finite computational domain $[0, P_{\max}] \times [c_{\min}, c_{\max}] \times [h_{\min}, h_{\max}]$.

Boundary equations for PIDE (2.21)

As $h \to h_{\min}$, from inequality (2.8), the characteristics are outgoing (or zero) in the h direction at $h = h_{\min}$, and we simply solve PIDE (2.21) along the $h = h_{\min}$ boundary, no further information is needed. Similarly, as $h \to h_{\max}$, inequality (2.9) implies that the characteristics are outgoing (or zero) in the h direction at $h = h_{\max}$. We can simply solve the pricing equation along the $h = h_{\max}$ boundary, no further information is needed.

Conditions (2.13-2.14) respectively imply that the characteristics are outgoing in the c direction at $c = c_{\min}$ and $c = c_{\max}$. As such, we can solve the PIDEs along the $c = c_{\min}$ and $c = c_{\max}$ boundaries without requiring further information.

Taking the limit of equation (2.21) as $P \to 0$, we obtain the boundary PDE

$$V_\tau = \mathcal{C}_0 V + \frac{1}{a}(f-c)G(c,h)V_h + \sup_{z\in Z(c)} \big(zV_c\big)\ ;\ \ P \to 0 \tag{2.30}$$

$$\text{with} \quad \mathcal{C}_0 V = \alpha K(t) V_P - rV. \tag{2.31}$$

Note that the integral terms disappear as we take the limit $P \to 0$ and interchange the integrals and the limit. Since $\alpha K(t) \geq 0$ in $\mathcal{C}_0 V$, we can solve (2.30) without requiring additional boundary conditions, as we do not need information from outside the computational domain $[0, P_{\max}]$.

As $P \to \infty$, we need to deal with two major issues. The first problem is that there is no obvious Dirichlet-type condition that can be imposed for P large. The second issue is that for any finite domain $[0, P_{\max}]$, the two integral terms require information from outside the computational domain.

We can resolve the issues using the localization strategy presented in [12, 13]. Specifically, we apply the commonly used boundary condition $V_{PP} \to 0$ [24, 27], which implies that

$$V \simeq x(h,c,\tau)P + y(h,c,\tau) \tag{2.32}$$

in the region $P \in [P_{\max} - \chi, P_{\max}]$ for a sufficiently large $P_{\max}$, where functions x and y are independent of P and χ is a positive constant. Substituting equation (2.32) into the pricing PIDE (2.21) results in

$$\begin{aligned} V_\tau = \mathcal{C}_1 V + \frac{1}{a}(f-c)G(c,h)V_h + \sup_{z\in Z(c)} \big(zV_c\big) + G(c,h)H(c,h)P \\ P \in [P_{\max} - \chi, P_{\max}], \end{aligned} \tag{2.33}$$

with

$$\mathcal{C}_1 V = \begin{cases} \frac{1}{2}\sigma^2 P^2 V_{PP} + \alpha(K(t) - P)V_P - rV & \text{if } P \in [P_{\max} - \chi, P_{\max}), \\ \alpha(K(t) - P)V_P - rV & \text{if } P = P_{\max}\ . \end{cases} \tag{2.34}$$

We leave the V_{PP} term in the operator $\mathcal{C}_1$, though we have assumed that $V_{PP} = 0$ as $P \to \infty$, so that the differential operator is formally parabolic. Note that the integral terms vanish in equation (2.33).

Following [13], we solve PIDE (2.21) in the region $P \in (0, P_{\max} - \chi)$, where the integrals in the PIDE are evaluated using a Fast Fourier Transform (FFT); we solve PDE (2.30) at $P = 0$ boundary; and solve PDE (2.33) in the region $P \in [P_{\max} - \chi, P_{\max}]$. By carefully choosing χ using the method presented in [13], we can provide sufficient data for the computation of integral terms and at the same time reduce the effect of FFT wrap-around. Note that we will choose $P_{\max}$ sufficiently large so that $P_{\max} \gg K(t)$, hence equation (2.33) can be solved at $P = P_{\max}$ without additional information.

Boundary conditions for equation (2.28)

Following a similar analysis as above, we can solve the HJB variational inequality (2.28) directly along the $h = h_{\min}$ and $h = h_{\max}$ boundaries without requiring further information.

Taking the limit of equation (2.28) as $c \to c_{\min}$, we obtain the boundary equation

$$\min\left\{ V_\tau - \mathcal{C}V - \mathcal{B}V - \frac{1}{a}(f-c)G(c,h)V_h - G(c,h)H(c,h)P, \right. \\ \left. V - \sup_{\delta \in (0, c_{\max}-c]} \left[V(P, c+\delta, h, \tau) - d_+ \right] \right\} = 0 \;\; ; \;\; c \to c_{\min}. \tag{2.35}$$

Taking the limit of equation (2.28) as $c \to c_{\max}$, we obtain the boundary equation

$$\min\left\{ V_\tau - \mathcal{C}V - \mathcal{B}V - \frac{1}{a}(f-c)G(c,h)V_h - G(c,h)H(c,h)P, \right. \\ \left. V - \sup_{\delta \in [c_{\min}-c, 0)} \left[V(P, c+\delta, h, \tau) - d_- \right] \right\} = 0 \;\; ; \;\; c \to c_{\max}. \tag{2.36}$$

Equations (2.35-2.36) can be solved directly without using further information from outside the computational domain.

Taking the limit of equation (2.28) as $P \to 0$, we obtain the boundary equation

$$\min\left\{ V_\tau - \mathcal{C}_0 V - \frac{1}{a}(f-c)G(c,h)V_h,\; V - \sup_{\delta \in [c_{\min}-c, 0)} \left[V(P, c+\delta, h, \tau) - d_- \right], \right. \\ \left. V - \sup_{\delta \in (0, c_{\max}-c]} \left[V(P, c+\delta, h, \tau) - d_+ \right] \right\} = 0; \;\; P \to 0. \tag{2.37}$$

Since $\alpha K(t) \geq 0$ in $\mathcal{C}_0 V$ given in (2.31), we can solve (2.37) without requiring additional boundary conditions.

As $P \to P_{\max}$, we also assume that V satisfies the linear form (2.32) in the region $P \in [P_{\max} - \chi, P_{\max}]$. Substituting (2.32) into the pricing equation (2.28) gives

$$\begin{aligned}\min\Big\{ & V_\tau - \mathcal{C}_1 V - \frac{1}{a}(f-c)G(c,h)V_h - G(c,h)H(c,h)P, \\ & V - \sup_{\delta\in[c_{\min}-c,0)} \big[V(P,c+\delta,h,\tau) - d_-\big], \\ & V - \sup_{\delta\in(0,c_{\max}-c]} \big[V(P,c+\delta,h,\tau) - d_+\big]\Big\} = 0; \qquad P \in [P_{\max}-\chi, P_{\max}].\end{aligned} \tag{2.38}$$

We will choose $P_{\max}$ sufficiently large so that $P_{\max} \gg K(t)$, hence equation (2.38) can be solved without requiring additional information. Similar to the way we handle the boundary conditions for PIDE (2.21), we solve equation (2.28) in the region $P \in (0, P_{\max} - \chi)$; solve equation (2.37) at $P = 0$ boundary; and solve equation (2.38) in the region $P \in [P_{\max}-\chi, P_{\max}]$ with χ chosen to reduce the effect of FFT wrap-around for the computation of integral terms.

