Management of Withdrawal Risk Through Optimal Life Cycle Asset Allocation

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Draft version: May 8, 2018

Abstract

Retirees who do not have defined benefit pension plans typically must fund spending from accumulated savings. This leads to the risk of depleting these savings, i.e. withdrawal risk. We analyze this risk through full life cycle optimal dynamic asset allocation, including the accumulation and decumulation phases. We pose the asset allocation strategy as a problem in optimal stochastic control. Various possible objective functions are tested and compared using metrics such as the probability of portfolio depletion, the median of the remaining portfolio value, and conditional value at risk (CVAR). The control problem is solved using a Hamilton-Jacobi-Bellman formulation, based on a parametric model of the underlying stochastic processes and a variety of objective functions. Monte Carlo simulations which use the optimal controls are presented to evaluate the performance metrics. These simulations are based on both the parametric model and bootstrap resampling of 91 years of historical data. Based primarily on the resampling tests, we conclude that target-based approaches which seek to establish a safety buffer of wealth at the end of the decumulation period appear to be superior to strategies which directly attempt to minimize risk measures such as the probability of portfolio depletion.

Keywords: Withdrawal risk, life cycle asset allocation, optimal control

1 Introduction

Nobel laureate William Sharpe has referred to decumulation (i.e. the use of savings to fund spending during retirement) as "the nastiest, hardest problem in finance" (Ritholz, 2017). Retirees are confronted with withdrawal risk and longevity risk, as well as additional uncertainties associated with unexpected inflation, the level of other sources of income such as government benefits, and the changing utility of income over time. Our focus is on withdrawal risk, which is the chance of running out of money, even when the retirement period is specified, due to the demand for constant income from a volatile portfolio. Withdrawal risk can be assessed in a variety of ways: the probability of ruin (i.e. depleting savings to zero), the magnitude of ruin, and the "waste" of leaving more of a legacy than intended.

We examine both the accumulation and decumulation phases of life cycle asset allocation. As an example, consider a typical defined contribution (DC) pension plan. The employer and employee

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each contribute a fraction of the employee salary each year to a (usually) tax-advantaged account. This represents a reasonably predictable stream of cash flows into the DC plan account, over a long period. For a typical labour force participant, there is a rapid increase of salary up to the age of 35, and thereafter a slow real increase (less than 2% per year) until retirement (Blake et al., 2014). If we consider a prototypical 35 year-old who has obtained stable employment, then the accumulation period would be about 30 years. Due to increases in longevity, it would seem prudent to plan for another 30 years of retirement. This 60 year life cycle makes DC plan holders truly long-term investors.

The total employee-employer contribution to the DC account during the accumulation period is usually in the range of 10-20% of salary. Recommended final salary replacement ratios (including additional government programs) are variously estimated as 40-70%. If we postulate 30 years of accumulation at 20% of salary, followed by 30 years of decumulation at 40% of final salary, it seems clear that this cannot be funded by low risk bond investments. This immediately raises the question of the optimal asset allocation to bonds and stocks, during both the accumulation and the decumulation phases.

Specifying a constant real withdrawal per year means that we are attempting so far as possible to create a defined benefit (DB) experience. We let the asset allocation change throughout the life cycle to minimize the adverse consequences. We can also view this strategy as an asset-liability matching (ALM) approach where, given a specific sequence of market returns, we determine the equity allocation that is most likely to meet the pension liability at each point in time.

During the decumulation phase, the retiree is faced with longevity risk and perhaps a bequest motive. Due to the pooling of risk and the earning of *mortality credits*, it is often suggested that annuities are good investments for the decumulation phase of retirement savings. However, it is well known that very few retail investors take advantage of annuities upon retirement (Peijnenburg et al., 2016). This is especially understandable in the current environment of extremely low real interest rates, which lead to meager annuity payouts. We therefore assume that our 35 year old DC plan holder has no plans to annuitize on retirement, and so adopts an asset allocation strategy which will be operational *to and through* the retirement date.

Popular investment vehicles during the accumulation phase are target date funds (TDFs). A standard TDF begins with a high allocation to equities, and moves to a higher weighting in bonds as retirement approaches. The fraction invested in equities over time is referred to as a *glide path*. Typically, these glide paths are *deterministic* strategies, i.e. the equity fraction is only a function of time to go. Total assets invested in US TDFs at the end of 2016 were over \$887 billion.[1] The rationale for the high initial equity allocation to stocks is often based on a human capital argument, i.e. a young DC plan holder has many years of bond-like cash flows from employment, and can take on a large equity risk in the DC account. As retirement approaches, the future income from employment diminishes, and hence the holder should switch to bonds. However, recent work calls into question the effectiveness of the TDF type of approach (see, e.g. Arnott et al., 2013; Graf, 2017; Westmacott and Daley, 2015; Forsyth et al., 2017; Forsyth and Vetzal, 2017b). For example, Forsyth et al. (2017) and Forsyth and Vetzal (2017b) show that for a fixed value of target expected wealth at the end of the accumulation period, there is always a constant weight strategy that achieves the same target expected wealth as a deterministic glide path with a similar cumulative standard deviation. More recently, deterministic strategies have also been suggested for *to and through* funds, i.e. both the accumulation and the decumulation phases (O'Hara and Daverman, 2017).

In this article we treat life cycle asset allocation as an optimal stochastic control problem. There

[1] Investment Company Fact Book (2017), available at `www.ici.org`.

is a large literature on maximizing various traditional utility functions (see, e.g. Blake et al. (2014) and the references therein). However, in our experience a typical retiree is concerned with such concrete issues as the probability of portfolio depletion and the size of a possible bequest. Therefore, we take the approach that we evaluate the appropriateness of an objective function in terms of these types of metrics. We attempt to design the objective function (which can be viewed as a type of utility function) so that it directly maximizes (or minimizes) quantities of interest. We view the choice of objective function strictly as a means to shape the probability density of the outcome of the investment process, not as an end in itself. Vigna (2014) argues that traditional utility functions are not dynamically mean variance efficient, and suggests that target-based objective functions are both efficient and lend themselves to intuitive interpretation by retail clients. We note that industry surveys suggest that retirees are extremely concerned with exhausting their savings.[2] Moreover, it is generally easier for practitioners to talk with clients about the risk of depleting their savings and/or the likely range of a bequest, as opposed to trying to determine the parameters of a utility function. As a result, we focus on metrics such as the probability of savings exhaustion, and the median and CVAR of the final portfolio value, instead of standard utility functions.

