Multi-period Mean CVAR Asset Allocation: Is it Advantageous to be Time Consistent?

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Abstract

We formulate the multi-period, time consistent mean-CVAR (Conditional Value at Risk) asset allocation problem in a form amenable to numerical computation. Our numerical algorithm can impose realistic constraints such as: no shorting, no-leverage, and discrete rebalancing. We focus on long term (i.e. 30 year) strategies, which would be typical of an investor in a Defined Contribution (DC) pension plan. A comparison with pre-commitment mean-CVAR strategies shows that adding the time consistent constraint compares unfavourably with the pure pre-commitment strategy. Since the pre-commitment strategy computed at time zero is identical to a time consistent strategy based on an alternative objective function, the pre-commitment mean-CVAR strategy is implementable in this case. Hence it would seem that there is little to be gained from enforcing time consistency.

Keywords: multi-period mean-CVAR, time consistent, pre-commitment, asset allocation
JEL codes: G11, G22
AMS codes: 91G, 65N06, 65N12, 35Q93

1 Introduction

Long term investors, such as those saving for retirement in a Defined Contribution (DC) pension are motivated by asset allocation strategies which are optimal under multi-period criteria. Initial work on multi-period strategies under mean-variance criteria were carried out in (Li and Ng, 2000; Zhou and Li, 2000). More recent issues are addressed in (Basak and Chabakauri, 2010; Bjork and Murgoci, 2010; Wang and Forsyth, 2011; Bjork and Murgoci, 2014; Bjork et al., 2014). It is important to distinguish between two categories of multi-period optimal control strategies.

Pre-commitment strategies (Li and Ng, 2000; Zhou and Li, 2000) are globally optimal when viewed from the initial time, but these strategies are not *time consistent*. Consider an asset allocation problem with a fixed stopping time T. We compute the optimal strategy, as a function of the state variables, at time zero. Now, suppose we re-compute the strategy at some later time t, $0 < t < T$. For a given state of the system, the strategy we compute at this later time may not agree with the strategy computed at time zero. This has led many investigators to label pre-commitment mean-variance strategies as *non-implementable*, since the investor has an incentive to deviate from the pre-commitment strategy at $t > 0$. However, in the mean-variance case, this objection is perhaps not well thought out. For any pre-commitment mean-variance optimal strategy, there exists

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a parameter W^*, such that this strategy is also optimal for the strategy which minimizes quadratic shortfall (Cui et al., 2012; Dang and Forsyth, 2016)

$$E\left[(\min(W_T - W^*,0))^2\right] \tag{1.1}$$

where W_T is the accumulated wealth at $t = T$, and $E[\cdot]$ is the expectation operator. For a fixed W^*, we can determine the optimal policy for objective function (1.1) using dynamic programming. Hence, if we fix W^*, the pre-commitment policy is the time consistent strategy for objective function (1.1). In fact, as noted in (Vigna, 2017)

> *"The equivalence between [pre-commitment] mean-variance criterion and the target-based approach is one of the characteristics that make the mean-variance preferences appealing with respect to other types of preferences...For the average pension fund member it is easier to select a wealth target rather than an abstract index."*

which makes the objective function (1.1) with a fixed target W^* very appealing in practice.

In an effort to force time consistency, while retaining the mean-variance objective function when viewed at times $t > 0$, several authors have developed techniques to ensure this property (Basak and Chabakauri, 2010; Bjork and Murgoci, 2010; Wang and Forsyth, 2011; Bjork and Murgoci, 2014; Bjork et al., 2014; Staden et al., 2018; Landriault et al., 2018). Since we can view time consistent mean-variance strategies as pre-commitment strategies with an additional constraint, it is immediately obvious that time-consistent strategies are not globally optimal as seen at time zero. In fact, as noted by Cong and Oosterlee (2016a,b), the pre-commitment mean-variance strategy is consistent with a fixed target, but not with a risk aversion attitude. Conversely, time consistent mean-variance strategies are consistent with a fixed risk aversion, but not with a fixed target. Further discussion concerning the equivalence of pre-commitment mean-variance and the time consistent policy which minimizes the target based objective function (1.1) can be found in (Vigna, 2014; Menoncin and Vigna, 2017). The merits and demerits of time consistent and pre-commitment policies are also discussed in Vigna (2017). Some other interesting problems with time consistency are noted in Bensoussan et al. (2019), in the case of a wealth dependent risk aversion parameter. Suffice to say, we cannot dismiss pre-commitment policies out of hand.

An interesting alternative for using variance to measure risk is Conditional Value at Risk (CVAR). CVAR at level α is simply the average of the worst α fraction of outcomes, hence is a measure of tail risk. Pre-commitment mean-CVAR strategies were developed in (Miller and Yang, 2017; Gao et al., 2017). As we shall see, although pre-commitment mean-CVAR strategies are formally time inconsistent, these strategies are identical at time zero to a linear target shortfall policy, with a fixed target. This alternative objective function generates a time consistent policy. Hence these strategies are in fact implementable. Under this alternative objective function, the investor has no incentive to deviate from the strategy computed at time zero.

The objective of this article is to formulate the time consistent mean-CVAR problem into a form which is amenable to computation. We do this by expanding the state space to include the local Value at Risk (VAR) as an independent variable. We develop a numerical technique for solving this problem. Since we use fully numerical methods, we can consider realistic constraints, such as no-leverage, no shorting and discrete rebalancing. We then compare the time consistent and pre-commitment mean-CVAR solutions for a long term investment problem. The stochastic process parameters are determined by fitting to 90 years of market data.

We should mention that our approach differs from the method in (Cui et al., 2019), where the authors reduce the time consistent mean-CVAR problem to solution of a convex program based on a finite Monte Carlo sampling of return paths. In particular, Cui et al. (2019) assumes a lump sum

investment, where the control (in terms of amount in each asset) is shown to be a piecewise linear function of wealth. The lump sum assumption does not hold in general for DC plan investors, who typically make periodic contributions to an investment portfolio.

Our main conclusion is that imposing a time consistent constraint on the solution of a mean-CVAR problem appears to result in an investment strategy with undesirable properties. On the other hand, based on the cumulative distribution function of the final wealth, the time consistent linear target shortfall strategy (which coincides with pre-commitment mean-CVAR at time zero) is superior in terms of tail risk reduction compared to the time consistent mean-CVAR policy.

2 Formulation

For simplicity we assume that there are only two assets available in the financial market, namely a risky asset and a risk-free asset. In practice, the risky asset would be a broad market index fund.

The investment horizon is T. S_t and B_t respectively denote the *amounts* invested in the risky and risk-free assets at time t, $t \in [0, T]$. In general, these amounts will depend on the investor's strategy over time, as well as changes in the unit prices of the assets. In the absence of an investor determined control (i.e. cash injections or rebalancing), all changes in S_t and B_t result from changes in asset prices. In this case (absence of control), we assume that S_t follows a jump diffusion process. Let $t^- = t - \epsilon, \epsilon \to 0^+$, i.e. t^- is the instant of time before t, and let ξ be a random number representing a jump multiplier. When a jump occurs, $S_t = \xi S_{t^-}$. Allowing discontinuous jumps lets allows us to explore the effects of severe market crashes on the risky asset holding. We assume that $\log(\xi)$ follows a double exponential distribution (Kou, 2002; Kou and Wang, 2004). If a jump occurs, p_{up} is the probability of an upward jump, while $1 - p_{up}$ is the chance of a downward jump. The density function for $y = \log(\xi)$ is

$$f(y) = p_{up}\eta_1 e^{-\eta_1 y}\mathbf{1}_{y\geq 0} + (1 - p_{up})\eta_2 e^{\eta_2 y}\mathbf{1}_{y<0}. \tag{2.1}$$

For future reference, note that

$$\begin{aligned} E[y = \log\xi] &= \frac{p_{up}}{\eta_1} - \frac{(1-p_{up})}{\eta_2} \ ; \ E[\xi] = \frac{p_{up}\eta_1}{\eta_1 - 1} + \frac{(1-p_{up})\eta_2}{\eta_2 + 1} \\ E[(\xi-1)^2] &= \frac{p_{up}\eta_1}{\eta_1 - 2} + \frac{(1-p_{up})\eta_2}{\eta_2 + 2} - 2\left(\frac{p_{up}\eta_1}{\eta_1 - 1} + \frac{(1-p_{up})\eta_2}{\eta_2 + 1}\right) + 1. \end{aligned} \tag{2.2}$$

In the absence of control, S_t evolves according to

$$\begin{aligned} \frac{dS_t}{S_{t^-}} &= (\mu - \lambda_\xi \kappa_\xi)\, dt + \sigma\, d\mathcal{Z} + d\left(\sum_{i=1}^{\pi_t}(\xi_i - 1)\right) \\ & \qquad \kappa_\xi = E[\xi - 1]\ , \end{aligned} \tag{2.3}$$

where μ is the (uncompensated) drift rate, σ is the volatility, $d\mathcal{Z}$ is the increment of a Wiener process, π_t is a Poisson process with positive intensity parameter λ_ξ, and ξ_i are i.i.d. positive random variables having distribution (2.1). Moreover, ξ_i, π_t, and Z are assumed to all be mutually independent.

