

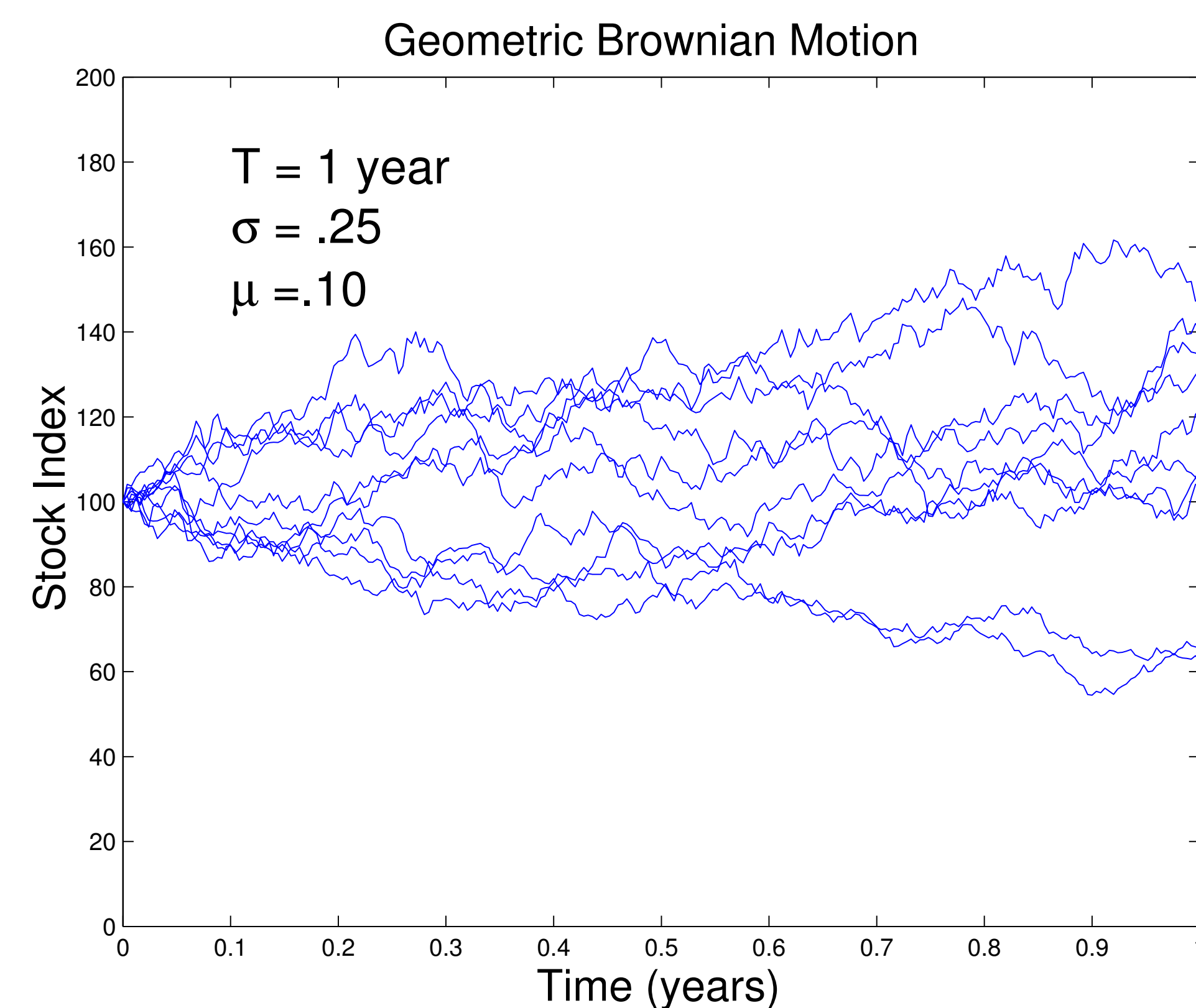
How to Get Rich When You Retire

Jian Wang and Peter Forsyth

David R. Cheriton School of Computer Science
University of Waterloo

What is Rich?

- Many studies have shown that people are happy at retirement if their wealth (W) is large compared to their annual salary (Y) the year before they retire.
- Assume that there are two possible investment choices at any time. One is a risk-free bond, and the other is a stock index.
- We assume that the investor contributes a fixed fraction q of her salary each year into her pension account.
- We assume the stock index follows a stochastic process (Geometric Brownian Motion). We also assume that the investor's salary is stochastic but correlated with the stock index.



- Our objective is to maximize the wealth-to-income ratio ($X = \frac{W}{Y}$) at retirement ($t = T$). Our control is the fraction (p) of total wealth W , which is invested in the stock index.