3 Numerical Algorithms

In [8], we develop a numerical scheme based on a semi-Lagrangian approach for solving the gas storage valuation problem—an optimal stochastic control problem with a bounded control. The scheme has advantage that it is more efficient than the existing numerical methods and satisfies sufficient conditions for convergence to the viscosity solution of the gas storage equation. In [7], we extend the scheme in [8] to price variable annuities with a guaranteed minimum withdrawal benefit—an optimal stochastic control problem with an unbounded control. Since the power plant valuation problem is characterized as a stochastic control problem with either a bounded or an unbounded control, respectively for the case when the ramping constraint is imposed or not, we can solve the power plant valuation problem using the schemes proposed in [8] and [7]. We will directly present the schemes in this section. Refer to [7, 8] for more motivation for the schemes.

Prior to introducing the numerical schemes, we introduce the following notation. We use an unequally spaced grid in P direction for the PDE discretization, represented by $[P_0, P_1, \ldots, P_{i_{\max}}]$ with $P_0 = 0$ and $P_{i_{\max}} = P_{\max}$. Similarly, we use unequally spaced grids in c and h directions, respectively denoted by $[c_0, c_1, \ldots, c_{j_{\max}}]$ and $[h_0, h_1, \ldots, h_{k_{\max}}]$ with $c_0 = c_{\min}$, $c_{j_{\max}} = c_{\max}$, $h_0 = h_{\min}$, and $h_{k_{\max}} = h_{\max}$. We denote by $0 = \Delta\tau < \ldots < N\Delta\tau = T$ the discrete timesteps. Let $\tau^n = n\Delta\tau$ denote the nth timestep. Let $V(P_i, c_j, h_k, \tau^n)$ denote the exact solution of the pricing equation when the electricity spot price is P_i, the outflow rate is c_j, the head is h_k and discrete time is τ^n. Let $V_{i,j,k}^n$ denote an approximation of the exact solution $V(P_i, c_j, h_k, \tau^n)$.

It will be convenient to define $\Delta P_{\max} = \max_i(P_{i+1} - P_i)$, $\Delta P_{\min} = \min_i(P_{i+1} - P_i)$, $\Delta c_{\max} = \max_j(c_{j+1} - c_j)$, $\Delta c_{\min} = \min_j(c_{j+1} - c_j)$, $\Delta h_{\max} = \max_k(h_{k+1} - h_k)$,

$\Delta h_{\min} = \min_k(h_{k+1} - h_k)$. We assume that there is a mesh size/timestep parameter ε such that

$$\begin{aligned} &\Delta P_{\max} = C_1\varepsilon\ ,\ \ \Delta c_{\max} = C_2\varepsilon\ ,\ \ \Delta h_{\max} = C_3\varepsilon\ ,\ \ \Delta\tau = C_4\varepsilon\ , \\ &\Delta P_{\min} = C_1'\varepsilon\ ,\ \ \Delta c_{\min} = C_2'\varepsilon\ ,\ \ \Delta h_{\min} = C_3'\varepsilon\ , \end{aligned} \tag{3.1}$$

where $C_1, C_1', C_2, C_2', C_3, C_3', C_4$ are constants independent of ε.

We use standard finite difference methods to discretize the operator $\mathcal{C}_0 V$, $\mathcal{C}V$ and $\mathcal{C}_1 V$ as given in (2.31), (2.22) and (2.34). Let $(\mathcal{C}_\varepsilon V)^n_{i,j,k}$ denote the discrete value of the differential operators $\mathcal{C}_0 V$, $\mathcal{C}V$, or $\mathcal{C}_1 V$ at a node (P_i, c_j, h_k, τ^n) so that $(\mathcal{C}_\varepsilon V)^n_{i,j,k}$ is an approximation for $(\mathcal{C}_0 V)^n_{i,j,k}$ if $P_i = 0$, an approximation for $(\mathcal{C}V)^n_{i,j,k}$ if $P_i \in (0, P_{\max} - \chi)$ and for $(\mathcal{C}_1 V)^n_{i,j,k}$ if $P_i \in [P_{\max} - \chi, P_{\max}]$. The operators (2.31), (2.22) and (2.34) can be discretized using central, forward, or backward differencing in the P direction to give

$$\begin{aligned} &(\mathcal{C}_\varepsilon V)^n_{i,j,k} \\ &= \begin{cases} \gamma_i^n V^n_{i-1,j,k} + \beta_i^n V^n_{i+1,j,k} \\ \qquad\qquad -\big(\gamma_i^n + \beta_i^n + r + \lambda_1 + \lambda_2(P_i)\big) V^n_{i,j,k} & \text{if } P_i \in (0, P_{\max} - \chi), \\ \gamma_i^n V^n_{i-1,j,k} + \beta_i^n V^n_{i+1,j,k} - (\gamma_i^n + \beta_i^n + r) V^n_{i,j,k} & \text{if } P_i \in [P_{\max} - \chi, P_{\max}), \\ \beta_i^n V^n_{i+1,j,k} - (\beta_i^n + r) V^n_{i,j,k} & \text{if } P_i = 0, \\ \gamma_i^n V^n_{i-1,j,k} - (\gamma_i^n + r) V^n_{i,j,k} & \text{if } P_i = P_{\max}, \end{cases} \end{aligned} \tag{3.2}$$

where γ_i^n and β_i^n are determined using an algorithm in [8]. The algorithm guarantees γ_i^n and β_i^n satisfy the following positive coefficient conditions:

$$\gamma_i^n \geq 0\ ,\ \ \beta_i^n \geq 0 \qquad\qquad i = 0, \ldots, i_{\max}\ ,\ \ n = 1, \ldots, N. \tag{3.3}$$

Let $(\mathcal{B}_\varepsilon V)^n_{i,j,k}$ be an approximation of the operator $\mathcal{B}V$ at a mesh node (P_i, c_j, h_k, τ^n). We compute two integrals in $(\mathcal{B}_\varepsilon V)^n_{i,j,k}$ using the FFT approach described in [11,13]: transforming discrete values $V^n_{i,j,k}$ to an equally spaced $\log P$ grid, carrying out an FFT on the data, computing the correlations resulting from approximation of the integrals using a Trapezoidal rule, and then transforming back to P coordinates.

Following [11,13], we use linear interpolation to transform discrete solution values from equally spaced $\log P$ grid to unequally spaced P grid (and vice versa), which introduces a second-order discretization error. In other words, if $\phi(P, c, h, \tau)$ is a smooth function on (P, c, h, τ) with $\phi^n_{i,j,k} = \phi(P_i, c_j, h_k, \tau^n)$, then we have

$$(\mathcal{B}_\varepsilon \phi)^n_{i,j,k} = (\mathcal{B}\phi)^n_{i,j,k} + O\big(\Delta P_{\max}^2\big). \tag{3.4}$$

See [11, 13] for more details of discretizing and computing the integral terms. Effectively, we can approximate $(\mathcal{B}V)^n_{i,j,k}$ by

$$\begin{aligned} &(\mathcal{B}_\varepsilon V)^n_{i,j,k} = \lambda_1 \sum_l b^1_{i,l} V^n_{l,k,j} + \lambda_2(P_i) \sum_l b^2_{i,l} V^n_{l,k,j} \\ &\qquad \text{with } 0 \leq b^1_{i,l} \leq 1\ ,\ \ \sum_l b^1_{i,l} \leq 1\ ,\ 0 \leq b^2_{i,l} \leq 1\ ,\ \ \sum_l b^2_{i,l} \leq 1. \end{aligned} \tag{3.5}$$

Note that $b^1_{i,l}$, $b^2_{i,l}$ satisfy

$$b^1_{i,l} = 0 \ , \ b^2_{i,l} = 0 \qquad \forall l, \ \forall P_i \in \{0\} \cup [P_{\max} - \chi, P_{\max}] \tag{3.6}$$

since boundary equations (2.33) and (2.38) imply that the integral terms vanish in the boundary regions $P = 0$ and $P \in [P_{\max} - \chi, P_{\max}]$.