We focus on jump diffusion models for long term equity dynamics since sudden drops in the equity index just before retirement can have a devastating impact on retirement portfolios. Since we consider discrete rebalancing, the jump process models the cumulative effects of large market

drops between rebalancing times. Previous studies show that stochastic volatility effects are small for the long term investor (Ma and Forsyth, 2016). This can be traced to the fact that stochastic volatility models are mean-reverting, with typical mean reversion times of less than one year.

In the absence of control, we assume that the dynamics of the amount B_t invested in the risk-free asset are

$$dB_t = rB_t\,dt, \tag{2.4}$$

where r is the (constant) risk-free rate.

Remark 2.1 (Parsimonious Model)**.** *Equations (2.3) and (2.4) are very simplified models of real stock and bond processes. However, tests of the controls determined using these parsimonious model processes on bootstrapped historical market data gives good results for a variety of objective functions (Forsyth et al., 2019). This suggests that the parsimonious model (2.3)-(2.4) seems sufficient for the purposes of generating an asset allocation strategy for the long term investor.*

We define the investor's total wealth at time t as

$$\text{Total wealth } \equiv W_t = S_t + B_t. \tag{2.5}$$

We impose the constraints that shorting stock and using leverage (i.e. borrowing) are not permitted, which would be typical of a DC plan retirement savings account.

Properties 2.1 (Constant coefficients)**.** *In the following, we will assume that the stochastic process parameters $r, \mu, \sigma, \lambda_\xi, p_{up}, \eta_1, \eta_2$ are constants, independent of (S,B,t). This also implies that κ_ξ is a constant as well.*

3 Notational Conventions

To avoid subscript clutter, in the following, we will occasionally use the notation $S_t \equiv S(t), B_t \equiv B(t)$ and $W_t \equiv W(t)$. Let the inception time of the investment be $t_0 = 0$. We consider a set $\mathcal{T}$ of pre-determined *rebalancing times*,

$$\mathcal{T} \equiv \{t_0 = 0 < t_1 < \cdots < t_M = T\}. \tag{3.1}$$

For simplicity, we specify $\mathcal{T}$ to be equidistant with $t_i - t_{i-1} = \Delta t = T/M$, $i = 1, \ldots, M$. At each rebalancing time t_i, $i = 0, 1, \ldots, M-1$, the investor (i) injects an amount of cash q_i into the portfolio, and then (ii) rebalances the portfolio. At $t_M = T$, the portfolio is liquidated. In the following, given a time dependent function $f(t)$, then we will use the shorthand notation

$$f(t_i^+) \equiv \lim_{\epsilon \to 0^+} f(t_i + \epsilon) \quad ; \quad f(t_i^-) \equiv \lim_{\epsilon \to 0^+} f(t_i - \epsilon) \ . \tag{3.2}$$

We assume that there are no taxes or other transaction costs, so that the condition

$$W(t_i^+) = W(t_i^-) + q_i \tag{3.3}$$

holds. Typically, DC plan savings are held in a tax advantaged account. With infrequent (e.g. yearly) rebalancing, we also expect transaction costs to be small.

We denote by $X(t) = (S(t), B(t))$, $t \in [0,T]$, the multi-dimensional controlled underlying process, and by $x = (s, b)$ the state of the system. Let the rebalancing control $p_i(\cdot)$ be the fraction invested in the risky asset at the rebalancing date t_i, i.e.

$$p_i\left(X(t_i^-)\right) = p\left(X(t_i^-), t_i\right) \quad = \quad \frac{S(t_i^+)}{S(t_i^+) + B(t_i^+)} \ . \tag{3.4}$$

Note that the controls depend on the state of the investment portfolio, before the rebalancing occurs, i.e. $p_i(\cdot) = p\left(X(t_i^-), t_i\right) = p\left(X_i^-, t_i\right)$, $t_i \in \mathcal{T}$, where $\mathcal{T}$ is the set of rebalancing times. More specifically, in our case, we find the optimal strategies amongst all strategies with constant wealth (after injection of cash), so that

$$\begin{aligned} p_i(\cdot) &= p(W(t_i^+), t_i) \\ & \quad W(t_i^+) = S(t_i^-) + B(t_i^-) + q_i \\ & \quad S(t_i^+) = S_i^+ = p_i(W_i^+)\ W_i^+ \ \ ; \ \ B(t_i^+) = B_i^+ = (1 - p_i(W_i^+))\ W_i^+ \ \ . \end{aligned} \tag{3.5}$$

Let $\mathcal{Z}$ represent the set of admissible values of the control $p_i(\cdot)$. An admissible control $\mathcal{P} \in \mathcal{A}$, where $\mathcal{A}$ is the admissible control set, can be written as

$$\mathcal{P} = \{p_i(\cdot) \in \mathcal{Z} \ : \ i = 0, \ldots, M-1\} \tag{3.6}$$

We also define $\mathcal{P}_n \equiv \mathcal{P}_{t_n} \subset \mathcal{P}$ as the tail of the set of controls in $[t_n, t_{n+1}, \ldots, t_{M-1}]$, i.e.

$$\mathcal{P}_n = \{p_n(\cdot), \ldots, p_{M-1}(\cdot)\} \ . \tag{3.7}$$

4 Definition of CVAR

Let $g(W_T)$ be the probability density function of wealth W_T at $t = T$. Let

$$\int_{-\infty}^{W_\alpha^*} g(W_T)\ dW_T = \alpha, \tag{4.1}$$

i.e. $Pr[W_T > W_\alpha^*] = 1 - \alpha$. We can interpret W_α^* as the Value at Risk (VAR) at level α. The Conditional Value at Risk (CVAR) at level α is then

$$\text{CVAR}_\alpha = \frac{\int_{-\infty}^{W_\alpha^*} W_T\ g(W_T)\ dW_T}{\alpha}, \tag{4.2}$$

which is the mean of the worst α fraction of outcomes. Typically $\alpha = .01, .05$. Note that the definition of CVAR in equation (4.2) uses the probability density of the final wealth distribution, not the density of *loss*. Hence, in our case, a larger value of CVAR (i.e. a larger value of average worst case terminal wealth) is desired.

Given an expectation under control $\mathcal{P}$, $E_{\mathcal{P}}[\cdot]$, as noted by Rockafellar and Uryasev (2000), CVAR_α can be alternatively written as

$$\begin{aligned} \text{CVAR}_\alpha &= \sup_{W^*} E_{\mathcal{P}}\left[W^* + \frac{1}{\alpha}(W_T - W^*)^-\right] \\ & \qquad\qquad (W_T - W^*)^- \equiv \min(W_T - W^*, 0) \ . \end{aligned} \tag{4.3}$$

The admissible set for W^* in equation (4.3) is over the set of possible values for W_T. Using this equivalent definition of CVAR_α, as noted by Miller and Yang (2017), the mean-CVAR problem can be expressed as (for a given scalarization parameter $\kappa > 0$)

$$\sup_{\mathcal{P}}\left\{\sup_{W^*} E_{\mathcal{P}}\left[W^* + \frac{1}{\alpha}(W_T - W^*)^- + \kappa W_T\right]\right\} \ . \tag{4.4}$$

In some cases, it is useful to interchange the sup sup in equation (4.4), as suggested in (Gao et al., 2017; Miller and Yang, 2017). This allows us to rewrite the objective function (4.4) as

$$\sup_{W^*}\left\{\sup_{\mathcal{P}} E_{\mathcal{P}}\left[W^* + \frac{1}{\alpha}(W_T - W^*)^- + \kappa W_T\right]\right\}, \tag{4.5}$$

and solve the inner optimization problem using an HJB equation (Dang and Forsyth, 2014; Forsyth and Labahn, 2019). Standard numerical methods can then be used to solve the outer optimization problem.