We consider the following stylized life cycle investment problem. We assume that the investor contributes a fixed real amount into a DC account for 30 years. The investor then desires a stream of fixed (real) cash flows for 30 years of retirement. This assumption of fixed real cash flows from employment income during the accumulation phase takes into account human capital effects in a quantitative manner, in an optimal control sense. By using a fixed, lengthy time for fixed cash outflows, we sidestep the issue of longevity risk. We recognize that this is a weakness of our analysis, but it appears to be a reasonable approach in the absence of any desire to annuitize. Since we have ruled out annuities, using a conservative estimate of longevity (30 years in this case) seems prudent.

We study a variety of objective functions. An obvious starting point is to minimize the probability of ruin, before the end of the decumulation phase. We then consider mean-CVAR strategies (Gao et al., 2017), as well as target-based approaches (Vigna, 2014) that correspond to multi-period mean variance strategies (Li and Ng, 2000; Dang et al., 2017).

We assume that the investment account contains only a stock index and a bond index. We model the real (inflation-adjusted) stock index as following a jump diffusion model (Kou and Wang, 2004). We fit the parameters of this model to monthly US data over the 1926:1-2016:12 period. We consider two markets in our simulation analysis. The *synthetic market* assumes that the stock and bond processes follow the models with constant parameters fit to the historical time series. Given an objective function, we determine optimal strategies by solving a Hamilton-Jacobi-Bellman equation in the synthetic market. We use a fully numerical approach, which allows us to impose realistic constraints: infrequent rebalancing (yearly) and no leverage/no-shorting constraints. The entire distribution function of the strategy is then determined by Monte Carlo simulations in the synthetic market. As a stress test, we apply these strategies to bootstrap resampling of the historical data, which we refer to as the *historical market*. The bootstrap tests make no assumptions about the actual processes followed by the stock and bond indexes. In some cases, we reject strategies which appear promising based on synthetic market results due to poor performance in the bootstrapped historical market.

2 Formulation

For simplicity we assume that there are only two assets available in the financial market, namely a risky asset and a risk-free asset. In practice, the risky asset would be a broad market index fund.

[2]See `https://www.allianzlife.com/about/news-and-events/news-releases/Generations-Ahead-Study-2017`.

For example, many wealth managers have funds which have a fixed weight of domestic and foreign equity markets.

The investment horizon (over both the accumulation and decumulation phases) is T. S_t and B_t respectively denote the *amounts* invested in the risky and risk-free assets at time t, $t \in [0, T]$. In general, these amounts will depend on the investor's strategy over time, including contributions, withdrawals, and portfolio rebalances, as well as changes in the unit prices of the assets. Suppose for the moment that the investor does not take any action with respect to the controllable factors, so that any change in the value of the investor's portfolio is due to changes in asset prices. We refer to this as the absence of control. In this case, we assume that S_t follows a jump diffusion process. Let $t^- = t - \epsilon, \epsilon \to 0^+$, i.e. t^- is the instant of time before t, and let ξ be a random number representing a jump multiplier. When a jump occurs, $S_t = \xi S_{t^-}$. Allowing discontinuous jumps lets us explore the effects of severe market crashes on the risky asset holding. We assume that $\log \xi$ follows a double exponential distribution (Kou and Wang, 2004). If a jump occurs, p_{up} is the probability of an upward jump, while $1 - p_{up}$ is the chance of a downward jump. The mean upward and downward log jump sizes are $1/\eta_1$ and $-1/\eta_2$ respectively. The density function for $y = \log \xi$ is

$$f(y) = p_{up}\eta_1 e^{-\eta_1 y}\mathbf{1}_{y \geq 0} + (1 - p_{up})\eta_2 e^{\eta_2 y}\mathbf{1}_{y<0}. \tag{2.1}$$

We note that

$$E[y = \log \xi] = \frac{p_{up}}{\eta_1} - \frac{(1 - p_{up})}{\eta_2} \; ; \qquad E[\xi] = \frac{p_{up}\eta_1}{\eta_1 - 1} + \frac{(1 - p_{up})\eta_2}{\eta_2 + 1}. \tag{2.2}$$

In the absence of control, S_t evolves according to

$$\frac{dS_t}{S_{t^-}} = (\mu - \lambda E[\xi - 1])\ dt + \sigma\, d\mathcal{Z} + d\left(\sum_{i=1}^{\pi_t}(\xi_i - 1)\right), \tag{2.3}$$

where μ is the (uncompensated) drift rate, σ is the volatility, $d\mathcal{Z}$ is the increment of a Wiener process, π_t is a Poisson process with positive intensity parameter λ, and ξ_i are i.i.d. positive random variables having distribution (2.1). Moreover, ξ_i, π_t, and Z are assumed to all be mutually independent.

We focus on jump diffusion models for long-term equity dynamics since sudden drops in the equity index can have a devastating impact on retirement portfolios, particularly during the decumulation phase. Since we consider discrete rebalancing, the jump process models the cumulative effects of large market movements between rebalancing times.[3]

In the absence of control, we assume that the dynamics of the amount B_t invested in the risk-free asset are

$$dB_t = rB_t\, dt, \tag{2.4}$$

where r is the (constant) risk-free rate. This is obviously a simplification of the actual bond market. However, long term real bond returns do not appear to follow any simple recognizable process. In any case, we will test our strategies in a bootstrapped historical market which introduces inflation shocks and stochastic interest rates.

We define the investor's total wealth at time t as

$$\text{Total wealth } \equiv W_t = S_t + B_t. \tag{2.5}$$

[3] A possible extension would be to incorporate stochastic volatility. However, previous work has shown that stochastic volatility effects are small for the long-term investor (Ma and Forsyth, 2016). This can be traced to the fact that stochastic volatility models are mean-reverting, with typical mean reversion times of less than one year.

Since we specify the real withdrawals during decumulation, the objective functions which we consider below are all defined in terms of terminal wealth W_T. In all cases, we impose the constraints that shorting stock and using leverage (i.e. borrowing) are not permitted, which would be typical of a retirement savings account.

3 Data, synthetic market, and historical market

The data used in this work was obtained from Dimensional Returns 2.0 under licence from Dimensional Fund Advisors Canada. In particular, we use the Center for Research in Security Prices (CRSP) Deciles (1-10) index. This is a total return value-weighted index of US stocks. We also use one month Treasury bill (T-bill) returns for the risk-free asset.[4] Both the equity returns and the Treasury bill returns are in nominal terms, so we adjust them for inflation by using the US CPI index. We use real indexes since long-term retirement saving should be attempting to achieve real (not nominal) wealth goals. All of the data used was at the monthly frequency, with a sample period of 1926:1 to 2016:12.