Mean-variance Approach

- Let $E(\cdot)$ and $Var(\cdot)$ be the expectation and variance operators respectively. Original Problem:

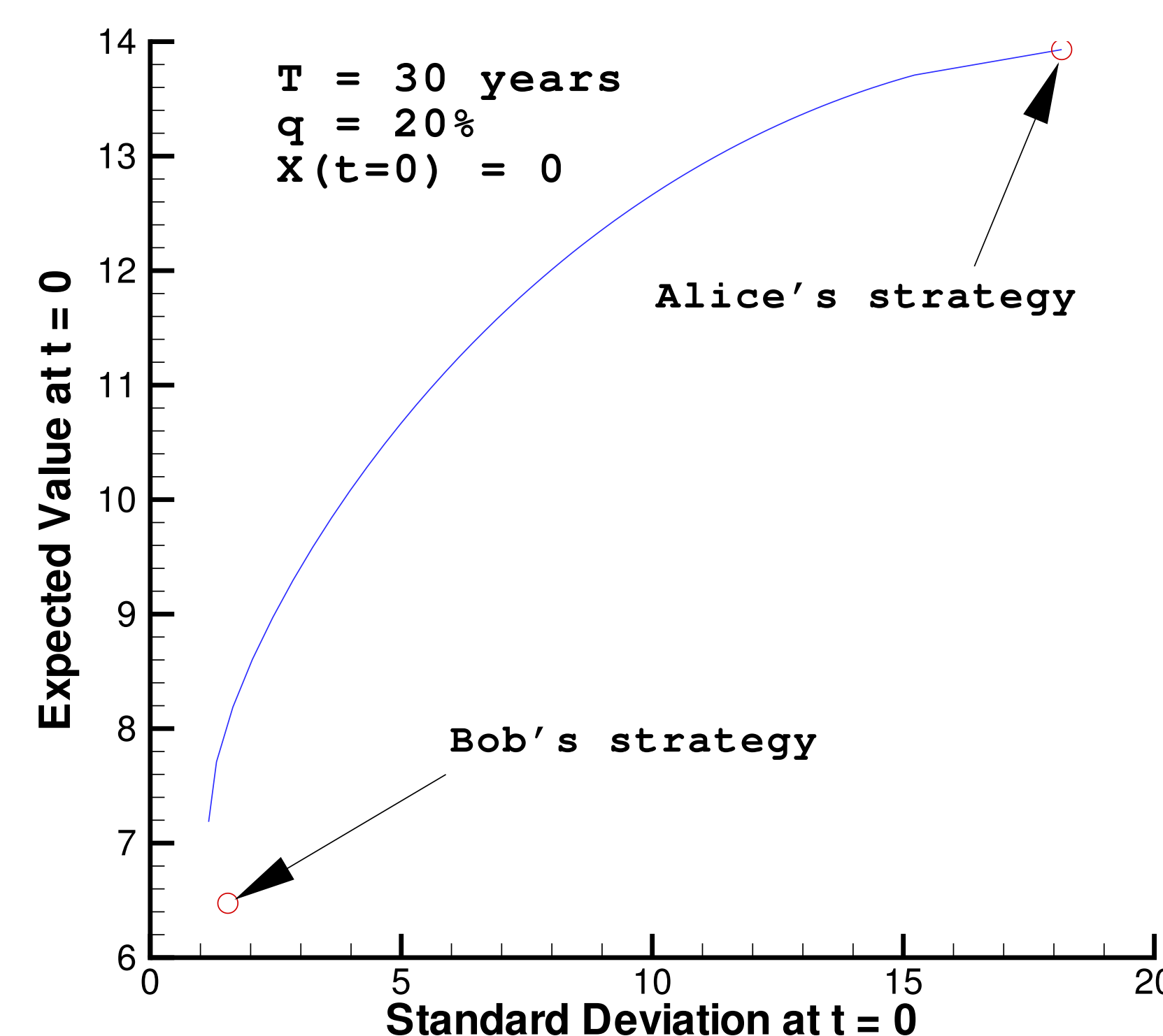
$$\begin{aligned} & \max_{p(t)} E^{t=0}(X_T) \\ & \text{subject to: } Var^{t=0}(X_T) = c \\ & X(t=0) = x \end{aligned}$$

- We can embed the original problem into a linear-quadratic (LQ) optimal stochastic control problem.

- We solve the LQ optimal stochastic control problem by solving numerically a non-linear Hamilton-Jacobi-Bellman (HJB) Partial Differential equation (PDE).