5 Time Consistent Mean-CVAR

With these notational conventions, for a given scalarization parameter $\kappa > 0$ and an intervention time t_n, we define the scalarized time consistent mean-CVAR problem ($TCMC_{t_n}(\kappa)$) in terms of the value function $J(s,b,t)$: as follows:

$$\begin{aligned}
(TCMC_{t_n}(\kappa)): \qquad J\left(s,b,t_n^-\right) &= \max_{\mathcal{P}_n \in \mathcal{A}} \sup_{W^*}\left\{E_{\mathcal{P}_n}^{X_n^+,t_n^+}\left[W^* + \frac{1}{\alpha}(W_T - W^*)^- + \kappa W_T \right.\right. \\
&\qquad \left.\left. \Big| X(t_n^-) = (s,b)\right]\right\} && (5.1)\\
\text{s.t. } \mathcal{P}_n &= \left\{p_n(\cdot),\mathcal{P}_{n+1}^*\right\} : \left\{p_n(\cdot),p_{n+1}^*(\cdot),\ldots,p_{M-1}^*(\cdot)\right\} && (5.2)\\
&\quad \text{where } \mathcal{P}_{n+1}^* \text{ is optimal for problem } \left(TCMC_{t_{n+1}}(\kappa)\right)\\
\text{subject to} \quad & \begin{cases} (S_t, B_t) \text{ follow processes (2.3)-(2.4)}; \quad t \notin \mathcal{T} \\ W_n^+ = s + b + q_n\,;\ X_n^+ = (S_n^+, B_n^+) \\ S_n^+ = p_n(\cdot)W_n^+\,;\ B_n^+ = (1 - p_n(\cdot))W_n^+ \\ p_n(\cdot) \in \mathcal{Z} = [0,1] \end{cases} && (5.3)
\end{aligned}$$

Remark 5.1 (Replacement of sup by max)**.** *Note that we have replaced* $\sup_{\mathcal{P}}$ *by* $\max_{\mathcal{P}}$ *in equation (5.1). Since the admissible value set* $\mathcal{Z}$ *is compact, this amounts to assuming continuity of the value function with respect to the controls. We can avoid this assumption by taking the* max *over the upper semi-continuous envelope of the value function, but this would add unpleasant heavy notation.*

Remark 5.2 (Time Consistent Constraint)**.** *Time consistency is enforced via the constraint (5.2) This approach, which explicitly enforces the time consistent constraint, is similar in spirit to the methods used in (Wang and Forsyth, 2011; Staden et al., 2018; Landriault et al., 2018) for the mean-variance case.*

Remark 5.3 (Constraints)**.** *Note that we enforce no-leverage, no shorting by requiring that* $p_n \in \mathcal{Z} = [0,1]$ *in equation (5.3). Due to the leverage constraint imposed in equation (5.3), this optimization problem is well-posed without adding an additional constraint on the terminal wealth(Gao et al., 2017).*

Remark 5.4 (Admissible set for W^*.)**.** *For all the examples in this work, we have that the initial wealth is non-negative,* $q_i \geq 0$ *and the constraints (5.3) are imposed. This implies that* $W_T, W^* \in [0,\infty)$.

5.1 Expanded State Space Formulation

In order to develop an numerical algorithm for problem ($TCMC_{t_n}(\kappa)$), we follow the usual strategy of embedding the original problem in a higher dimensional space. We lift the state space to $\hat{X} = (s,b,W^*)$, and define an auxiliary function $V(s,b,W^*,t)$, which is given by

$$\begin{aligned} V\left(s,b,W^*,t_n^-\right) &= \left\{ E_{\mathcal{P}_n}^{\hat{X}_n^+,t_n^+}\left[W^* + \frac{1}{\alpha}(W_T - W^*)^- + \kappa W_T \right.\right. \\ &\qquad \left.\left. \Big| \hat{X}(t_n^-) = (s,b,W^*) \right] \right\} \end{aligned} \tag{5.4}$$

$$\text{subject to} \quad \begin{cases} (S_t, B_t) \text{ follow processes (2.3)-(2.4)}; \quad t \notin \mathcal{T} \\ W_n^+ = s + b + q_n\,;\ X_n^+ = (S_n^+, B_n^+) \\ S_n^+ = p_n(\cdot)W_n^+\,;\ B_n^+ = (1 - p_n(\cdot))W_n^+ \end{cases} \tag{5.5}$$

The optimal control $p_n(w)$ at time t_n is then determined from

$$p_n(w) \quad = \arg\max_{p' \in \mathcal{Z}} \left\{ \sup_{W^*} V(w\ p', w\ (1-p'), W^*, t_i^+) \right\}\ . \tag{5.6}$$

The solution is advanced (backwards) across time t_n by

$$\begin{aligned} V(s,b,W^*,t_n^-) &= V\big(\ w^+\ p_n(w^+), w^+\ (1 - p_n(w^+)\ ), W^*, t_n^+\ \big) \\ & \qquad w^+ = s + b + q_n\ . \end{aligned} \tag{5.7}$$

At $t = T$, we have

$$V(s,b,W^*,T^+) \quad = \quad W^* + \frac{1}{\alpha}\big((s+b) - W^*\big)^- + \kappa(s+b)\ . \tag{5.8}$$

For $t \in (t_{n-1}, t_n)$, there are no external cash flows, discounting, or controls applied, hence the tower property gives for $h < (t_n - t_{n-1})$

$$V(s,b,W^*,t) \quad = \quad E\left[V(S(t+h), B(t+h), W^*, t+h) \Big| S(t) = s, B(t) = b \right] \quad ; \quad t \in (t_{n-1}, t_n - h)\ .] \tag{5.9}$$

Applying Ito's Lemma assuming processes (2.3)-(2.4), and letting $h \to 0$ gives the PIDE for $V(s,b,W^*,t)$ for $t \in (t_{n-1}, t_n)$

$$V_t + \frac{\sigma^2 s^2}{2} V_{ss} + (\mu - \lambda_\xi \kappa_\xi) V_s - \lambda_\xi V + rbV_b + \int_{-\infty}^{+\infty} V(e^y s, b, q, t) f(y)\ dy\ = 0\ . \tag{5.10}$$

Remark 5.5 (Form of PIDE)**.** *Between rebalancing dates, W^* can be regarded as a constant parameter, i.e. as in equation (5.9). Interdependence of the solution on W^* comes about indirectly due to the control equation (5.6). This means that the PIDE (5.10) coupled with control equations (5.6-5.7) falls outside of the scope of viscosity solution theory (Crandall et al., 1992), hence we cannot use the results in Barles and Souganidis (1991) to prove convergence of a numerical scheme.*

Proposition 5.1 (Equivalence of formulation (5.4-5.10))**.** *Define*

$$J\left(s,b,t_n^-\right) = \sup_{W^*} V(s,b,W^*,t_n^-)\ , \tag{5.11}$$

then formulation (5.5-5.10) is equivalent to Problem ($TCMC_{t_n}(\kappa)$).

Proof. Replace $V(s,b,W^*,t_n^-)$ in equation (5.11) by the expressions in equations (5.4-5.10), and apply these expressions recursively backwards in time, using condition (5.8) at $t = T$. We then obtain equations (5.1- 5.3). □

5.2 Computation of $E[W_T]$

Given an initial wealth W_0, and the optimal controls $\mathcal{P}^*$, then the above method can be used to determine

$$J_0 = J\left(0,W_0,t_0{}^-\right) \;=\; \sup_{W^*}\left\{ E^{X_0^+,t_0^+}_{\mathcal{P}^*}\left[W^* + \frac{1}{\alpha}(W_T - W^*)^- + \kappa W_T\right] \Bigg| X(t_0^-) = (0,W_0)\right\} . \tag{5.12}$$