In our tests, we consider a *synthetic* and an *historical* market. The synthetic market is generated by assuming processes (2.3) and (2.4). We fit the parameters to the historical data using the methods described in Appendix A. We then use these parameters to determine optimal strategies and carry out Monte Carlo computations. As a test of robustness, we also carry out tests using bootstrap resampling of the actual historical data, which we call the historical market. In this case, we make no assumptions about the underlying stochastic processes. We use the stationary block resampling method described in Appendix B. A crucial parameter for block bootstrap resampling is the expected blocksize. We carry out our tests using a range of expected blocksizes. Although the absolute performance of variance strategies is mildly sensitive to the choice of blocksize, the relative performance of the various strategies appears to be insensitive to blocksize. See Appendix B for more discussion.

4 Investment scenario

Let the inception time of the investment be $t_0 = 0$. We consider a set $\mathcal{T}$ of pre-determined *rebalancing times*,

$$\mathcal{T} \equiv \{t_0 = 0 < t_1 < \cdots < t_M = T\}. \tag{4.1}$$

For simplicity, we specify $\mathcal{T}$ to be equidistant with $t_i - t_{i-1} = \Delta t = T/M$, $i = 1, \ldots, M$. At each rebalancing time t_i, $i = 0, 1, \ldots, M$, the investor (i) injects an amount of cash q_i into the portfolio, and then (ii) rebalances the portfolio. At $t_M = T$, the portfolio is liquidated. If $q_i < 0$, this corresponds to cash withdrawals. Let $t_i^- = t_i - \epsilon$ ($\epsilon \to 0^+$) be the instant before rebalancing time t_i, and $t_i^+ = t_i + \epsilon$ be the instant after t_i. Let $p(t_i^+, W_i^+) = p_i$ be the fraction in the risky asset at t_i^+.

Table 4.1 shows the parameters for our investment scenario. This corresponds to an individual with a constant salary of \$100,000 per year (real) who saves 20% of her salary for 30 years, then withdraws 40% of her final real salary for 30 years in retirement. The target salary replacement level of 40% is at the lower end of the recommended range. We assume that government benefits will increase this to a more desirable level. We do not consider escalating the (real) contribution during the accumulation phase (which also impacts the desired replacement ratio), although this

[4] We have also carried out tests using a 10 year US treasury as the bond asset (Forsyth and Vetzal, 2017a). The results are qualitatively similar to those reported in this paper.

Investment horizon (years)	60
Equity market index	Value-weighted CRSP deciles 1-10 US market index
Risk-free asset index	1-month T-bill
Initial investment W_0	0.0
Real investment each year	20.0 ($0 \leq t_i \leq 30$), -40.0 ($31 \leq t_i \leq 60$)
Rebalancing interval (years)	1
Market parameters	See Appendix A

TABLE 4.1: *Input data for examples. Cash is invested at $t_i = 0,1,\ldots,30$ years, and withdrawn at $t_i = 31,32,\ldots,60$ years. Units for real investment: thousands of dollars.*

is arguably more realistic. Assuming flat contributions and withdrawals, we can interpret the above scenario as an investment strategy which allows real withdrawals of twice as much as real contributions. We shall see that this rather modest objective still entails significant risk. As indicated in Table 4.1, we assume yearly rebalancing.[5]

5 Constant weight strategies and linear glide paths

Let p denote the fraction of total wealth that is invested in the risky asset, i.e.

$$p = \frac{S_t}{S_t + B_t}. \tag{5.1}$$

A *deterministic* glide path restricts the admissible strategies to those where $p = p(t)$, i.e. the strategy depends only on time and cannot take into account the actual value of W_t at any time. Clearly this is a very restrictive assumption, but it is commonly used in TDFs.

We consider two cases: $p(t) = const.$ and a linear glide path

$$p(t) = p_{\max} + (p_{\min} - p_{\max})\,\frac{t}{T}. \tag{5.2}$$

Note that this is a *to and through* strategy, since $t = 0$ is the time of initiation of the accumulation phase, while $t = T$ is the time at the end of the decumulation phase.

Monte Carlo simulations were carried out for the scenario given in Table 4.1. We run these simulations in the synthetic market, assuming processes (2.3) and (2.4), with parameters given in Appendix A. We consider constant weight strategies and a linear glide path (5.2). The results are shown in Table 5.1. Here, 5% CVAR (Conditional Value at Risk) refers to mean of the worst 5% of the outcomes.[6]

The results in Table 5.1 show the high risks associated with deterministic strategies. Note the very high dispersion of final wealth as indicated by the large standard deviations and the large differences between the means and medians. Consistent with the findings reported for the accumulation phase by Forsyth et al. (2017) and Forsyth and Vetzal (2017b), the results here for the entire life cycle for a linear glide path are similar to the results for a constant weight strategy having the same time-averaged weighting in stocks (i.e. $p = .40$ in this case). It is interesting to note that while the high constant weighting in equities ($p = 0.8$) has a much higher dispersion of final wealth compared to lower allocations, the $p = 0.8$ strategy has a smaller probability of

[5] More frequent rebalancing has little effect for long-term (> 20 years) investors (Forsyth and Vetzal, 2017c).

[6] See Appendix C for a precise definition of CVAR as used in this work.

Strategy	$Median[W_T]$	$Mean[W_T]$	$std[W_T]$	$Pr[W_T < 0]$	5% CVAR
Glide path	935	1385	1795	.15	−483
$p = .40$	992	1542	2093	.16	−482
$p = .60$	2922	5422	8882	.093	−516
$p = .80$	6051	14832	34644	.082	−592

TABLE 5.1: *Synthetic market results for deterministic strategies, assuming the scenario given in Table 4.1. W_T denotes real terminal wealth after 60 years, measured in thousands of dollars. Statistics based on 6.4×10^5 Monte Carlo simulation runs. The constant weight strategies have equity fraction p. The glide path is linear with $p_{\max} = .80$ and $p_{\min} = 0.0$.*

Strategy	$\hat{b}$	$Median[W_T]$	$Mean[W_T]$	$std[W_T]$	$Pr[W_T < 0]$	5% CVAR
$p = .40$	0.5	900	1337	1683	.16	−490
$p = .60$	0.5	2767	4592	6251	.085	−488
$p = .80$	0.5	5893	12120	21278	.071	−540
$p = .40$	1.0	955	1367	1637	.16	−493
$p = .60$	1.0	2896	4614	5814	.081	−466
$p = .80$	1.0	6075	12028	18991	.068	−514
$p = .40$	2.0	961	1339	1530	.15	−461
$p = .60$	2.0	2931	4248	4955	.07	−389
$p = .80$	2.0	6151	10865	15023	.054	−411
$p = .40$	5.0	965	1306	1451	.14	−438
$p = .60$	5.0	2890	4068	4326	.051	−275
$p = .80$	5.0	5986	9768	12543	.034	−190

TABLE 5.2: *Historical market results for constant proportion strategies with equity fraction p, assuming the scenario given in Table 4.1. W_T denotes real terminal wealth after 60 years, measured in thousands of dollars. Statistics based on 10,000 stationary block bootstrap resamples of the historical data from 1926:1 to 2016:12. $\hat{b}$ is the expected blocksize, measured in years.*

ruin (i.e. $Pr[W_T < 0]$) and larger median value of terminal wealth compared to the lower equity allocation strategies. The downside for the $p = .8$ case compared to the $p = .6$ case is an increase in the tail risk (5% CVAR).