3.1 Numerical scheme for pricing equation (2.21)

Following [8], we discretize the terms

$$\frac{DV}{D\tau} \equiv V_\tau - \frac{1}{a}(f - c)G(c,h)V_h - zV_c \tag{3.7}$$

in PIDE (2.21) using a semi-Lagrangian timestepping. Let $\zeta^{n+1}_{i,j,k}$ denote the value of the control variable z at the mesh node $(P_i, c_j, h_k, \tau^{n+1})$. Then we can approximate the value of $\frac{DV}{D\tau}$ at $(P_i, c_j, h_k, \tau^{n+1})$ by the following:

$$\left(\frac{DV}{D\tau}\right)^{n+1}_{i,j,k} = \frac{1}{\Delta\tau}\big(V^{n+1}_{i,j,k} - V^n_{i,\hat{j},\hat{k}}\big) + \text{truncation error}, \tag{3.8}$$

where $V^n_{i,\hat{j},\hat{k}}$ is an approximation of $V\big(P_i, c^n_{\hat{j}}, h^n_{\hat{k}}, \tau^n\big)$ obtained by linear interpolation with $c^n_{\hat{j}}$ and $h^n_{\hat{k}}$ given by

$$c^n_{\hat{j}} = c_j + \zeta^{n+1}_{i,j,k}\Delta\tau, \tag{3.9}$$

$$h^n_{\hat{k}} = \min\left[\max\left[h_k + \frac{f - c_j}{a}G(c_j, h_k)\Delta\tau, \ h_{\min}\right], h_{\max}\right]. \tag{3.10}$$

Following [8], the control $\zeta^{n+1}_{i,j,k}$ must satisfy the constraint $\zeta^{n+1}_{i,j,k} \in Z(c_j)$, where $Z(c_j)$ is the set of controls satisfying constraint (2.20) and conditions (2.15-2.16). Moreover, to prevent the value of $c^n_{\hat{j}}$ from going outside of the domain $[c_{\min}, c_{\max}]$, we need to impose further constraints on $\zeta^{n+1}_{i,j,k}$. Let $Z_j \subseteq Z(c_j)$ denote the set of values of $\zeta^{n+1}_{i,j,k} \in Z(c_j)$ such that the resulting $c^n_{\hat{j}}$ computed from (3.9) is bounded within $[c_{\min}, c_{\max}]$. We regard all elements in Z_j as admissible controls. Note that Z_j is a closed region and is determined only by the grid node c_j.

Equation (3.10) guarantees that the value of $h^n_{\hat{k}}$ will never go outside of the domain $[h_{\min}, h_{\max}]$. As shown in Lemma 4.4, when the mesh size/timestep parameter ε is sufficiently small, (3.10) reduces to $h^n_{\hat{k}} = h_k + \frac{f - c_j}{a}G(c_j, h_k)\Delta\tau$.

Given the notation above, at any discrete mesh node $(P_i, c_j, h_k, \tau^{n+1})$, $n \geq 0$, PIDE (2.21) and the associated boundary equations (2.30) and (2.33) can be discretized as

$$V^{n+1}_{i,j,k} = \sup_{\zeta^{n+1}_{i,j,k} \in Z_j} V^n_{i,\hat{j},\hat{k}} + \Delta\tau(\mathcal{C}_\varepsilon V)^{n+1}_{i,j,k} + \Delta\tau(\mathcal{B}_\varepsilon V)^n_{i,j,k} + \Delta\tau G(c_j, h_k)H(c_j, h_k)P_i. \tag{3.11}$$

We can rewrite the discrete equation (3.11) at a node $(P_i, c_j, h_k, \tau^{n+1})$, $n \geq 0$, as

$$
\begin{aligned}
&\mathcal{G}_{i,j,k}^{n+1}\big(\varepsilon, V_{i,j,k}^{n+1}, \{V_{l,j,k}^{n+1}\}_{l\neq i}, \{V_{i,j,k}^{n}\}\big) \\
&\equiv \inf_{\zeta_{i,j,k}^{n+1} \in Z_j} \left[\frac{V_{i,j,k}^{n+1} - V_{i,\hat{j},\hat{k}}^{n}}{\Delta\tau} \right] - (\mathcal{C}_\varepsilon V)_{i,j,k}^{n+1} - (\mathcal{B}_\varepsilon V)_{i,j,k}^{n} - G(c_j, h_k) H(c_j, h_k) P_i \\
&= 0,
\end{aligned}
\tag{3.12}
$$

where $\{V_{l,j,k}^{n+1}\}_{l\neq i}$ is the set of values $V_{l,j,k}^{n+1}$, $l \neq i$, $l = 0, \ldots, i_{\max}$, and $\{V_{i,j,k}^{n}\}$ is the set of values $V_{i,j,k}^{n}$, $i = 0, \ldots, i_{\max}$, $j = 0, \ldots, j_{\max}$, $k = 0, \ldots, k_{\max}$.

Remark 3.1. *Based on our assumption of the mesh size/timestep parameters (3.1), conditions (2.15-2.16) and (2.20) imply that*

$$
\min\{\zeta_{i,j,k}^{n+1}\} = \begin{cases} 0 & \text{if } c_j = c_{\min}, \\ O(x^\theta) & \text{if } c_j > c_{\min} \text{ and} \\ & \quad c_j - c_{\min} = O(x),\ x \ll 1, \\ z_{\min} & \text{otherwise.} \end{cases}
\tag{3.13}
$$

and

$$
\max\{\zeta_{i,j,k}^{n+1}\} = \begin{cases} 0 & \text{if } c_j = c_{\max}, \\ O(x^\theta) & \text{if } c_j < c_{\max} \text{ and} \\ & \quad c_{\max} - c_j = O(x),\ x \ll 1, \\ z_{\max} & \text{otherwise.} \end{cases}
\tag{3.14}
$$

Since θ is any positive constant, we can set θ such that $\theta|\log\varepsilon| \ll 1$ for all values of ε chosen for practical purposes, which implies that $\varepsilon^\theta \approx 1$. This means that the numerical implementation assuming that control z has the behavior in (3.13-3.14) is, for all practical purposes, the same as an implementation assuming that

$$
\begin{aligned}
Z(c_j) &= [0, z_{\max}] \ ; \ c_j = c_{\min} \\
&= [z_{\min}, z_{\max}] \ ; \ c_{\min} < c_j < c_{\max} \\
&= [z_{\min}, 0] \ ; \ c_j = c_{\max} \ .
\end{aligned}
$$

Note that the set of admissible values $Z_j \subseteq Z(c_j)$ in equation (3.12) ensures that $c_{\hat{j}}^n \in [c_{\min}, c_{\max}]$.

Following a discussion similar to the above, we can show that from a practical point of view, the implementation assuming that function $G(c,h)$ satisfies (2.10) is the same as an implementation assuming that $G(c_j,h_k) = 0$ if $c_j > f$, $h_k = h_{\min}$ or if $c_j < f$ and $h_k = h_{\max}$, and $G(c_j,h_k) = 1$ at all other mesh nodes.

Remark 3.2. *In (3.11), we evaluate the integral terms $\mathcal{B}V$ explicitly at timestep τ^n, instead of implicitly at timestep τ^{n+1}, so that no Policy-type iteration is required to solve the linear system resulting from the scheme (3.11) at each timestep. As shown in [12] and Section 4,* *the scheme is still unconditionally stable and monotone. Such an explicit evaluation results in a first order error in time, which is asymptotically identical to the error in space generated by linear interpolating $V^n_{i,\hat{j},\hat{k}}$. In the following scheme (3.18) for the pricing equation (2.28), we also explicitly evaluate the integral terms.*

Numerical tests show that there is no advantage in terms of convergence as $\varepsilon \to 0$ if we use an implicit discretization of the integral terms.