We can determine CVAR_α from

$$\text{CVAR}_\alpha = J_0 - \kappa E^{X_0^+,t_0^+}_{\mathcal{P}^*}[W_T] \;, \tag{5.13}$$

which means we need to compute $E^{X_0^+,t_0^+}_{\mathcal{P}^*}[W_T]$ separately. This is easily done. Define the function

$$U(s,b,T^+) \;=\; (s+b) \tag{5.14}$$

where at $t_n \in \mathcal{T}$

$$\begin{aligned} U(s,b,t_n^-) \;&=\; U\big(\; w^+\; p_n(w^+), w^+\; (1-p_n(w^+)\;), t_n^+\;\big) \\ & \quad w^+ = s+b+q_n \;. \end{aligned} \tag{5.15}$$

with $p_n(w)$ being the optimal control from equation (5.6). For $t \in (t_{n-1},t_n)$, U satisfies the PIDE

$$U_t + \frac{\sigma^2 s^2}{2}U_{ss} + (\mu - \lambda_\xi \kappa_\xi)U_s - \lambda_\xi U + rbU_b + \int_{-\infty}^{+\infty} U(e^y s,b,q,t)f(y)\;dy \;= 0 \;, \tag{5.16}$$

so that

$$E^{X_0^+,t_0^+}_{\mathcal{P}^*}[W_T] \;=\; U(0,W_0,t_0^-) \;. \tag{5.17}$$

6 Pre-commitment Mean-CVAR

For a given scalarization parameter κ and intervention times t_n, the pre-commitment mean-CVAR problem $(PCMC_{t_n}(\lambda))$ is given in terms of the value function $J(s,b,t_n^-)$

$$(PCMC_{t_n}(\kappa)): \qquad J\left(s,b,t_n^-\right) \;=\; \max_{\mathcal{P}_n\in\mathcal{A}}\sup_{W^*}\left\{ E^{X_n^+,t_n^+}_{\mathcal{P}_n}\left[W^* + \frac{1}{\alpha}(W_T - W^*)^- + \kappa W_T \Bigg| X(t_n^-) = (s,b)\right]\right\} \tag{6.1}$$

$$\text{subject to} \qquad \begin{cases} (S_t,B_t) \text{ follow processes (2.3)-(2.4)}; \quad t \notin \mathcal{T} \\ W_n^+ = s+b+q_n\;;\; X_n^+ = (S_n^+,B_n^+) \\ S_n^+ = p_n(\cdot)W_n^+\;;\; B_n^+ = (1-p_n(\cdot))W_n^+ \\ p_\ell(\cdot) \in \mathcal{Z} = [0,1]\;;\; \ell = n,\ldots,M-1 \end{cases} \tag{6.2}$$

Remark 6.1 (Relation to $(TCMC(\kappa))$)**.** *Note that compared to equations (5.1-5.3) we have dropped the time consistent constraint in equations (6.1-6.2).*

Remark 6.2 (Interchange $\max_{\mathcal{P}_n \in \mathcal{A}} \sup_{W^*} \ldots$)**.** *. Observe that we can interchange the* $\max\sup$ *in equation (6.1), as in equation (4.5), to obtain*

$$J\left(s,b,t_n^-\right) = \max_{\mathcal{P}_n \in \mathcal{A}} \sup_{W^*} \left\{ E_{\mathcal{P}_n}^{X_n^+,t_n^+} \left[W^* + \frac{1}{\alpha}(W_T - W^*)^- + \kappa W_T \middle| X(t_n^-) = (s,b) \right] \right\} \tag{6.3}$$

$$= \sup_{W^*} \max_{\mathcal{P}_n \in \mathcal{A}} \left\{ E_{\mathcal{P}_n}^{X_n^+,t_n^+} \left[W^* + \frac{1}{\alpha}(W_T - W^*)^- + \kappa W_T \middle| X(t_n^-) = (s,b) \right] \right\} . \tag{6.4}$$

6.1 Expanded State Space Formulation

In order to develop a numerical algorithm for problem ($PCMC_{t_n}(\kappa)$), we again embed the original problem in a higher dimensional space. This method is essentially the method in (Miller and Yang, 2017). We will also use the form of the value function as defined in equation (6.4). We lift the state space to $\hat{X} = (s,b,W^*)$, and define an auxiliary function $V(s,b,W^*,t)$, which is given by

$$V\left(s,b,W^*,t_n^-\right) = \left\{ E_{\mathcal{P}_n}^{\hat{X}_n^+,t_n^+} \left[W^* + \frac{1}{\alpha}(W_T - W^*)^- + \kappa W_T \middle| \hat{X}(t_n^-) = (s,b,W^*) \right] \right\} \tag{6.5}$$

$$\text{subject to} \quad \begin{cases} (S_t, B_t) \text{ follow processes (2.3)-(2.4)}; \quad t \notin \mathcal{T} \\ W_n^+ = s + b + q_n \,;\; X_n^+ = (S_n^+, B_n^+) \\ S_n^+ = p_n(\cdot) W_n^+ \,;\; B_n^+ = (1 - p_n(\cdot)) W_n^+ \end{cases} \tag{6.6}$$

The optimal control $p_n(w, W^*)$ at time t_n is then determined from

$$p_n(w, W^*) = \arg\max_{p' \in \mathcal{Z}} \left\{ V(w\ p', w\ (1 - p'), W^*, t_i^+) \right\} . \tag{6.7}$$

The solution is advanced (backwards) across time t_n by

$$V(s,b,W^*,t_n^-) = V\big(\ w^+\ p_n(w^+,W^*), w^+\ (1 - p_n(w^+,W^*)\ ), W^*, t_n^+\ \big)$$
$$w^+ = s + b + q_n\ . \tag{6.8}$$

Remark 6.3 (Dependence on W^*)**.** *Since we do not impose a time consistent constraint in equations (6.1-6.2), this has the effect of decoupling the solution as a function of* W^**. This can be seen by examining equation (6.7) in contrast to equation (5.7).*

At $t = T$, we have

$$V(s,b,W^*,T^+) = W^* + \frac{1}{\alpha}\big((s+b) - W^*\big)^- + \kappa(s+b)\ . \tag{6.9}$$

The usual argument gives the PIDE for $V(s,b,W^*,t)$ for $t \in (t_{n-1}, t_n)$

$$V_t + \frac{\sigma^2 s^2}{2} V_{ss} + (\mu - \lambda_\xi \kappa_\xi) V_s - \lambda_\xi V + rbV_b + \int_{-\infty}^{+\infty} V(e^y s, b, q, t) f(y)\ dy\ = 0\ . \tag{6.10}$$

Proposition 6.1 (Equivalence of formulation (6.5-6.10))**.** *Define*

$$J\left(s,b,t_n^-\right) = \sup_{W'} V(s,b,W',t_n^-)\ , \tag{6.11}$$

then formulation (6.5-6.10) is equivalent to Problem ($PCMC_{t_n}(\kappa)$)*.*

Proof. Replace $V(s,b,W',t_n^-)$ in equation (6.11) by the expressions in equations (6.5-6.10), use equation (6.9), and recursively work backwards in time, then we obtain equations (6.1 - 6.2), by interchanging the sup max in the final step. □

Remark 6.4 (Time inconsistency)**.** *Define*

$$W^*(t_n) = \arg\max_{W'} \Big(V(s,b,W',t_n^-) \Big)^{u^*} \quad , \tag{6.12}$$

where $(\cdot)^{u^*}$ refers to the upper semi-continuous envelope of the argument, as a function of W' with fixed (s,b). Then equation (6.11) shows that $W^*(t_n)$ depends on t_n. This is the source of the time inconsistency. The pre-commitment strategy is determined from equation (6.11) at t_0.

Remark 6.5 (Viscosity solution)**.** *Since we can regard W^* as fixed in equations (6.5-6.10) this equation does fall under the scope of viscosity solution theory.*

Proposition 6.2 (Pre-commitment strategy equivalence to time consistent policy for an alternative objective function)**.** *The pre-commitment mean-CVAR strategy $\mathcal{P}^*$ determined by solving $J(0,W^0,t_0^-)$ (with $W^*(t_0)$ from equation (6.12)) is the time consistent strategy for the equivalent problem $TCEQ$ (with fixed $W^*(t_0)$), with value function $J(s,b,t)$ defined by*

$$(TCEQ_{t_n}(\kappa\alpha)): \qquad J\left(s,b,t_n^-\right) = \sup_{\mathcal{P}_n \in \mathcal{A}} \left\{ E_{\mathcal{P}_n}^{X_n^+,t_n^+} \left[(W_T - W^*(t_0))^- + (\kappa\alpha) W_T \middle| X(t_n^-) = (s,b) \right] \right\} , \tag{6.13}$$

Proof. This follows since we can regard $W^*(t_0)$ as a constant in objective function (6.5), and $\alpha > 0$, which is then equivalent to equation (6.13). With a fixed value of $W^*(t_0)$, the objective function (6.13) is a simple expectation, hence we can determine $\mathcal{P}^*$ by dynamic programming, which is clearly time consistent. □

To be precise, we define an *implementable strategy* in terms of the controls relevant to this paper:

Definition 6.1 (Implementable strategy)**.** *A strategy is implementable if there is no incentive to deviate from the strategy computed at the initial time. More precisely, let $p^{t_\ell}(w,t_m)$ be the optimal control at time t_m, computed at time t_ℓ, $t_m \geq t_\ell$, $t_m, t_\ell \in \mathcal{T}$. An implementable strategy is such that*

$$p^{t_q}(w,t_n) = p^{t_\ell}(w,t_n) \quad ; \quad \forall t_n, \ \forall w \quad ; \quad t_n \geq t_q \geq t_\ell \quad . \tag{6.14}$$

Corollary 6.1 (Alternative objective function TCEQ: an implementable strategy)**.** *The following linear target shortfall strategy ($TCEQ$), based on pre-commitment mean-CVAR, is implementable*