Table 5.2 shows the results for constant proportion strategies based on bootstrap resampling of the historical market, for a range of expected blocksizes.[7] Since we sample simultaneously from the stock and bond historical time series, the choice of blocksize is not obvious (see Appendix B). A reasonable choice would appear to be an expected blocksize of $\simeq 2$ years. Nevertheless, the ranking of the three constant weight strategies is preserved across all blocksizes, i.e. the higher allocation to equities is superior (in terms of $Pr[W_T < 0]$) compared to the smaller allocation to equities. Note that the historical backtests show that the probability of ruin for a typical suggested equity weighting of .6 is in the range $.05 - .09$ depending on the assumed expected blocksize.

[7]Results for the linear glide path are again similar to the constant proportion case with $p = .40$ and have been excluded from Table 5.2 to save space.

6 Adaptive strategies: overview

We will attempt to improve on deterministic strategies by allowing the rebalancing strategy to now depend on the accumulated wealth, i.e. $p_i = p_i(W_i^+, t_i)$. We will specify an objective function, and compute the optimal controls in the synthetic market. This involves the numerical solution of a Hamilton-Jacobi-Bellman (HJB) equation to determine the controls. We use the numerical methods from Dang and Forsyth (2014; 2016) and Forsyth and Labahn (2017), and refer the reader to these sources for a detailed description of the HJB equation and solution techniques. We emphasize that, given an objective function, solving the HJB equation gives the provably optimal strategy in the constant parameter synthetic market. The following several sections consider various possible objective functions in this context.

7 Minimize probability of ruin

Many retirees place a premium on reducing the probability of ruin, i.e. portfolio depletion. Therefore, as a first attempt at defining a suitable objective function, we directly minimize probability of ruin. A similar objective function for the accumulation phase of DC plans has been suggested in Tretiakova and Yamada (2011). Consider a level of terminal wealth W^{min}. We wish to solve the following optimization problem:

$$\min_{\{(p_0,c_0),\ldots,(p_{M-1},c_{M-1})\}} Pr\left[W_T < W^{min}\right]$$

$$\text{subject to} \begin{cases} (S_t, B_t) \text{ follow processes (2.3)-(2.4)}; \quad t \notin \mathcal{T} \\ W_i^+ = W_i^- + q_i - c_i\,;\ S_i^+ = p_i W_i^+\,;\ B_i^+ = W_i^+ - S_i^+\,; \quad t \in \mathcal{T} \\ p_i = p_i(W_i^+, t_i)\,;\ 0 \le p_i \le 1 \\ c_i = c_i(W_i^- + q_i, t_i)\,;\ c_i \ge 0 \end{cases} . \tag{7.1}$$

We recognize objective function (7.1) as minimizing the probability that the terminal wealth W_T will be less than W^{min}. If $W^{min} = 0$, then this will minimize the probability of portfolio depletion.

In problem (7.1), we withdraw surplus cash $c_i(W_i^- + q_i, t_i)$ from the portfolio if investing in the risk-free asset ensures that $W_T \ge W^{min}$. More precisely, let

$$Q_\ell = \sum_{j=\ell+1}^{j=M-1} e^{-r(t_j - t_\ell)} q_j \tag{7.2}$$

be the discounted future contributions as of time t_ℓ. If

$$(W_i^- + q_i) > W^{min} e^{-r(T-t_i)} - Q_i, \tag{7.3}$$

then an optimal strategy is to (i) withdraw surplus cash $c_i = W_i^- + q_i - \left(W^{min} e^{-r(T-t_i)} - Q_i\right)$ from the portfolio; and (ii) invest the remainder $\left(W^{min} e^{-r(T-t_i)} - Q_i\right)$ in the risk-free asset. This is an optimal strategy in this case since $Pr[W_T < W^{min}] = 0$, which is the minimum of problem (7.1).

In the following, we will refer to $c_i > 0$ as *surplus cash*. We assume that any surplus cash is invested in the risk-free asset. Of course, it is also possible to invest it in the risky asset. Some experiments with this alternative approach showed a large effect on $E[W_T]$, but very little impact on $Median[W_T]$, $Pr[W_T < 0]$, and CVAR. Hence we assume that surplus cash is invested in the risk-free asset for simplicity.

$\hat{b}$	$Median[W_T]$	$Mean[W_T]$	$std[W_T]$	$Pr[W_T < 0]$	5% CVAR
		Synthetic market, $W^{min} = 0$			
NA	3.67	20.7	88.3	.0195	−223
		Historical market, $W^{min} = 0$			
0.5	187	199	103	.07	−353
1.0	207	236	105	.068	−346
2.0	228	283	86	.053	−245
5.0	260	341	59	.034	−121
		Historical market, $W^{min} = 200$			
0.5	412	417	141	.043	−335
1.0	434	456	145	.040	−337
2.0	456	512	118	.030	−183
5.0	492	579	85	.017	−9.3

TABLE 7.1: *Optimal control determined by solving problem* (7.1)*, i.e.* $\min Pr[W_T < W^{min}]$ *in the synthetic market, with* W^{min} *as indicated, assuming the scenario in Table 4.1.* W_T *denotes real terminal wealth after* 60 *years, measured in thousands of dollars. Statistics for the synthetic market case are based on* 6.4×10^5 *Monte Carlo simulation runs. Statistics for the historical market cases are based on* 10,000 *stationary block bootstrap resamples of the historical data from 1926:1 to 2016:12.* $\hat{b}$ *is the expected blocksize, measured in years. Surplus cash is included in the mean, median, CVAR and probability of ruin, but excluded from the standard deviation.*

In our summary statistics, we will include surplus cash in measures such as $E[W_T]$, but we will exclude it from the standard deviation $std[W_T]$ since this is supposed to be a measure of risk. Along any path where surplus cash is generated, we have no probability of ruin. But including the surplus cash in $std[W_T]$ will generally increase $std[W_T]$, which seems counter-informative since there is no risk (in the sense of ruin) along this path. In any case, we do not believe that $std[W_T]$ is a very useful risk measure for these types of problems, due to the highly skewed distribution of terminal wealth.