3.2 Numerical scheme for pricing equation (2.28)

In order to develop a discretization scheme for the impulse control case, we will proceed in a heuristic fashion. In a later section, we will show rigorously that the resulting discretization is consistent with the impulse control problem (2.28).

We can consider the impulse control problem to be the limiting case where the ramping rate $\zeta^{n+1}_{i,j,k}$ in (3.9) is unbounded, hence $c^n_{\hat{j}}$ in (3.9) can be any value between $c_{\min}$ and $c_{\max}$. Therefore, we rewrite (3.9) as

$$c^n_{\hat{j}} = c_j + \delta^{n+1}_{i,j,k}, \tag{3.15}$$

where the control variable $\delta^{n+1}_{i,j,k} \in [c_{\min} - c_j, c_{\max} - c_j]$ so that $c^n_{\hat{j}} \in [c_{\min}, c_{\max}]$. We denote by Δ_j the admissible control set

$$\Delta_j = [c_{\min} - c_j, c_{\max} - c_j]. \tag{3.16}$$

According to (2.28), if $\delta^{n+1}_{i,j,k} < 0$, then a fixed cost d_- is charged; if $\delta^{n+1}_{i,j,k} > 0$, then a fixed cost d_+ is charged. Therefore, we use $d(\delta)$ to represent the fixed cost as a function of δ given by

$$d(\delta) = \begin{cases} d_- & \text{if } \delta < 0, \\ 0 & \text{if } \delta = 0, \\ d_+ & \text{if } \delta > 0. \end{cases} \tag{3.17}$$

Using (3.15-3.17), we can generalize the scheme (3.11) to the following discretization for the impulse control case

$$\begin{aligned} V^{n+1}_{i,j,k} = &\sup_{\delta^{n+1}_{i,j,k} \in \Delta_j} \left[V^n_{i,\hat{j},\hat{k}} - d(\delta^{n+1}_{i,j,k}) \right] + \Delta\tau (\mathcal{C}_\varepsilon V)^{n+1}_{i,j,k} + \Delta\tau (\mathcal{B}_\varepsilon V)^n_{i,j,k} \\ &+ \Delta\tau G(c_j, h_k) H(c_j, h_k) P_i, \end{aligned} \tag{3.18}$$

where $V^n_{i,\hat{j},\hat{k}}$ is an approximation of $V\left(P_i, c^n_{\hat{j}}, h^n_{\hat{k}}, \tau^n\right)$ obtained by linear interpolation with $c^n_{\hat{j}}$ and $h^n_{\hat{k}}$ given by (3.15) and (3.10).

For the purpose of proving consistency of the scheme, following [7], we separate the control region Δ_j in (3.16) into three subregions: $\Delta_j = [c_{\min} - c_j, 0) \cup \{0\} \cup (0, c_{\max} - c_j]$, where we adopt the convention that $(\alpha, \beta] = \emptyset$ and $[\alpha, \beta) = \emptyset$ if $\alpha = \beta$. We will write equation (3.18) in terms of these three subregions. Let us define

$$\begin{aligned} \mathcal{H}^{n+1}_{i,j,k}\big(\varepsilon, V^{n+1}_{i,j,k}, \{V^{n+1}_{l,j,k}\}_{l \neq i}, \{V^n_{i,j,k}\}\big) = &\frac{V^{n+1}_{i,j,k} - V^n_{i,j,\hat{k}}}{\Delta\tau} - (\mathcal{C}_\varepsilon V)^{n+1}_{i,j,k} - (\mathcal{B}_\varepsilon V)^n_{i,j,k} \\ &- G(c_j, h_k) H(c_j, h_k) P_i\ , \end{aligned} \tag{3.19}$$

where $V^n_{i,j,\hat{k}}$ is an approximation of $V(P_i, c_j, h^n_{\hat{k}}, \tau^n)$. We denote (assuming $c_j > c_{\min}$)

$$\begin{aligned}
&\mathcal{I}^{n+1}_{i,j,k}\big(\varepsilon, V^{n+1}_{i,j,k}, \{V^{n+1}_{l,j,k}\}_{l\neq i}, \{V^n_{i,j,k}\}\big) \\
&= V^{n+1}_{i,j,k} - \sup_{\delta^{n+1}_{i,j,k}\in[c_{\min}-c_j,0)} \Big[V^n_{i,\hat{j},\hat{k}} - d_-\Big] - \Delta\tau(\mathcal{C}_\varepsilon V)^{n+1}_{i,j,k} \\
&\quad - \Delta\tau(\mathcal{B}_\varepsilon V)^n_{i,j,k} - \Delta\tau G(c_j, h_k) H(c_j, h_k) P_i
\end{aligned} \tag{3.20}$$

where $V^n_{i,\hat{j},\hat{k}}$ is an approximation of $V(P_i, c^n_{\hat{j}}, h^n_{\hat{k}}, \tau^n)$, and (assuming $c_j < c_{\max}$)

$$\begin{aligned}
&\mathcal{J}^{n+1}_{i,j,k}\big(\varepsilon, V^{n+1}_{i,j,k}, \{V^{n+1}_{l,j,k}\}_{l\neq i}, \{V^n_{i,j,k}\}\big) \\
&= V^{n+1}_{i,j,k} - \sup_{\delta^{n+1}_{i,j,k}\in(0,c_{\max}-c_j]} \Big[V^n_{i,\hat{j},\hat{k}} - d_+\Big] - \Delta\tau(\mathcal{C}_\varepsilon V)^{n+1}_{i,j,k} \\
&\quad - \Delta\tau(\mathcal{B}_\varepsilon V)^n_{i,j,k} - \Delta\tau G(c_j, h_k) H(c_j, h_k) P_i.
\end{aligned} \tag{3.21}$$

Note that within (3.19-3.21), the fixed cost term $d(\delta^{n+1}_{i,j,k})$ in (3.18) is replaced by the representation given in (3.17) based on the subregion where the control $\delta^{n+1}_{i,j,k}$ resides. Given the definitions of $\mathcal{H}, \mathcal{I}, \mathcal{J}$, we can write scheme (3.18) in an equivalent way at a node $(P_i, c_j, h_k, \tau^{n+1})$, $n \geq 0$, as

$$\mathcal{G}^{n+1}_{i,j,k}\big(\varepsilon, V^{n+1}_{i,j,k}, \{V^{n+1}_{l,j,k}\}_{l\neq i}, \{V^n_{i,j,k}\}\big) \equiv \begin{cases} \min\Big\{\mathcal{H}^{n+1}_{i,j,k}, \mathcal{I}^{n+1}_{i,j,k}, \mathcal{J}^{n+1}_{i,j,k}\Big\} & \text{if } c_{\min} < c_j < c_{\max}, \\ \min\Big\{\mathcal{H}^{n+1}_{i,j,k}, \mathcal{I}^{n+1}_{i,j,k}\Big\} & \text{if } c_j = c_{\max}, \\ \min\Big\{\mathcal{H}^{n+1}_{i,j,k}, \mathcal{J}^{n+1}_{i,j,k}\Big\} & \text{if } c_j = c_{\min}, \end{cases}$$
$$= 0. \tag{3.22}$$

Remark 3.3. *In the impulse control case, although the ramping rate is of bang-bang type, i.e., $\zeta^{n+1}_{i,j,k} \in \{-\infty, 0, \infty\}$, our numerical results indicate that the outflow rate $c^n_{\hat{j}}$ resulting from (3.15) is not of bang-bang type, i.e., $c^n_{\hat{j}}$ can be any value between $c_{\min}$ and $c_{\max}$. This is due to the nonlinear revenue structure resulting from the nonlinear function $H(c, h)$ in (2.3).*