- *At $t = t_0$, solve for the pre-commitment control from equations (6.1-6.2). As a by-product, we obtain $W^*(t_0)$.*
- *Using this fixed value of $W^*(t_0)$, we solve problem (6.13) for all $t > t_0$.*

Remark 6.6 (Time consistent problem TCEQ)**.** *We can alternatively regard problem $(TCEQ_{t_n}(\kappa\alpha))$ in equation (6.13) as the fundamental objective function. In order to determine an appropriate value for $W^*(t_0)$,we solve the pre-commitment problem at time zero. This generates $W^*(t_0)$ from equation (6.12). Since the pre-commitment solution is only used at time zero to determine $W^*(t_0)$, the time inconsistency of $PCMC$ is irrelevant for problem $(TCEQ_{t_n}(\kappa\alpha))$.*

Remark 6.7 (Intuitive appeal of TCEQ)**.** *Problem* ($TCEQ_{t_n}(\kappa\alpha)$) *is a target based objective function, which should have great intuitive appeal to DC plan investors (Vigna, 2017).* $W^*(t_0)$ *can be interpreted as a disaster level of terminal real wealth.*

We learn from Bjork and Murgoci (2010) that for any time inconsistent problem, where we force time consistency, there is a different utility function which produces a time consistent problem having the same controls. In other words, enforcing time consistency means that the investor is not preferences consistent to his original preferences (Vigna, 2017). In the mean-CVAR case, we can see that the time inconsistent pre-commitment problem has the same controls as the time consistent problem ($TCEQ_{t_n}(\kappa\alpha)$). It is interesting to speculate on the relationship of these two results.

6.2 Computation of $E[W_T]$

As for the Problem TCMC, we need to determine $E[W_T]$ in order to recover CVAR from the value function. Define the function

$$U(s,b,W^*,T^+) \quad = \quad (s+b) \tag{6.15}$$

where at $t_n \in \mathcal{T}$

$$\begin{aligned} U(s,b,W^*,t_n^-) \quad &= \quad U\big(\ w^+\ p_n(w^+,W^*), w^+\ (1-p_n(w^+,W^*)\ ), t_n^+\ \big) \\ & \qquad w^+ = s+b+q_n\ . \end{aligned} \tag{6.16}$$

with $p_n(w,W^*)$ being the optimal control from equation (6.7). For $t \in (t_{n-1}^+, t_n^-)$, U satisfies the PIDE

$$U_t + \frac{\sigma^2 s^2}{2} U_{ss} + (\mu - \lambda_\xi \kappa_\xi) U_s - \lambda_\xi U + rbU_b + \int_{-\infty}^{+\infty} U(e^y s, b, q, t) f(y)\ dy\ = 0\ , \tag{6.17}$$

so that

$$E_{\mathcal{P}^*}^{X_0^+, t_0^+}[W_T] \quad = \quad U(0, W_0, W^*(t_0), t_0^-)\ . \tag{6.18}$$

7 Scaling Property of the Time Consistent Mean-CVAR control

We consider the degenerate case where a lump sum investment is made, and no cash is injected at rebalancing times. In other words, at $t=0$, $(s,b)=(0,W_0)$, with W_0 being the initial lump sum, and $q_i = 0, \forall i$. For ease of analysis, we will use the formulation of problem $TCMC_{t_n}(\kappa)$ as given in Section 5.1. Before stating our main result, the following Lemmas will be useful.

Lemma 7.1 (Properties of solution of equation (5.10))**.** *If Property 2.1 holds, then, given a scalar* $\lambda > 0$*, if*

$$V(\lambda s, \lambda b, \lambda W^*, t_i^-) \quad = \quad \lambda V(s, b, W^*, t_i^-)\ , \tag{7.1}$$

then

$$V(\lambda s, \lambda b, \lambda W^*, t_{i-1}^+) \quad = \quad \lambda V(s, b, W^*, t_{i-1}^+)\ . \tag{7.2}$$

Proof. Changing variables in PIDE (5.10) to $x = \log s$, and noting that the transformed PIDE now has constant coefficients (from Property 2.1), then we can write the solution of PIDE (5.10) as

$$V(s, b, W^*, t_{i-1}^+) = \int_{-\infty}^{\infty} g\left(\log s - x', \Delta t\right) V(e^{x'}, be^{r\Delta t}, W^*, t_i^-)\, dx'$$
$$\Delta t = t_i - t_{i-1} \ , \tag{7.3}$$

where $g(x - x', \Delta t)$ is the Green's function of PIDE (5.10), excluding the rbV_b term (Garroni and Menaldi, 1992; Forsyth and Labahn, 2019), after transforming to $x = \log s$ coordinates. Let $x' = \log s'$ in equation (7.3) to give

$$V(s, b, W^*, t_{i-1}^+) = \int_{0}^{\infty} g\left(\log(s/s'), \Delta t\right) V(s', be^{r\Delta t}, W^*, t_i^-)\, \frac{ds'}{s'} \ . \tag{7.4}$$

For $\lambda > 0$ we have

$$\begin{aligned}
V(\lambda s, \lambda b, \lambda W^*, t_{i-1}^+) &= \int_{0}^{\infty} g\left(\log(\lambda s/s'), \Delta t\right) V(s', \lambda be^{r\Delta t}, \lambda W^*, t_i^-)\, \frac{ds'}{s'} \ . \\
&= \int_{0}^{\infty} g\left(\log(s/s''), \Delta t\right) V(\lambda s'', \lambda be^{r\Delta t}, \lambda W^*, t_i^-) \frac{ds''}{s''} \qquad (s' = \lambda s'') \\
&= \lambda \int_{0}^{\infty} g\left(\log(s/s'), \Delta t\right) V(s', be^{r\Delta t}, W^*, t_i^-)\, \frac{ds'}{s'} \qquad (\text{from } (7.1)) \\
&= \lambda V(s, b, W^*, t_{i-1}^+) \ . \qquad (7.5)
\end{aligned}$$

□

Across rebalancing dates $t_n^+ \to t_n^-$, we have the following result:

Lemma 7.2. *Suppose the time consistent solution satisfies equations (5.7) where $p_n(w)$ is given by equation (5.6). If $q_n = 0$, and, for any $\lambda > 0$*

$$V(\lambda s, \lambda b, \lambda W^*, t_n^+) = \lambda V(s, b, W^*, t_n^+) \ , \tag{7.6}$$

then

$$p_n(w) = p_n(\lambda w) \ , \tag{7.7}$$
$$V(\lambda s, \lambda b, \lambda W^*, t_n^-) = \lambda V(s, b, W^*, t_n^-) \ . \tag{7.8}$$

Proof. From equation (5.6)

$$\begin{aligned}
p_n(w) &= \underset{p' \in \mathcal{Z}}{\arg\max} \left\{ \sup_{W^*} V(w\ p', w\ (1 - p'), W^*, t_i^+) \right\} \\
&= \underset{p' \in \mathcal{Z}}{\arg\max} \left\{ \sup_{W^*} \lambda V(w\ p', w\ (1 - p'), W^*, t_i^+) \right\} \qquad (\lambda > 0) \\
&= \underset{p' \in \mathcal{Z}}{\arg\max} \left\{ \sup_{W^*} V(\lambda w\ p', \lambda w\ (1 - p'), \lambda W^*, t_i^+) \right\} \qquad (\text{from}(7.6)) \\
&= \underset{p' \in \mathcal{Z}}{\arg\max} \left\{ \sup_{W'} V(\lambda w\ p', \lambda w\ (1 - p'), W', t_i^+) \right. \qquad (W' = \lambda W^* \ ; \ W^* \in [0,\infty)) \\
&= p_n(\lambda w) \ . \qquad (7.9)
\end{aligned}$$

Next, we have, from equation (5.7), noting that $w = s + b$

$$\begin{aligned} V(\lambda s, \lambda b, \lambda W^*, t_n^-) &= V\big(\ \lambda w\ p_n(\lambda w), \lambda w\ (1 - p_n(\lambda w)\ ), \lambda W^*, t_n^+\ \big) \\ &= V\big(\ \lambda w\ p_n(w), \lambda w\ (1 - p_n(w)\ ), \lambda W^*, t_n^+\ \big) \qquad \text{(from 7.9)} \\ &= \lambda V\big(\ w\ p_n(w), w\ (1 - p_n(w)\ ), W^*, t_n^+\ \big) \qquad \text{(from (7.6))} \\ &= \lambda V(s, b, W^*, t_n^-)\ . \end{aligned} \tag{7.10}$$

□

We now have our final result.