We begin by computing and storing the optimal controls from solving problem (7.1) with $W^{min} = 0$. In other words, we try to minimize the probability of portfolio depletion before year 60. To assess this strategy, we use these controls as input to a Monte Carlo simulation in the synthetic market. Recall that in this case the simulated paths will have exactly the same statistical properties as those assumed when generating the optimal controls. The results are shown in the first row of Table 7.1. In this idealized setting, the final wealth distribution has a median that is almost zero, but also about a 2% chance of being less than zero. Figure 7.1 plots the cumulative distribution function of W_T for this case. The sharp increase in the distribution function near $W_T = 0$ suggests that this strategy will be very sensitive to the asset market parameters. Figure 7.2 shows the percentiles of the total wealth (panel (a)) and the optimal fraction invested in equities (panel (b)) as a function of time. Figure 7.2(a) shows greater dispersion between the 5th and 95th percentiles during the accumulation phase ($t \leq 30$) than during the decumulation phase ($30 < t \leq 60$). From Figure 7.2(b), the median fraction invested in the risky stock index is surprisingly low, essentially de-risking completely by the end of the accumulation period.

We next test this strategy with $W^{min} = 0$ in the historical market. This implies using the

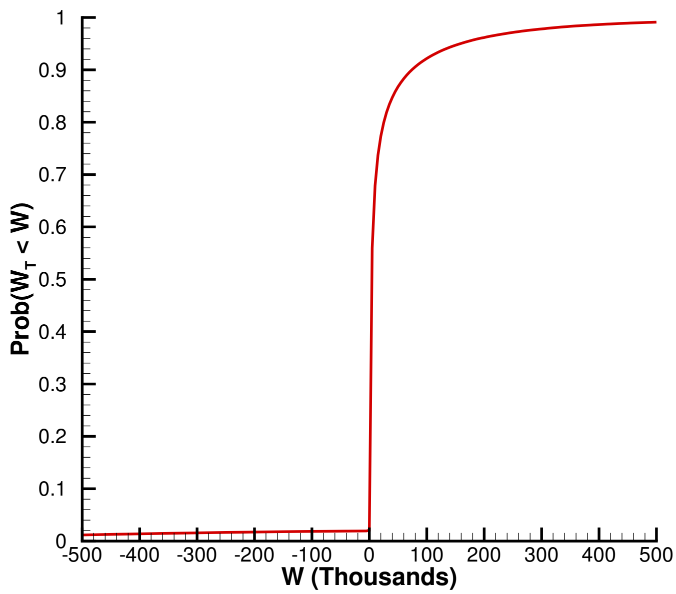
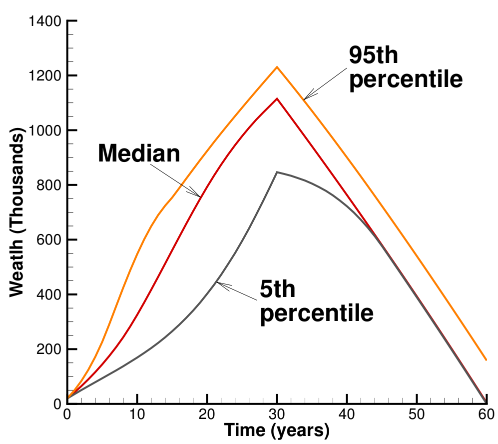
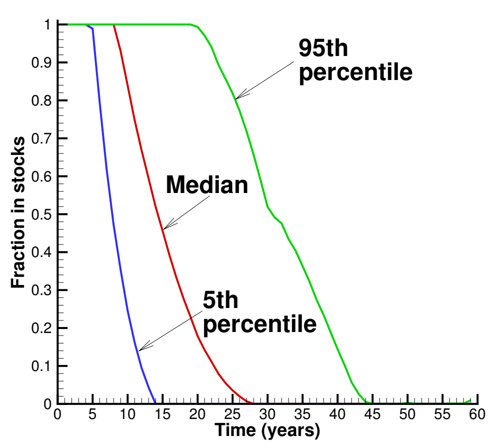


FIGURE 7.1: *Cumulative distribution function. Optimal control determined by solving problem* (7.1)*, i.e.* $\min Pr[W_T < W^{min}]$ *in the synthetic market, with* $W^{min} = 0$*, assuming the scenario in Table 4.1. Distribution computed from* 6.4×10^5 *Monte Carlo simulation runs in the synthetic market. Surplus cash is included in the distribution function.*



(a) *Percentiles of accumulated wealth.*



(b) *Percentiles of optimal fraction in equities.*

FIGURE 7.2: *Percentiles of real wealth and the optimal fraction invested in equities. Optimal control computed by solving problem* (7.1)*, i.e.* $\min Pr[W_T < W^{min}]$ *in the synthetic market, with* $W^{min} = 0$*, assuming the scenario in Table 4.1. Statistics based on* 6.4×10^5 *Monte Carlo simulation runs in the synthetic market. Surplus cash is included in the real wealth percentiles.*

same optimal controls as above, but instead simulating by bootstrap resampling of the historical data over the 1926:1 to 2016:12 period (see Appendix B). Results for several different expected blocksizes $\hat{b}$ ranging from 0.5 years to 5.0 years are provided in the second to fifth rows of Table 7.1. These results differ substantially from the synthetic market case: $Median[W_T]$ and $Mean[W_T]$ are markedly higher in the historical market, but so are the risk measures $std[W_T]$, $Pr[W_T < 0]$, and 5% CVAR (except if $\hat{b} = 5$ years). Since we are directly trying to minimize $Pr[W_T < 0]$, it is worth emphasizing that this ruin probability is higher than in the synthetic market by a factor of more than 3 for the two shortest expected blocksizes. Even when $\hat{b} = 5$ years, the ruin probability is almost 75% higher in the historical market. These results are consistent with our earlier discussion regarding Figure 7.1: the very sharp increase in the cumulative distribution function at $W_T = 0$ for the synthetic market implies that performance is unlikely to be robust to departures from the statistical properties of the idealized synthetic market, which is exactly what happens in the historical market.