3.3 Solving the local optimization problems

In scheme (3.11) we need to solve a discrete local optimization problem

$$\sup_{\zeta^{n+1}_{i,j,k}\in Z_j} V^n_{i,\hat{j},\hat{k}} \tag{3.23}$$

at a mesh node $(P_i, c_j, h_k, \tau^{n+1})$. We solve problem (3.23) using an approach similar to that given in [7], which we briefly describe as follows. If we fix a mesh node $(P_i, c_j, h_k, \tau^{n+1})$, then from (3.9-3.10), $h^n_{\hat{k}}$ is fixed and $c^n_{\hat{k}}$ varies according to different values of $\zeta^{n+1}_{i,j,k}$, where all the values of $c^n_{\hat{k}}$ form a closed region, denoted by

$$C_j = \left\{ c^n_{\hat{j}} \mid c^n_{\hat{j}} = c_j + \zeta^{n+1}_{i,j,k} \Delta\tau,\ \forall \zeta^{n+1}_{i,j,k} \in Z_j \right\}. \tag{3.24}$$

We first sample a sequence of values from the region C_j, denoted by $\hat{C}_j$, where $\hat{C}_j$ includes the lower and upper bounds of region C_j as well as all the discrete grid nodes in the c direction residing within the region. We then evaluate $V^n_{i,\hat{j},\hat{k}}$ for all elements in the sequence $\hat{C}_j$ and return as output the maximum among the set of computed values. In other words, we solve an alternative problem

$$\sup_{c^n_{\hat{j}} \in \hat{C}_j} V^n_{i,\hat{j},\hat{k}}. \tag{3.25}$$

Following [7], we can prove the following results, showing that the solutions to problems (3.23) and (3.25) are consistent.

Proposition 3.4. *Let $\phi(P, c, h, \tau)$ be a smooth function with $\phi^n_{i,j,k} = \phi(P_i, c_j, h_k, \tau^n)$. Then the optimization procedure introduced above results in*

$$\begin{aligned} \sup_{c^n_{\hat{j}} \in \hat{C}_j} \phi^n_{i,\hat{j},\hat{k}} &= \sup_{\zeta^{n+1}_{i,j,k} \in Z_j} \phi^n_{i,\hat{j},\hat{k}} + O(\varepsilon^2) \\ &= \sup_{\zeta^{n+1}_{i,j,k} \in Z_j} \phi\big(P_i, \hat{c}^n_j, \hat{h}^n_j, \tau^n\big) + O(\varepsilon^2). \end{aligned} \tag{3.26}$$

We can also use the approach described above to solve the local optimization problem

$$\sup_{\delta^{n+1}_{i,j,k} \in \Delta_j} \left[V^n_{i,\hat{j},\hat{k}} - d(\delta^{n+1}_{i,j,k}) \right] \tag{3.27}$$

for scheme (3.18).

Remark 3.5. *Note that the discretizations (3.11) and (3.18) are virtually identical. As discussed above, in both cases, we reduce the local optimization problems to a grid search using a set of values $\hat{C}_j$. Hence, we use essentially the same discretization for either the finite control case or the impulse control case. Only the set of admissible controls and the inclusion/exclusion of the fixed cost term is different in each case.*

Remark 3.6. *Since $\zeta^{n+1}_{i,j,k}$ in discretization (3.11) is bounded, then according to (3.9), at each mesh node (P_i, c_j, h_k, τ^n) we need to perform only a finite number of linear interpolations to solve problem (3.25). More precisely, in view of Remark 2.1, we need only examine the upper and lower bounds of Z_j and $c^n_{\hat{j}} = c_j$.*

Moreover, since Z_j is independent of the timestep τ^n, we can precompute the interpolation weights to avoid binary searches for linear interpolations at each timestep.

As for problem (3.18), however, we need to perform $O(1/\varepsilon)$ interpolations at each mesh node since we have to examine $O(1/\varepsilon)$ grid nodes in the c direction according to (3.15). Nevertheless, we can still save the cost for binary searches at each timestep by precomputing the interpolation weights.

4 Properties of the Numerical Schemes

Provided a strong comparison result for the pricing PDE/PIDE applies, [2, 5] demonstrate that a numerical scheme will converge to the viscosity solution of the equation if it is l_∞ stable, monotone, and consistent. In this section, we will prove the convergence of our numerical schemes (3.11) and (3.18) (or equivalently, schemes (3.12) and (3.22)) to the viscosity solution of the pricing equations (2.21) and (2.28) respectively by verifying these three properties.

4.1 l_∞ Stability

Definition 4.1 (l_∞ stability)**.** *Discretizations (3.11) and (3.18) are l_∞ stable if*

$$\|V^{n+1}\|_\infty \leq C_5 \tag{4.1}$$

for $0 \leq n \leq N-1$ as $\Delta\tau \to 0$, $\Delta P_{\min} \to 0$, $\Delta c_{\min} \to 0$, $\Delta h_{\min} \to 0$, where C_5 is a constant independent of $\Delta\tau$, $\Delta P_{\min}$, $\Delta c_{\min}$, $\Delta h_{\min}$. Here $\|V^{n+1}\|_\infty = \max_{i,j,k} |V^{n+1}_{i,j,k}|$.

Lemma 4.2 (l_∞ stability)**.** *If discretizations (3.11) and (3.18) satisfy the positive coefficient condition (3.3) and conditions (3.5-3.6) and if linear interpolation is used to compute $V^n_{i,\hat{j},\hat{k}}$, then schemes (3.11) and (3.18) satisfy*

$$\|V^{n+1}\|_\infty \leq \|V^0\|_\infty + T \cdot P_{\max} \cdot \|H\|, \tag{4.2}$$

where

$$\|H\| = \max_{j,k} |H(c_j, h_k)|. \tag{4.3}$$

Therefore, discretizations (3.11) and (3.18) are l_∞ stable according to Definition 4.1.

Proof. The proof directly follows from applying the maximum principle to the discrete equations (3.11) and (3.18). We omit the details here. Readers can refer to [12, Theorem 5.5] and [16] for complete stability proofs of the semi-Lagrangian fully implicit scheme for American Asian options and that of finite difference schemes for controlled HJB equations, respectively. □

4.2 Consistency

Let us define a vector $\mathbf{x} = (P, c, h, \tau)$ and let $DV(\mathbf{x})$ and $D^2V(\mathbf{x})$ be the first and second order derivatives of $V(\mathbf{x})$, respectively. Let $IV(\mathbf{x})$ be the integral terms in the pricing equation. Let $\bar{\Omega} = [0, P_{\max}] \times [c_{\min}, c_{\max}] \times [h_{\min}, h_{\max}] \times [0, T]$ be the closed domain in which our problem is defined. Then following the steps in [7], we can rewrite the pricing problem (2.21), (2.29), (2.30), (2.33) or the problem (2.28), (2.29), (2.35-2.38) into one equation as follows:

$$F\big(D^2V(\mathbf{x}), DV(\mathbf{x}), IV(\mathbf{x}), V(\mathbf{x}), \mathbf{x}\big) = 0 \quad \text{for all } \mathbf{x} = (P, c, h, \tau) \in \bar{\Omega}\ . \tag{4.4}$$

The authors of [2, 5] define the consistency of a numerical scheme to a possible discontinuous viscosity solution of a nonlinear PDE, which in our case is given as follows.