Theorem 7.1. *If a lump sum investment is made (i.e. $q_n = 0, \forall n$), Property 2.1 holds, and the time consistent mean-CVAR strategy is determined by equations (5.5-5.11), then the optimal control for Problem $TCMC_{t_n}(\kappa)$ at each time t_n is independent of wealth w, that is*

$$p_n(w) = p(t_n)\ ;\ n = 0, \ldots, M-1, \tag{7.11}$$

which also implies that $V(\lambda s, \lambda b, \lambda W^*, t) = \lambda V(s,b,W^*,t)$.

Proof. From equation (5.8) we have, for constant $\lambda > 0$,

$$V(\lambda\ s, \lambda\ b, \lambda\ W^*, T^+) \quad = \quad \lambda V(s, b, W^*, T^+)\ . \tag{7.12}$$

Apply Lemmas 7.2 and 7.1 recursively. Then we have that equation (7.9) holds $\forall n$. Hence, for any $\lambda > 0$

$$p_n(\lambda w) = p_n(w) = p(t_n)\ , \tag{7.13}$$

and $V(\lambda s, \lambda b, \lambda W^*, t) = \lambda V(s,b,W^*,t)$. □

Remark 7.1 (Significance of Theorem 7.1)**.** *In our numerical examples, we will consider only the practical case where the initial investment is zero, and the investor adds a fixed amount (real) to the portfolio at each rebalancing date, which is at odds with the assumptions of Theorem 7.1. However, suppose at time $t_n \in \mathcal{T}$,*

$$w = (s + b) \gg \sum_{i=n}^{i=M-1} e^{-r(T-t_i)} q_i\ . \tag{7.14}$$

In other words, we examine points in the state space where the future discounted value of the cash injections is small compared to the current wealth. In this case, we can expect that $p_n(w)$ is only weakly dependent on wealth.

If we are, in fact, interested in a pure lump sum investment, then Theorem 7.1 can be used to reduce the dimensionality of the problem, with resulting computational efficiency.

Remark 7.2 (Result in (Cui et al., 2019))**.** *For the lump sum investment case, Cui et al. (2019) show that the* amount *of wealth invested in the risky asset is a piecewise linear function of the current wealth. The problem in Cui et al. (2019) is posed in terms of specifying a minimum value of $E[W_T]$, which introduces a non-scaled variable into the problem. In contrast, we use the scalarization method in equation (5.1), which does not introduce any non-scaled variables. In any case, from the analysis above, we can see that a simple form for the control is unlikely to exist for the case of periodic contributions to the portfolio, which we verify in our numerical computations.*

8 Numerical Methods

8.1 Time consistent Mean-CVAR

For Problem ($TCMC_{t_n}(\kappa)$), our starting point is the formulation in Section 5.1. We discretize the state space using n_x equally spaced nodes in the $x = \log s$ direction, on a finite computational domain $[x_{\min}, x_{\max}]$. We discretize the domain $[0, b_{\max}]$ using an unequally spaced grid with n_b nodes, and similarly, we discretize the domain $[0, W^*_{\max}]$ using n_w nodes (unequally spaced). We use the Fourier methods discussed in Forsyth and Labahn (2019) to solve PIDE (5.10) between rebalancing times. To avoid wrap-around errors, we extend the computational domain for $x < x_{\min}$ and $x > x_{\max}$ and assume a constant value for the solution in the extended domain as described in Forsyth and Labahn (2019). This localization error can be made small by selecting $|x_{\min}|, x_{\max}$ sufficiently large. In the b and W^* directions, we localize the problem by capping the solution values at the $(b_{\max}, W^*_{\max})$ values. The error in regions of interest can be made small by selecting sufficiently large values of $(b_{\max}, W^*_{\max})$.

At rebalancing times $t_n \in \mathcal{T}$, we discretize $p \in [0,1]$ using n_b equally spaced nodes, and then evaluate the right hand side of equation (5.6) using linear interpolation. We then solve the optimization problem (5.6) using exhaustive search over the discretized p values and the discretized W^* grid.

Once the optimal control is determined, we then use this control to determine the solution for $E[W_T]$ in Section 5.2. Similarly, we use a Fourier method to advance the solution between rebalancing times.

8.2 Pre-commitment Mean-CVAR

For Problem ($PCMC_{t_n}(\kappa)$), we start with the formulation in Section 6.1. We will use the approach described in Miller and Yang (2017). This method is based on equation (6.4). We solve the outer optimization problem, maximization with respect to W^*, by solving a sequence of inner optimization problems, which require optimizing with respect to the rebalancing controls $\mathcal{P}$.

For the inner optimization problems (i.e. we regard W^* as fixed) we proceed as follows. We discretize in the $x = \log s$ direction using n_x equally spaced nodes on the domain $[x_{\min}, x_{\max}]$, and n_b nodes in the b direction on the domain $[0, b_{\max}]$. We use the same Fourier methods as described in Section 8.1 to advance the solution between rebalancing times.

At rebalancing times $t_n \in \mathcal{T}$, we discretize $p \in [0,1]$ using n_b equally spaced nodes, and then evaluate the right hand side of equation (6.16) using linear interpolation. We then solve the optimization problem (6.16) using exhaustive search over the discretized p values (recall that W^* is fixed in this case).

The outer optimization problem in equation (6.4) can be written in terms of $V(s,b,W^*,t)$ as

$$J(0, W_0, t_0^-) = \sup_{W'} V(0, W_0, W', t_0^-) \tag{8.1}$$

where each evaluation of $V(\cdot)$ requires solution of problem (6.5).

We carry out the maximization in equation (8.1) by using a sequence of grids $n_x \times n_b$ to solve problem (6.5). On the coarsest grid, we discretize W^* and solve problem (6.5) for each discrete value of W^*. We then determine the $\max_{W^*}$ by exhaustive search. We use this value of W^* as a starting point for a one dimensional optimization algorithm on a sequence of finer grids. The solution on the coarse grid is inexpensive, and the fine grid optimization solutions do not require many iterations since we have a good starting estimate.

μ	σ	λ	p_{up}	η_1	η_2
.0884	.1451	.3370	.2581	4.681	5.600

TABLE 9.1: *Estimated annualized parameters for the double exponential jump diffusion model given in equation* (2.1) *applied to the value-weighted CRSP Deciles (1-10) index, deflated by the CPI. Sample period 1926:1 to 2017:12.*

Note that there is no guarantee that we have found the global maximum since the problem is not guaranteed to be convex.[1] However, we have made a few tests by carrying a grid search on the finest grid, which suggests that we do indeed have the globally optimal solution.

9 Numerical Example

9.1 Market Parameters

The data and the method used to fit the parameters for process (2.3) are described in (Dang and Forsyth, 2016; Forsyth and Vetzal, 2017; Forsyth et al., 2019). Briefly, we fit the parameters for process (2.3) from monthly market data for the sample period of 1926:1 to 2017:12. We use the Center for Research in Security Prices (CRSP) Deciles (1-10) index. This is a total return value-weighted index of US stocks. We also use one month Treasury bill (T-bill) returns, over the period 1926:1 to 2017:12, for the risk-free asset. We adjust the returns for inflation by using the US CPI index, so all returns are real.

Table 9.1 provides the resulting annualized parameter estimates for the double exponential jump diffusion given in equation (2.1). The average real one-month T-bill rate for the period 1926:1-2017:12 was $r = .00464$.

9.2 Investment Scenario

Studies have shown that earnings for a typical employee increase rapidly until the age of 35, then increase slowly thereafter, until a few years before retirement, and then decreasing as fewer hours are worked in the transition to retirement (Cocco et al., 2005; Ruppert and Zanella, 2015).

As a motivating example, we consider a 35 year old investor saving for retirement in a defined contribution (DC) pension plan. We assume that the investor has a constant (real) salary of 100,000 per year, and the total employee-employer contribution to a tax advantaged DC plan account is 20% of (real) salary per year. The investor plans to retire at age 65.

To be more precise, in our modelling context, we assume that the investor has zero initial wealth, and injects 20,000 per year (real) into the portfolio at times $t = 0, 1, \ldots, 29$ years. The investment horizon is $T = 30$ years, with annual rebalancing. Further details are given in Table 9.2.

10 Tests of convergence

In Table 10.1 we show the results for the scenario in Table 9.2, but using a default strategy of rebalancing to a constant weight of $p = 0.4$ in equities at each rebalancing date.

[1] In Miller and Yang (2017), the pre-commitment mean-CVAR problem is posed in terms of $\log W_T$, which is then shown to result in a convex outer optimization over W^*. However, we have posed the problem in terms of W_T, which seems more natural, since *"you can only spend dollars, not returns"*.