The instability here can be traced to the use of bootstrap historical real interest rates. For example, if the case with $\hat{b} = 2.0$ years is repeated using the fixed average historical real interest rate (i.e. $r = .004835$) for all time periods, but with the bootstrapped historical stock returns, then $Pr[W_T < 0] = .013$ compared to the value of .053 in Table 7.1. In this case, since $W^{min} = 0$ under the objective function (7.1), any errors in prediction of the real bond return become magnified, due to the very rapid de-risking. It could be argued that the use of bootstrapped real bond returns is very pessimistic with a blocksize of 2.0 years. Effectively, this simulates a market where the investor de-risks rapidly after the accumulation phase, but then the strategy fails due to real interest rate shocks.

In an effort to determine a more robust strategy, we experimented with setting $W^{min} > 0$, so as to provide a buffer of wealth as insurance against misspecification of real interest rates. The last four rows of Table 7.1 show the results obtained by computing and storing the optimal strategy from solving problem (7.1) with $W^{min} = 200$ in the synthetic market and then using this strategy in bootstrap resampling tests. As expected, this strategy is much more stable in terms of the probability of ruin compared to the $W^{min} = 0$ case. By any measure, the bootstrap results for $W^{min} = 200$ are superior to the those obtained with $W^{min} = 0$.[8]

We can summarize our attempts to minimize probability of ruin as follows. Although at first glance it would appear that minimizing the probability of negative terminal wealth (i.e. portfolio depletion) is a reasonable objective, our tests call this into question. Clearly, aiming for zero final wealth is too sensitive to modelling parameters to be useful. This sensitivity appears to be solely due to the use of bootstrapped bond return data and not due to the bootstrapped equity return data. Due to rapid de-risking, this strategy is sensitive to real interest rate shocks along any paths with early allocation to the bond index. The bootstrap resampling approach introduces random (and potentially large) real interest rate shocks into the market, which occur more often as the expected blocksize gets smaller. It could be argued that this is unduly pessimistic, but we contend that this is a useful stress test. This sensitivity to real interest rate shocks is ameliorated somewhat by setting the final wealth target to be a non-zero amount. However, comparing the historical market results in Tables 5.2 (constant weight allocations) and 7.1 (minimizing probability of ruin), it seems that the median terminal wealth is reduced significantly in order to reduce the probability of portfolio depletion.

[8] Experiments with larger values of W^{min} increased $Pr[W_T < 0]$ in the bootstrap tests.

8 Mean-CVAR optimization

As another possible objective, we consider minimizing the mean of the worst α fraction of outcomes, which is the conditional value at risk (CVAR). We define CVAR in terms of terminal wealth, not losses, so we want to maximize CVAR.[9]

Let $\mathcal{P} = \{p_0, p_1, \ldots, p_{M-1}\}$ be the set of controls at $t \in \mathcal{T}$. In the mean-CVAR case, we will not allow cash withdrawals. Let CVAR_α denote the CVAR at level α. For a fixed value of α and a scalar κ, the mean-CVAR optimization problem is:

$$\max_{\mathcal{P}} \quad E^{\mathcal{P}}\left[\text{CVAR}_\alpha + \kappa W_T\right]$$

$$\text{subject to} \begin{cases} (S_t, B_t) \text{follow processes (2.3)-(2.4)}; \quad t \notin \mathcal{T} \\ W_i^+ = W_i^- + q_i\,; \; S_i^+ = p_i W_i^+\,; \; B_i^+ = W_i^+ - S_i^+\,; \quad t \in \mathcal{T} \\ p_i = p_i(W_i^+, t_i)\,; \; 0 \le p_i \le 1 \end{cases}, \tag{8.1}$$

where we use the notation $E^{\mathcal{P}}[\cdot]$ to emphasize that the expectation is computed using the control $\mathcal{P}$. We give a brief description of the algorithm used to solve problem (8.1) in Appendix C. Due to the leverage constraint imposed in equation (8.1), this optimization problem is well-posed without adding an additional funding level constraint on the terminal wealth (Gao et al., 2017).

Note that problem (8.1) is underspecified if $\kappa = 0$. By setting κ to a small positive number, e.g. $\kappa = 10^{-8}$, we can force the following strategy. Let W_α^* be the VAR at level α (see Appendix C). Along any path where we can achieve $W_T > W_\alpha^*$ with certainty by investing some amount in bonds, we then invest the remainder in stocks. More precisely, if

$$(W_i^- + q_i) > W_\alpha^* e^{-r(T-t_i)} - Q_i, \tag{8.2}$$

where Q_i is defined in equation (7.2), then the optimal strategy is to invest $W_\alpha^* e^{-r(T-t_i)} - Q_i$ in bonds and $(W_i^- + q_i) - W_\alpha^* e^{-r(T-t_i)} - Q_i$ in stocks. Effectively, we are maximizing CVAR_α (i.e. minimizing risk) with the tie-breaking strategy that if our wealth is large enough, then we invest the amount required to attain $W_T > W_\alpha^*$ in bonds and the excess in stocks. Conversely, if we set κ to a small negative number, then the optimal strategy along any path where equation (8.2) holds will be to switch all accumulated wealth to bonds.

Table 8.1 shows the results. In the synthetic market, $Median[W_T]$, $Pr[W_T < 0]$, and 5% CVAR are the same for both $\kappa = \pm 10^{-8}$, but $Mean[W_T]$ and $std[W_T]$ are dramatically different. This indicates that the large mean of terminal wealth for $\kappa = +10^{-8}$ is due to small probability paths with extremely large values of W_T. The bootstrap (i.e. historical market) results are generally worse than the synthetic market results, except for an expected blocksize of 5 years. The 5th, 50th, and 95th percentiles of W_T for the bootstrap tests are shown in Figure 8.1(a) for the case $\kappa = +10^{-8}$. Note the U-shape of the 95th percentile. This is due to the fact that on any path where the wealth satisfies equation (8.2), the optimal strategy is to invest the surplus in stocks since this will maximize expected terminal wealth. Contrast this with Figure 8.1(b), which shows the results when $\kappa = -10^{-8}$. Recall that this forces the strategy to invest in bonds along any path where the wealth satisfies equation (8.2).