Definition 4.3 (Consistency)**.** *The schemes $\mathcal{G}_{i,j,k}^{n+1}$ given in (3.12) and (3.22) are consistent with the pricing problem (2.21), (2.29), (2.30), (2.33) and the pricing problem (2.28), (2.29), (2.35-2.38), respectively, if for all $\hat{\mathbf{x}} = (\hat{P}, \hat{c}, \hat{h}, \hat{\tau}) \in \bar{\Omega}$ and any function $\phi(P, c, h, \tau)$ having bounded derivatives of all orders in $(P, c, h, \tau) \in \bar{\Omega}$ with $\phi_{i,j,k}^{n+1} = \phi(P_i, c_j, h_k, \tau^{n+1})$ and $\mathbf{x} = (P_i, c_j, h_k, \tau^{n+1})$, we have*

$$\limsup_{\substack{\varepsilon\to 0\\ \mathbf{x}\to\hat{\mathbf{x}}\\ \xi\to 0}} \mathcal{G}_{i,j,k}^{n+1}\big(\varepsilon, \phi_{i,j,k}^{n+1}{+}\xi, \{\phi_{l,j,k}^{n+1}{+}\xi\}_{l\neq i}, \{\phi_{i,j,k}^{n}{+}\xi\}\big) \le F^*\big(D^2\phi(\hat{\mathbf{x}}), D\phi(\hat{\mathbf{x}}), I\phi(\hat{\mathbf{x}}), \phi(\hat{\mathbf{x}}), \hat{\mathbf{x}}\big), \tag{4.5}$$

and

$$\liminf_{\substack{\varepsilon\to 0\\ \mathbf{x}\to\hat{\mathbf{x}}\\ \xi\to 0}} \mathcal{G}_{i,j,k}^{n+1}\big(\varepsilon, \phi_{i,j,k}^{n+1}{+}\xi, \{\phi_{l,j,k}^{n+1}{+}\xi\}_{l\neq i}, \{\phi_{i,j,k}^{n}{+}\xi\}\big) \ge F_*\big(D^2\phi(\hat{\mathbf{x}}), D\phi(\hat{\mathbf{x}}), I\phi(\hat{\mathbf{x}}), \phi(\hat{\mathbf{x}}), \hat{\mathbf{x}}\big), \tag{4.6}$$

where F^* and F_* in (4.5-4.6) are upper semi-continuous (usc) and lower semi-continuous (lsc) envelopes of the function F defined in (4.4), respectively. See, for example, [7] for the definition of usc and lsc envelopes.

We first show the following properties of the schemes (3.11) and (3.18).

Lemma 4.4. *If the function $G(c, h)$ satisfies condition (2.10) and $\Delta\tau$ satisfies the assumption (3.1), then by taking ε sufficiently small, (3.10) becomes*

$$h_{\hat{k}}^{n} = h_k + \frac{f - c_j}{a} G(c_j, h_k)\Delta\tau. \tag{4.7}$$

If the control $\zeta_{i,j,k}^{n+1}$ satisfies condition (2.15-2.16) and $\Delta\tau$ satisfies the assumption (3.1), then by taking ε sufficiently small, we have

$$Z_j = [z_{\min}, z_{\max}]. \tag{4.8}$$

Proof. The proof follows from the steps in [8, Lemma B.1] by showing that (4.7) follows if $\Delta\tau < Const.$ and $\Delta\tau = o(\varepsilon^{1-\nu})$, and that (4.8) holds if $\Delta\tau < Const.$ and $\Delta\tau = o(\varepsilon^{1-\theta})$, where $Const.$ represents a positive constant. Since $\nu > 0$, $\theta > 0$, these conditions are weaker than the assumption $\Delta\tau = C_4\varepsilon$ in (3.1) and will be satisfied if ε is sufficiently small. □

Lemma 4.5 (Consistency)**.** *Suppose that the mesh size and timestep size satisfy assumption (3.1). Then the discretizations (3.12) and (3.22) are consistent as defined in Definition 4.3. In particular, if the value function is sufficiently smooth, and assuming that linear interpolation is used to compute $V_{i,\hat{j},\hat{k}}^{n}$, then the global discretization errors of both schemes are $O(\varepsilon)$.*

Proof. The proof follows the lines in [7, 8] and the results in Lemma 4.4. We omit the details here. □

4.3 Monotonicity

The following result shows that schemes (3.12) and (3.22) are monotone according to the definition in [2,5].

Lemma 4.6 (Monotonicity). *If discretizations (3.12) and (3.22) satisfy the positive coefficient condition (3.3) and conditions (3.5-3.6) and if linear interpolation is used to compute $V^n_{i,\hat{j},\hat{k}}$, then discretizations (3.12) and (3.22) are monotone according to the definition in [2,5], i.e.,*

$$\begin{aligned} &\mathcal{G}^{n+1}_{i,j,k}\big(\varepsilon, V^{n+1}_{i,j,k}, \{X^{n+1}_{l,j,k}\}_{l\neq i}, \{X^n_{i,j,k}\}\big) \\ &\leq \mathcal{G}^{n+1}_{i,j,k}\big(\varepsilon, V^{n+1}_{i,j,k}, \{Y^{n+1}_{l,j,k}\}_{l\neq i}, \{Y^n_{i,j,k}\}\big); \ \textit{for all } X^n_{i,j,k} \geq Y^n_{i,j,k}, \forall i,j,k,n. \end{aligned} \tag{4.9}$$

Proof. Inequality (4.9) can be easily verified for all mesh nodes $(P_i, c_j, h_k, \tau^{n+1})$ for schemes (3.12) and (3.22) (e.g. see [16]). □

4.4 Convergence

In order to prove the convergence of our schemes using the results in [2,5,6], we need to assume the following strong comparison result, as defined in [2,5], for equations (2.21) and (2.28).

Assumption 4.7. *For either the localized pricing problem (2.21), (2.29), (2.30), (2.33) or the localized pricing problem (2.28), (2.29), (2.35-2.38), if u and v are an upper semi-continuous (usc) subsolution and a lower semi-continuous (lsc) supersolution of the problem, respectively, then*

$$u \leq v \quad \textit{on } \Omega_{in} = (0, P_{\max}) \times (c_{\min}, c_{\max}) \times (h_{\min}, h_{\max}) \times (0, T]. \tag{4.10}$$

The strong comparison result is proved for other similar (but not identical) bounded stochastic control problems in [3,4] and impulse control problems in [1,20,22,26]. From Lemmas 4.2, 4.5 and 4.6 and Assumption 4.7, using the results in [2,5,6], we can obtain the following convergence result:

Theorem 4.8 (Convergence to the viscosity solution). *Assuming that discretizations (3.12) and (3.22) satisfy all conditions required for Lemmas 4.2, 4.5 and 4.6, and that Assumption 4.7 is satisfied, then schemes (3.12) and (3.22) converge to the viscosity solutions of the pricing problem (2.21), (2.29), (2.30), (2.33) and the pricing problem (2.28), (2.29), (2.35-2.38), respectively.*

5 Numerical Results

In this section, we conduct numerical experiments for the hydroelectric power plant valuation problem. We use "dollars per megawatt-hour" ($/MWhr), "meter" ($m$), "hour" ($hr$),

Parameter	Value	Parameter	Value
α	0.4	$\bar{\lambda}_2$	0.85 hr^{-1}
K_0	27 \$/MWhr	μ^1	0.3
β	15	ψ_0^1	0
t_0	7.7π	ψ_1^1	3.2
σ	0.2 $hr^{-1/2}$	μ^2	0.4
λ_1	0.01 hr^{-1}	ψ_0^2	-3.6
P_0	100 \$/MWhr	ψ_1^2	0

TABLE 5.1: *Values of electricity spot price parameters in (2.17-2.19). The values of α, K_0, β, t_0, σ, P_0, $\bar{\lambda}_2$ are chosen from [24].*

"cubic meters per second" (m^3/s), and "cubic meters per second per hour" ($m^3/s - hr$) as the default units for electricity spot price, water head, time, inflow/outflow rate, and ramping rate, respectively. We will value the cash flows over a one week cycle ($T = 1$ week), similar to the approach in [24]. We will use the terminal condition (2.29), as in [24].