Investment Parameters	
Expiry time T	30 years
Initial wealth	0
Rebalancing frequency	yearly
Cash injection $\{q_i\}_{i=0,\ldots,29}$	20,000
Model Parameters	
Real interest rate r	.00464
Equity process parameters	Table 9.1
Numerical Parameters	
$x_{\max}$	$\log(10^5)+8$
$x_{\min}$	$\log(10^5)-8$
$b_{\max} = W^*_{\max}$	5×10^8

TABLE 9.2: *Investment scenario and model parameters.*

N_{sim}	$E[W_T]$	CVAR (5%)	$Median[W_T]$
1.6×10^5	1160 (2.7)	577	1083
6.4×10^5	1162 (1.3)	598	1084
2.56×10^6	1162 (.7)	598	1084

TABLE 10.1: *Convergence test, rebalance to constant weight $p = .4$ in stocks. Parameters in Table 9.2. N_{sim} is the number of Monte Carlo simulations. The numbers in brackets are the standard errors at the 99% confidence level. Units: thousands of dollars (real).*

Table 10.2 shows a convergence test for solution of the pre-commitment mean-CVAR Problem ($PCMC_{t_n}(\kappa)$), equations (6.1-6.2). The optimal controls are computed and stored, and then used as input to Monte Carlo simulations. The Monte Carlo results are also shown in Table 10.2. We choose the parameter κ in equation (6.1) so that $Median[W_T]$ is approximately the same as for the constant weight $p = 0.4$ strategy ($\kappa = 0.1$). In addition, we set $\alpha = .05$.

Table 10.3 shows a convergence test for solution of the time consistent mean-CVAR Problem ($TCMC_{t_n}(\kappa)$). equations (5.2-5.3). The optimal controls are computed and stored, and then used as input to Monte Carlo simulations. The Monte Carlo results are also shown in Table 10.3. We choose the parameter κ in equation (5.2) so that $Median[W_T]$ is approximately the same as for the constant weight $p = 0.4$ strategy ($\kappa = 2.5$). Again, we set $\alpha = .05$. Note that in contrast to the pre-commitment results, convergence is considerably slower here, as the mesh is refined.

For both pre-commitment and time consistent cases, we carried out tests replacing $[x_{\min}, x_{\max}]$ by $[x_{\min}-2, x_{\max}+2]$, b_{max} by $b_{\max}\times 10$, and for the time consistent case, we replaced $W^*_{\max}$ by $W^*_{\max}\times 10$. In all cases, this resulted in changes to the solution in at most the fifth digit.

Remark 10.1 (Slower convergence: time consistent case)**.** *In the pre-commitment case, we can*

HJB Equation				Monte Carlo		
Grid	$E[W_T]$	CVAR (5%)	W^*	$E[W_T]$	CVAR (5%)	$Median[W_T]$
512×333	2503	674.6	785.5	2485 (6.2)	679.2	1143
1024×665	2452	680.8	801.4	2447 (6.2)	682.0	1080
2048×1329	2434	682.3	806.8	2433 (6.2)	682.6	1067

TABLE 10.2: *Convergence test, pre-commitment mean-CVAR. Parameters in Table 9.2. The Monte Carlo method used* 2.56×10^6 *simulations. The numbers in brackets are the standard errors at the 99% confidence level.* $\kappa = 0.1, \alpha = .05$. *Grid refers to the grid used to solve the HJB PDE:* $n_x \times n_b$, *where* n_x *is the number of nodes in the* $\log S$ *direction, and* n_b *is the number of nodes in the* B *direction. Units: thousands of dollars (real).*

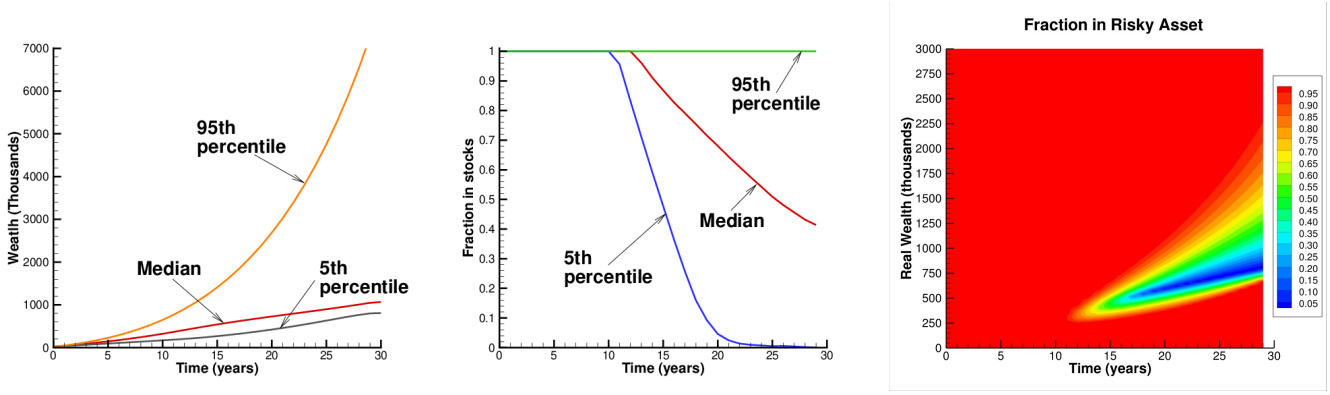
HJB Equation			Monte Carlo		
Grid	$E[W_T]$	CVAR (5%)	$E[W_T]$	CVAR (5%)	$Median[W_T]$
$256 \times 309 \times 309$	1746	495.7	1728 (1.9)	502.6	1381
$512 \times 617 \times 617$	1455	519.6	1450 (1.9)	520.4	1168
$1024 \times 1233 \times 1233$	1343	529.8	1341 (1.9)	530.0	1079

TABLE 10.3: *Convergence test, time consistent mean-CVAR. Parameters in Table 9.2. The Monte Carlo method used* 2.56×10^6 *simulations.* $\kappa = 2.5, \alpha = .05$. *Grid refers to the grid used to solve the HJB PDE:* $n_x \times n_b n_w$, *where* n_x *is the number of nodes in the* $\log S$ *direction,* n_b *is the number of nodes in the* B *direction, and* n_w *is the number of nodes in the* W^* *direction. The numbers in brackets are the standard errors at the 99% confidence level. Units: thousands of dollars (real).*

rewrite equations (6.7-6.8) as

$$\begin{aligned} V(s,b,W^*,t_n^-) &= \max_{p' \in \mathcal{Z}} V\big(\ w^+\ p', w^+\ (1-p'\ ), W^*, t_n^+\ \big) \\ & \qquad w^+ = s + b + q_n\ , \end{aligned} \tag{10.1}$$

which means that the value function $V(s,b,W^*,t_n^-)$ *is maximized at each point in* (s,b,W^*) *space. Intuitively, this means that even if the control is comparatively inaccurate, the value function solution is still reasonably accurate, since the control is an extreme point of the right hand side of equation (10.1). In contrast, for the time consistent case, we can see from equations (5.6-5.7) that in general, the value function* $V(s,b,W^*,t_n^-)$ *does not maximize the right hand side of equation (5.7) at all points in* (s,b,W^*) *space. Hence the extreme point property is lost, and we can expect slower convergence as the mesh is refined.*



(a) *Percentiles of accumulated wealth.*

(b) *Percentiles of optimal fraction in equities.*

(c) *Optimal control heat map.*

FIGURE 11.1: *Pre-commitment mean-CVAR, parameters in Table 9.2. Optimal control computed solving Problem 5.1-5.3. Percentiles of real wealth and the optimal fraction invested in equities. Optimal control computed by solving problem Statistics based on* 2.56×10^6 *Monte Carlo simulation runs.*

11 Numerical Results

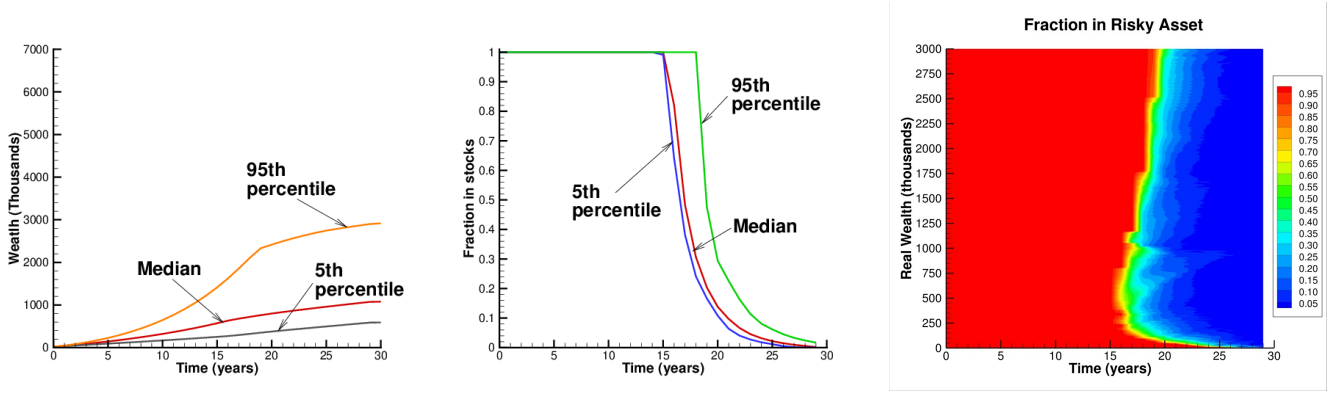
In the following, for the time consistent and pre-commitment policies, we compute the controls using the finest grids in Tables 10.2 and 10.3. All Monte Carlo computations used 2.56×10^6 simulations.