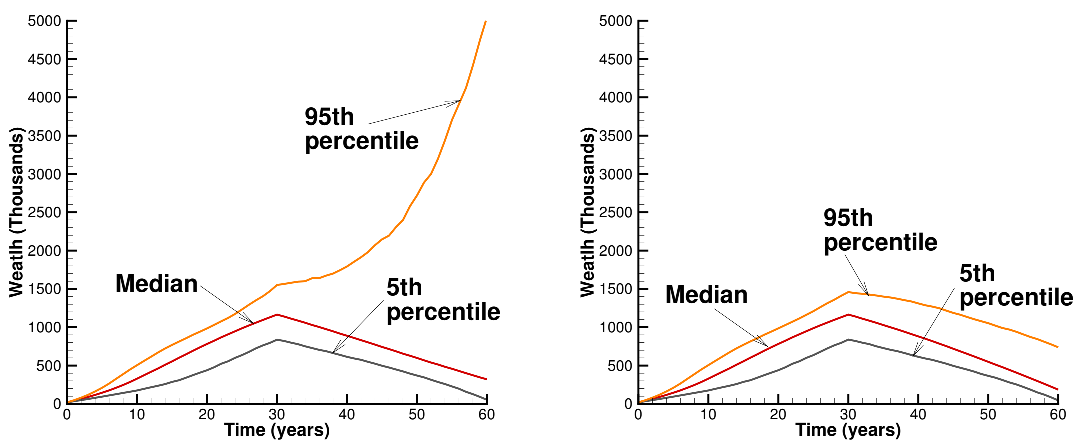
9 Quadratic shortfall with expected value constraint

By now it seems clear that directly minimizing a measure of the risk of ruin is not a good strategy, since the results are not very stable under the bootstrap tests. Even in the synthetic market tests,

[9] See Appendix C for a precise definition.

$\hat{b}$	κ	$Median[W_T]$	$Mean[W_T]$	$std[W_T]$	$Pr[W_T < 0]$	5% CVAR
			Synthetic market			
NA	10^{-8}	132	733	13844	.027	−185
NA	-10^{-8}	132	137	142	.027	−185
			Historical market			
0.5	10^{-8}	240	855	2957	.047	−283
0.5	-10^{-8}	165	193	182	.048	−285
1.0	10^{-8}	270	1053	2943	.046	−286
1.0	-10^{-8}	172	218	219	.048	−291
2.0	10^{-8}	320	1223	3343	.036	−184
2.0	-10^{-8}	186	259	253	.038	−189
5.0	10^{-8}	409	1434	3222	.024	−74
5.0	-10^{-8}	215	310	292	.025	−81

TABLE 8.1: *Optimal control determined by solving mean-CVAR problem* (8.1) *with* $\alpha = .05$ *in the synthetic market, assuming the scenario in Table 4.1.* W_T *denotes real terminal wealth after* 60 *years, measured in thousands of dollars. Statistics for the synthetic market cases are based on* 6.4×10^5 *Monte Carlo simulation runs. Statistics for the historical market cases are based on* 10,000 *stationary block bootstrap resamples of the historial data from 1926:1 to 2016:12.* $\hat{b}$ *is the expected blocksize, measured in years.* κ *specifies the asset allocation along paths where* $W_T > W^*_\alpha$ *with certainty; see equation* (8.2) *and accompanying discussion.*



(a) $\kappa = +10^{-8}$.

(b) $\kappa = -10^{-8}$.

FIGURE 8.1: *Percentiles of real wealth in the historical market. Optimal control determined by solving mean-CVAR problem* (8.1) *with* $\alpha = .05$ *and* $\kappa = \pm 10^{-8}$ *in the synthetic market, assuming the scenario in Table 4.1. Statistics based on* 10,000 *bootstrap resamples of the historical data from 1926:1 to 2016:12 with expected blocksize* $\hat{b} = 2$ *years.* κ *specifies the asset allocation along paths where* $W_T > W^*_\alpha$ *with certainty; see equation* (8.2) *and accompanying discussion.*

we can see that there is a very large cost incurred in terms of the median terminal wealth to reduce the probability of ruin by a small amount. It seems plausible to attempt to target a reasonable value of terminal wealth, and then to minimize the size of the shortfall. A natural candidate objective function in this case is minimizing the quadratic shortfall with respect to a target level of final wealth (W^*), as suggested in Menoncin and Vigna (2017). Writing this problem more formally:

$$\min_{\{(p_0,c_0),\ldots,(p_{M-1},c_{M-1})\}} E\left[(\min(W_T - W^*, 0))^2\right]$$

$$\text{subject to} \begin{cases} (S_t, B_t) \text{follow processes (2.3)-(2.4)}; \quad t \notin \mathcal{T} \\ W_i^+ = W_i^- + q_i - c_i\,;\ S_i^+ = p_i W_i^+\,;\ B_i^+ = W_i^+ - S_i^+\,; \quad t \in \mathcal{T} \\ p_i = p_i(W_i^+, t_i)\,;\ 0 \le p_i \le 1 \\ c_i = c_i(W_i^- + q_i, t_i)\,;\ c_i \ge 0 \end{cases} . \tag{9.1}$$

We can interpret problem (9.1) as minimizing the quadratic penalty for shortfall with respect to the target W^*. As in Section 7, we allow surplus cash withdrawals over and above the scheduled injections/withdrawals q_i. An optimal strategy is to withdraw

$$c_i = \max\left(W_i^- + q_i - \left(W^* e^{-r(T-t_i)} - Q_i\right), 0\right) \tag{9.2}$$

from the portfolio and invest the remainder in the bond index (Dang and Forsyth, 2016). Recall that Q_i is defined in equation (7.2). In addition, the following result due to Zhou and Li (2000) implies that problem (9.1) simultaneously minimizes two measures of risk: expected quadratic shortfall and variance.

Proposition 9.1 (Dynamic mean variance efficiency)**.** *The solution to problem* (9.1) *is multi-period mean variance optimal.*

Remark 9.1 (Time consistency)**.** *There is considerable confusion in the literature about pre-commitment mean-variance strategies. These strategies are commonly criticized for being time inconsistent (Basak and Chabakauri, 2010; Björk et al., 2014). However, the pre-commitment optimal policy can be found by solving problem* (9.1)*) using dynamic programming with a fixed* W^**, which is clearly time consistent. Hence, when determining the time consistent optimal strategy for problem* (9.1)*, we obtain the optimal mean variance pre-commitment solution as a by-product. Vigna (2017) and Menoncin and Vigna (2017) provide further insight into this. As noted by Cong and Oosterlee (2016), the pre-commitment strategy can be seen as a strategy consistent with a fixed investment target, but not with a risk aversion attitude. Conversely, a time consistent strategy has a consistent risk aversion attitude, but it is not consistent with respect to an investment target. We contend that consistency with a target is more useful for life cycle investment strategies.*

We determine W^* in problem (9.1) by enforcing the constraint

$$E[W_T] = W^{spec}. \tag{9.3}$$

Computationally, we do this by embedding problem (9.1) in a Newton iteration where we solve the equation $(E[W_T] - W^{spec}) = 0$ for W^*. Note that adjusting W^{spec} allows us to indirectly adjust $Median[W_T]$. We choose $W^{spec} = 1000$.[10] Our rationale for this choice is that it gives an average allocation to the stock index of about 0.42.[11] Moreover, it results in a median final wealth that is

[10]Recall that units are thousands of dollars, so this corresponds to real terminal wealth of $1,000,000.