Prior to illustrating results, we list the parameter values in our experiments. Following [17], we assume the logarithm of the random jump sizes J_1 and J_2 in (2.17) satisfies a truncated version of an exponential distribution with parameters μ, ψ_0 and ψ_1, $\psi_0 < \psi_1$. The corresponding probability density function is given by

$$p(x; \mu, \psi_0, \psi_1) = \begin{cases} \frac{\mu \exp(-\mu x)}{\exp(-\mu \psi_0) - \exp(-\mu \psi_1)} & \text{if } x \in [\psi_0, \psi_1], \\ 0 & \text{otherwise.} \end{cases} \tag{5.1}$$

Function (5.1) reveals that the value of x is bounded between ψ_0 and ψ_1. We use $p(\log J_1; \mu^1, \psi_0^1, \psi_1^1)$ and $p(\log J_2; \mu^2, \psi_0^2, \psi_1^2)$ to represent the probability density function for $\log J_1$ and $\log J_2$, respectively, where $\psi_0^1 = \psi_1^2 = 0$, $\psi_1^1 > 0$ and $\psi_0^2 < 0$. These parameter values are given in Table 5.1, where the values of $\mu^1, \psi_0^1, \psi_1^1$ are chosen similar to those calibrated from the time series of the electricity spot price in [17] and the values of $\mu^2, \psi_0^2, \psi_1^2$ are calculated such that $E[J_1] \cdot E[J_2] = 1$. This approximately reflects the idea that after an up-jump immediately followed by a down-jump (this approximates a typical price spike), on average the spot price will return to its starting value before the spike occurs. The values of ψ_0^1, ψ_1^1, ψ_0^2, ψ_0^2 indicate that $J_1 \geq 1$ and $0 < J_2 \leq 1$.

Table 5.1 also provides values of other electricity spot price parameters in equations (2.17-2.19).

The other parameters for our experiments are given in Table 5.2. We set $c_{\min} = 40\ m^3/s$ if the minimum flow rate constraint is imposed; otherwise we set $c_{\min} = 0$. According to (2.2), the values of g, ρ, $c_{\max}$, $h_{\max}$ imply that the theoretical maximum electricity power is $H_m(c_{\max}, h_{\max}) = 138.18 \times 10^6$ Watt $= 138.18$ MW. The values of $c_{\min}$, $c_{\max}$, $z_{\max}$ in the table imply that it takes $\frac{c_{\max} - c_{\min}}{z_{\max}} \approx 18$ hours to move the outflow rate from the minimum to the maximum when the minimum flow rate and ramping constraints are imposed. The values of a, f, $c_{\max}$, $h_{\min}$ and $h_{\max}$ indicate that reducing the head from

Parameter	Value	Parameter	Value
g	9.8 m/s^2	r	0.05 annually
ρ	1000 kg/m^3	T	1 week
k_1	85%	$h_{\min}$	90 m
k_2	120 MW	$h_{\max}$	94 m
a	1.8×10^6 m^2	$c_{\min}$	40 m^3/s
f	60 m^3/s	$c_{\max}$	150 m^3/s
d_-	10^{-8}	$z_{\min}$	-6 $m^3/s - hr$
d_+	10^{-8}	$z_{\max}$	6 $m^3/s - hr$

TABLE 5.2: *Other input parameters used to price the value of the hydroelectric power plant. Parameters g, ρ, k_1, k_2, a, f are shown in (2.2-2.5). r is the annual riskless interest rate and T is the operational time interval. $c_{\min}$ is the minimum outflow rate in case the minimum flow rate constraint is imposed; otherwise $c_{\min} = 0$. Parameters d_- and d_+ are the fixed costs in equation (2.28).*

its highest value to the lowest value will take at least $\frac{a(h_{\max}-h_{\min})}{3600(c_{\max}-f)} \approx 22$ hours. We choose small fixed costs $d_- = d_+ = 10^{-8}$ for the impulse control problem (2.28), which implies that the operator can switch between $z = -\infty$ and $z = +\infty$ almost freely due to the negligible costs associated with the operation.

After presenting the parameters, we first carry out a convergence analysis for the hydroelectric power plant valuation with/without operational constraints. Table 5.3 shows the convergence results with respect to different mesh size/timestep parameters when $P = K_0 = 27$ \$/MWhr (the average of daily electricity spot price), $c = 100$ m^3/s, $h = 92$ m and $t = 0$. The convergence ratio in the table is defined as the ratio of successive changes in the solution, as the timestep and mesh size are reduced by a factor of two. A ratio of two indicates first order convergence. As shown in Table 5.3, our schemes are able to achieve a first-order convergence for both the pricing equation (2.21) with operating constraints incorporated and the impulse control equation (2.28) without incorporating operating constraints, as the convergence ratios are approximately two. Table 5.3 also implies that imposing the operational constraints reduces power plant value by more than 37%.

Recall that the computational domain has been localized in the P direction to $[0, P_{\max}]$. Initially, we set $P_{\max} \approx 7 \times 10^5$ \$/MWhr. We repeated the computations with $P_{\max} = 7 \times 10^6$ \$/MWhr. All the numerical results at $t = 0$, $P = K_0 = 27$ \$/MWhr, $c = 100$ m^3/s, $h = 92$ m are identical (to the number of digits shown) to those in the Table 5.3 for all refinement levels. As a result, all the subsequent results will be reported using $P_{\max} \approx 7 \times 10^5$ \$/MWhr, since the corresponding solution error incurred by the domain localization is negligible.

Figure 5.1 plots the optimal operational strategies for the hydroelectric power plant as a function of outflow rate c and electricity spot price P at the current time $t = 0$ when $h = 92$ m, where the minimum flow rate and ramping rate constraints are imposed. The

P nodes	c nodes	h nodes	Timesteps	Value	Ratio
With minimum flow rate and ramping constraints					
66	12	5	337	187207	n.a.
131	23	9	673	194484	n.a.
261	45	17	1345	199182	1.55
521	89	33	2689	201848	1.76
1041	177	65	5377	203393	1.73
Without minimum flow rate or ramping constraint					
66	16	5	337	288084	n.a.
131	31	9	673	306584	n.a.
261	61	17	1345	317034	1.77
521	121	33	2689	322391	1.95

TABLE 5.3: *Convergence study for the value of the hydroelectric power plant at* $t = 0$, $P = K_0 = 27$ *\$/MWhr,* $c = 100\, m^3/s$, $h = 92\, m$ *with/without incorporating operational constraints. Input parameter data are given in Tables 5.1-5.2. Due to CPU time consideration, we do not report the value with respect to the finest mesh/timestep size for the case without incorporating the operational constraints.*

figure shows that the control strategy is of the bang-bang type: the optimal ramping rate is either $z_{\min}$, 0 or $z_{\max}$. According to Remark 2.1, $z = z_{\max}$ implies $V_c > 0$ in pricing equation (2.21), while $z = z_{\min}$ implies $V_c < 0$ in (2.21). The region where $z = 0$ corresponds to the following three cases:

- if $c \neq c_{\min}$ and $c \neq c_{\max}$ in this region, then $z = 0$ indicates $V_c = 0$ in the region. As shown in Remark 2.1, when $V_c = 0$, the optimal z can be any value residing in $Z(c)$. Our numerical algorithm will choose $z = 0$ in this case, since there is no advantage in changing the flow rate. This choice will be consistent with the impulse control formulation with an infinitesimal fixed cost.