Figures 11.1 and 11.2 compare the pre-commitment and time consistent mean-CVAR strategies in terms of: (a) percentiles of accumulated wealth, (b) percentiles fraction in equities and (c) control heat maps. Contrast Figure 11.1(c) with Figure 11.2(c). We can see that the time consistent heat map is somewhat poorly defined (contours not sharply delineated), which is expected from Remark 10.1. In particular, from Remark 7.1, we expect that for large values of W, then the time consistent heat map contours should become straight vertical lines (i.e. control independent of W). Due to storage limitations, it was not possible to compute the time consistent solution with a finer grid (the finest grid in Table 10.3 had $\simeq 10^9$ nodes).

As a point of comparison, we show the percentiles of accumulated wealth for the constant weight $p = 0.4$ case in Figure 11.3.

A more precise comparison of all three strategies is shown in Figure 11.4, which shows the cumulative distributions functions for the cumulative wealth W_T. By design, all three strategies have the approximately the same $Median[W_T]$, which can be verified by noting that all curves intersect at $Prob(W_T < W) = 0.5$. The investor has contributed a total of 600,000 (real) over the thirty years. Therefore, any values of $W < 600{,}000$ should be regarded as a poor result.

The time consistent strategy is dominated by the constant weight strategy for W below the median, which is a poor result given that TCMC is attempting to minimize tail risk. The pre-commitment strategy has a sharp decrease in the cumulative distribution function for $W < 800$, which is reflected in the fact that the CVAR for the pre-commitment strategy is the largest of all three strategies (recall that with definition (4.2) for CVAR, a larger CVAR value has less risk). In terms of reduction of tail risk, as observed at $t = 0$, the pre-commitment mean-CVAR strategy is clearly superior to the other strategies. Rather surprisingly, the time consistent mean-CVAR strategy appears to be the least effective of all the three strategies (smallest CVAR of all strategies). The pre-commitment strategy dominates the other strategies for wealth levels $W < 800{,}000$ and



(a) *Percentiles of accumulated wealth.*

(b) *Percentiles of optimal fraction in equities.*

(c) *Optimal control heat map.*

FIGURE 11.2: *Time consistent mean-CVAR, parameters in Table 9.2. Percentiles of real wealth, optimal fraction invested in equities, and control heat map. Optimal control computed by solving Problem 6.1-6.2. Statistics based on* 2.56×10^6 *Monte Carlo simulation runs in the synthetic market.*

$W > 1{,}100{,}000$.

11.1 Summary of Numerical Results

From Remark 7.1, and Figures 11.2(b) -11.2(c), we can see that the time consistent control is only weakly dependent on the current wealth (over a wide range of wealth values). Hence we would expect that the time consistent strategy behaves similarly to a deterministic strategy. This is consistent with the results in Figure 11.4. From Forsyth and Vetzal (2019), we learn that deterministic strategies offer virtually no improvement over constant weight strategies (for fixed $E[W_T]$).

From a practical point of view, it would appear that applying a time consistent constraint to multi-period mean-CVAR policies results in a strategy which is counter-intuitive. At time zero, we have some idea of what we desire as a minimum final wealth. Fixing this shortfall target for all $t > 0$ makes intuitive sense. If we have a billion dollars, we are probably only concerned with the probability of a final wealth being less than, say 50 million dollars. On the other hand, if we have one million dollars, then we are probably concerned about ending up with less than 500,000. In other words, our intuitive shortfall target is *not* a constant proportion of current wealth. This intuition contrasts with time consistent strategies, which adjust the effective shortfall target in response to current wealth. This does not appear to generate a reasonable strategy. On the other hand, pre-commitment mean-CVAR strategies at time zero are equivalent to a time consistent strategy which fixes the shortfall target at the initial time. This equivalent objective function (linear target shortfall) appears to produce an intuitively reasonable strategy.

12 Conclusions

Pre-commitment strategies have been widely criticized for being *"non-implementable."* However, in the mean-CVAR case, we know from Corollary 6.1, that since pre-commitment mean-CVAR is equivalent to a linear target shortfall strategy with a fixed shortfall target W^*, then this strategy is time consistent in terms of this alternative objective function, hence is implementable.

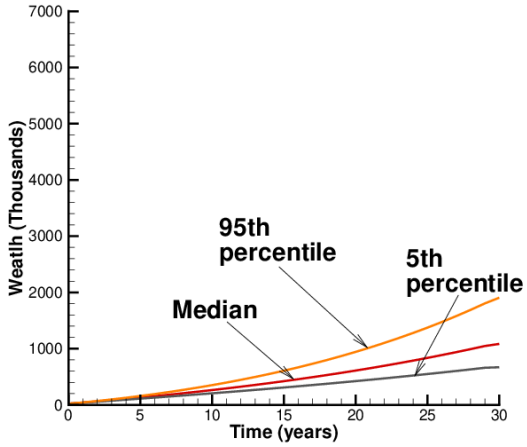


FIGURE 11.3: *Constant weight $p = 0.4$ in equities, parameters in Table 9.2. Percentiles of wealth.*

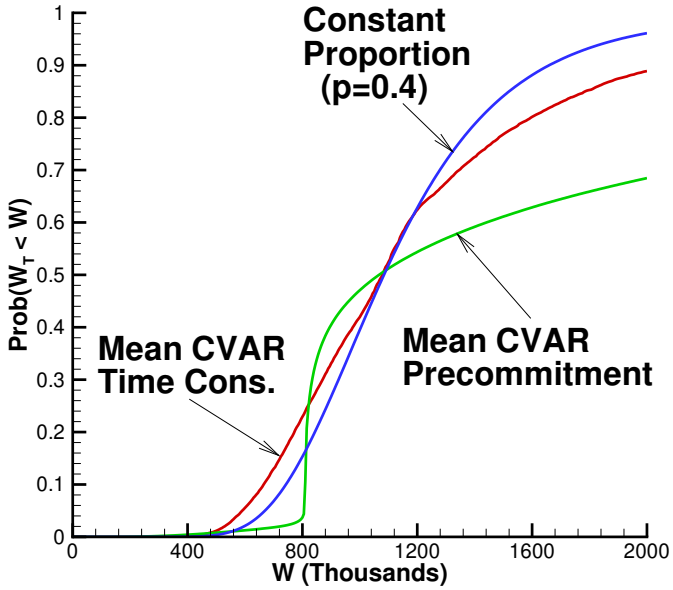


FIGURE 11.4: *Comparison of cumulative distribution functions for W_T.*

Since all strategies in this study can be considered to be time consistent (when viewed in terms of the appropriate objective function), we can then rank the various strategies in terms of the cumulative distribution functions of the terminal wealth. Forcing a time consistent constraint on a mean-CVAR strategy produces a final distribution function which has a larger left tail risk than a simple constant weight strategy (with the same median value of final wealth).

Note that forcing a time consistent constraint for the mean-variance problem has been shown to have undesirable consequences in some cases (Bensoussan et al., 2019). Hence, it is perhaps not surprising to see a similar effect for the mean-CVAR objective function.

Time consistent mean-CVAR policies behave in a manner very similar to deterministic (i.e. only a function of time) strategies. This offers little (if any) improvement compared to the standard constant weight strategy.

In contrast, the pre-commitment mean-CVAR strategy (which coincides with the time consistent linear target shortfall strategy at time zero) does minimize the left tail risk, compared to the other strategies.

Finally, in agreement with (Vigna, 2017; Bensoussan et al., 2019), our results indicate that simply forcing a time consistent constraint onto a pre-commitment policy, without considering the economic ramifications, may lead to strategies with undesirable characteristics.

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Declarations of Interest

The author has no conflicts of interest to declare.

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