Efficient Frontier

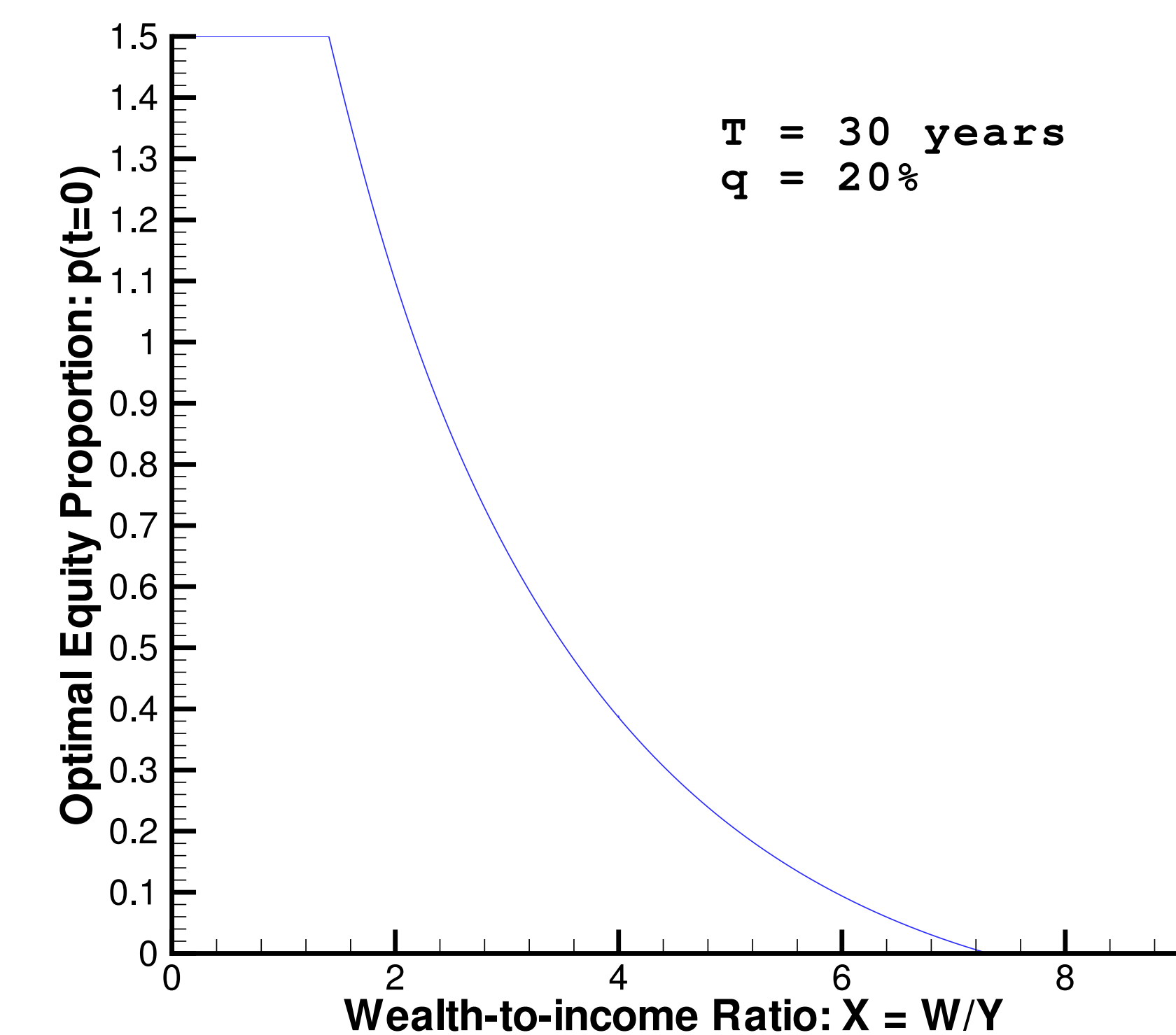
- The solution of the optimal control problem can be shown as an efficient frontier. For a given level of risk (standard deviation), the optimal strategy generates the maximum expected wealth-to-income ratio at retirement.



- Extreme case one: Investor Bob's strategy. Bob is very conservative. He does not believe in the stock market, so he invests entirely in bonds. This strategy is not efficient.
- Extreme case two: Investor Alice is an aggressive investor. She invests entirely in stocks and expects a high return. Her strategy is efficient but very risky.

Investment Strategy

- Suppose that the investor cannot short the stock index and that she can borrow at most 50% of her total wealth to invest in the stock index. The investment strategy at $t = 0$ is:



- As wealth-to-income ratio increases, the proportion invested in the stock index decreases. When wealth-to-income ratio is large enough, the plan holder would not invest in the stock index at all. This means that once the plan holder has enough wealth for retirement, she would prefer to keep the wealth and would not take any more risk.

Monte-Carlo Simulations

- Suppose $T = 30$, $q = 20\%$ and $X(t=0) = 0$. Suppose the investor specifies a standard deviation of 1.7659. We solve this optimal stochastic control problem and store the optimal strategy. The optimal strategy generates an expected wealth-to-income ratio of 8.3647 at $t = 0$.
- We then carry out Monte-Carlo simulations based on the optimal strategy we stored. 10,000 simulations are used. The distribution of outcomes is shown below.