- if $c = c_{\max}$ in the region, then $z = 0$ represents $V_c \geq 0$. This is because if $V_c > 0$, then Remark 2.1 implies that it is optimal to choose $z = \max\{Z(c)\}$, and $\max\{Z(c)\} = 0$ in this region due to condition (2.14).

- if $c = c_{\min}$ in the region, then $z = 0$ indicates $V_c \leq 0$ since condition (2.13) implies that $\min\{Z(c)\} = 0$ in this region.

From the figure, we can observe that for most outflow rates, it is optimal to ramp up at the maximum rate when electricity price is high so that more electricity will be generated and sold to the spot market; similarly, it is optimal to ramp down at the maximum rate when electricity price is low in order to store water in the reservoir for producing electricity in the future. Meanwhile, the ramp-up region expands when the outflow rate decreases, since a lower outflow rate gives more incentive for the ramp-up operation. Similarly, the ramp-down region expands when the outflow rate increases.

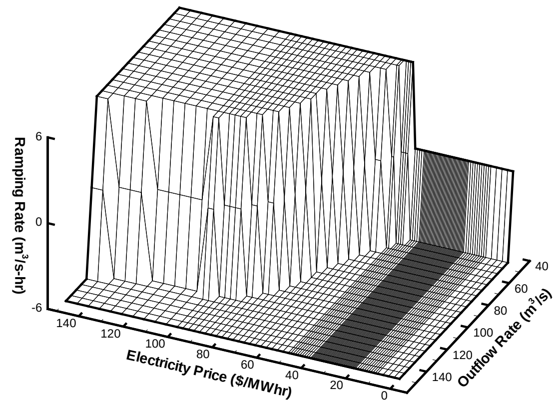

FIGURE 5.1: *Optimal operational strategy as a function of outflow rate c and electricity spot price P when $t = 0$, $h = 92\ m$. The minimum flow rate and ramping constraints are imposed. Input parameter data are given in Tables 5.1-5.2.*

Figure 5.2 plots the optimal control strategy that evolves over time as a function of electricity price when $c = 100\ m^3/s$ and $h = 92\ m$. In order to see the patterns more clearly, we use $T = 4$ days for this plot. The figure clearly reveals the daily trend of the electricity price: the control pattern is repeated every day. The plot also shows that the optimal control z forms three regions in which $z = z_{\max}$, $z = 0$ and $z = z_{\min}$, respectively. When $t \to T$, it is optimal to keep ramping up (i.e., to increase the outflow rate c) in order to produce as much profit as possible. This is, of course, an artifact of the terminal condition (2.29). Another possibility would be to impose a penalty unless the hydro plant was returned to its original state at the beginning of the weekly cycle. In fact, such penalties are common in leasing gas storage facilities [8]. However, to keep things simple we will impose condition (2.29).

Finally, we study the implication of the operational restrictions on the value of the hydroelectric power plant. For both cases with and without the minimum flow rate constraint, Table 5.4 lists values of the hydroelectric power plant with respect to different ramping rate constraints. The table shows that imposing the ramping constraint at $|z_{\min}| = |z_{\max}| = 6\ m^3/s - hr$ reduces the value by 20-32%, while imposing the minimum flow rate constraint reduces the value by 9-22%. Therefore, we conclude that it is important to take the operational constraints into consideration in order to accurately price the power plant cash flows.

Table 5.4 also indicates that as the maximum ramp-up and ramp-down rates $|z_{\max}|$, $|z_{\min}|$ increase, the value of the power plant converges to the value resulting from eliminating the ramping rate constraint (i.e., setting $|z_{\max}| = |z_{\min}| = \infty$). The convergence ap-

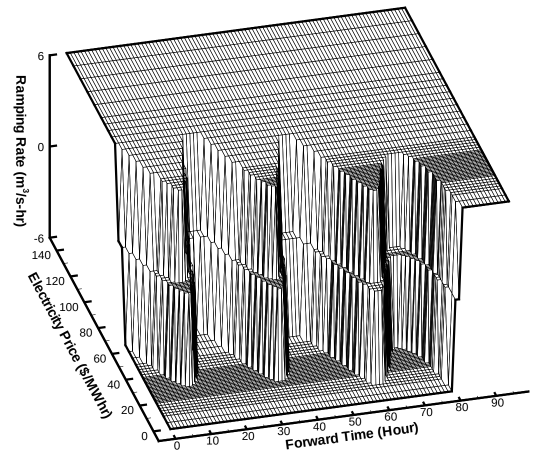

FIGURE 5.2: *Optimal operational strategy as a function of forward time t and electricity spot price P when $c = 100\ m^3/s$ and $h = 92\ m$. The minimum flow rate and ramping constraints are imposed. In order to be able to see some of the daily patterns, we use $T = 4$ days in this case. Other input parameters are given in Tables 5.1-5.2.*

pears to be rapid. In fact, under the minimum flow rate constraint, as $|z_{\max}| = |z_{\min}| = 96$ $m^3/s - hr$ (i.e., switching between the minimum outflow rate $c_{\min}$ and maximum outflow rate $c_{\max}$ can be finished in about 1.5 hours), the value of the power plant is very close to that obtained without any ramping constraint.

Conclusion

In this paper, we determine the risk neutral value of the cash flows to a hydroelectric power plant over a one week cycle under a stochastic control framework. We take into consideration operational constraints such as ramping and minimum flow rate constraints required for environmental protection.

We formulate the power plant valuation problem under a ramping constraint as a bounded stochastic control problem, resulting in an HJB PIDE. We also formulate the valuation problem without the ramping restriction as an impulse control problem, resulting in an HJB variational inequality.

We develop a consistent numerical scheme for solving both the HJB PIDE for the bounded control problem and the HJB variational inequality for the impulse control problem. Our discretization scheme is essentially the same for either the bounded control case or the impulse control case. This makes software implementation very straightforward, and ensures that we obtain the correct limits as the ramping rate becomes unbounded. We prove

Maximum Ramp-up/-down Rates $\lvert z_{\max}\rvert$, $\lvert z_{\min}\rvert$	Value	
	With MFR	Without MFR
$6\ m^3/s-hr$	2.0×10^5	2.2×10^5
$12\ m^3/s-hr$	2.2×10^5	2.5×10^5
$24\ m^3/s-hr$	2.3×10^5	2.8×10^5
$48\ m^3/s-hr$	2.4×10^5	3.0×10^5
$96\ m^3/s-hr$	2.5×10^5	3.1×10^5
$\infty\ m^3/s-hr$	2.5×10^5	3.2×10^5

TABLE 5.4: *Comparison on values of the hydroelectric power plant with respect to different values of ramp-up and ramp-down rates. The ∞ corresponds to the case without imposing the ramping rate constraint. The column "With MFR" represents the power plant values under the minimum flow rate constraint (i.e., setting $c_{\min}=40\ m^3/s$); the column "Without MFR" represents the values without the minimum flow rate constraint (i.e., setting $c_{\min}=0$). The values in the table are accurate up to the first two digits. Other input parameters are given in Tables 5.1-5.2.*

the convergence of the numerical scheme to the viscosity solution of each pricing problem, provided a strong comparison result holds. Numerical results indicate that our scheme can achieve first order convergence.

We also study the implication of the operational restrictions on the value of a hydroelectric power plant. We observe through an example that both the ramping and the minimum flow rate constraints can considerably affect the value of the power plant, where imposing either constraint can reduce the value by 9-32%, while imposing both constraints will reduce the value by more than 37%. Therefore, we conclude that it is important to take the operational constraints into consideration in order to accurately price the power plant cash flows.

Our numerical experiments also show that as the maximum ramp-up and ramp-down rates increase, the value of the hydroelectric power plant rapidly converges to the value obtained without imposing any constraints.

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