[11]This is the time average of the median value of the equity weight p.

$\hat{b}$	$Median[W_T]$	$Mean[W_T]$	$std[W_T]$	$Pr[W_T < 0]$	5% CVAR
		Synthetic market			
NA	1123	1032	354	.042	−377
		Historical market			
0.5	1144	1096	344	.041	−345
1.0	1155	1134	334	.038	−311
2.0	1169	1198	290	.026	−112
5.0	1200	1280	234	.015	+154

TABLE 9.1: *Optimal control determined by solving problem* (9.1) *(quadratic shortfall) with* $E[W_T] =$ 1000 *(excluding surplus cash) in the synthetic market, assuming the scenario in Table 4.1.* W_T *denotes real terminal wealth after* 60 *years, measured in thousands of dollars. Statistics for the synthetic market case are based on* 6.4×10^5 *Monte Carlo simulations. Statistics for the historical market cases are based on* 10,000 *stationary block bootstrap resamples of the historical data from 1926:1 to 2016:12.* $\hat{b}$ *is the expected blocksize, measured in years. Surplus cash is included in the mean, median, CVAR, and probability of ruin, but excluded from the standard deviation.*

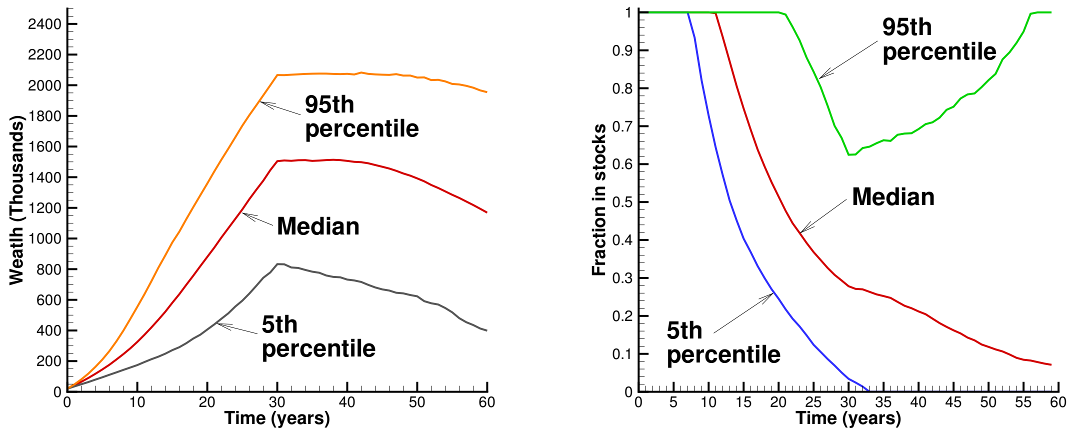
roughly comparable in the synthetic market to that seen earlier in Table 5.1 for the case with a constant equity weight of $p = 0.4$.[12]

Table 9.1 presents the results. Note that the constraint in equation (9.3) is the mean without surplus cash, while the means reported in this table include surplus cash. However, the average value of surplus cash is not very large ($1032 - 1000$ in the synthetic market). Unlike for the previous objective functions considered, in this quadratic shortfall case the results in the historical market are generally superior to those in the synthetic market.

Figure 9.1 shows the percentiles of the wealth (panel (a)) and the fraction invested in stocks (panel (b)) for the historical market with expected blocksize $\hat{b} = 2.0$ years. In Figure 9.1(a), the 5th percentile represents a very poor outcome. However, in this case there is still a reasonably large buffer of remaining wealth at the end of 60 years. Figure 9.1(b) shows that the optimal strategy for this quadratic shortfall objective starts out with 100% invested in the equity index over the first several years. If market returns are very favourable during that period, there will be a sharp fall in the equity fraction (e.g. the 5th percentile case), to the point of possibly being completely de-risked for the last 25 years of the 60 year horizon. The median case illustrates the same de-risking, but to a lesser extent (approximately 10% invested in the equity index over the last decade). On the other hand, the 95th percentile maintains the initial 100% allocation to equities for much longer, starts to de-risk, but then turns around with an increasing allocation to equities over approximately the last 25 years. It appears that withdrawals coupled with poor returns require higher equity exposures in order to reach the target.

Overall, it seems that these strategies, which can be interpreted as minimizing the expected quadratic shortfall with respect to a target, with an expected value constraint, are fairly robust. The ruin probabilities in the historical market are $Pr[W_T < 0] \simeq .03$ ($\hat{b} = 2$), which would certainly be acceptable in practice. Recall that in the synthetic market, the best possible strategy gives

[12] We experimented with other ways of specifying W^*. For example, rather than using the value which resulted in $E[W_T] = 1000$, we determined the value which minimized $Pr[W_T < 0]$. Although this looked promising in the synthetic market, its performance in the historical market tests was worse compared to the strategy which set $E[W_T] = 1000$.



(a) *Percentiles of accumulated wealth.* (b) *Percentiles of optimal fraction in equities.*

FIGURE 9.1: *Percentiles of real wealth and the optimal fraction invested in equities. Optimal control computed by solving the quadratic shortfall problem* (9.1) *with the constraint that* $E[W_T] = 1000$ *in the synthetic market, assuming the scenario in Table 4.1. Statistics based on* 10,000 *stationary block bootstrap resamples of the historical data from 1926:1 to 2016:12. Expected blocksize* $\hat{b} = 2$ *years.*

$Pr[W_T < 0] \simeq .02$. The quadratic shortfall strategies give up only a small amount in terms of probability of failure.[13] In return we have a good chance of a large bequest (or a safety buffer for longevity), i.e. $Median[W_t] > 1{,}000$.

10 Some alternative strategies

We now briefly discuss some other strategies which we have considered. First, we have tested strategies where we replace the objective function in the quadratic shortfall problem (9.1) by

$$\min_{\{(p_0,c_0),\ldots,(p_{M-1},c_{M-1})\}} E\left[(\min(W_T - W^*, 0))^\beta\right], \tag{10.1}$$

for powers $\beta \in \{1,3,4\}$, in addition to the $\beta = 2$ case considered in detail in Section 9. Similar results were obtained for all choices of β, with $\beta = 2$ having a slight edge.

Another target-based objective function has been recently suggested in Zhang et al. (2017). This is the *sharp target* objective. It seeks to maximize expected terminal wealth over a specified target range, where the upper end of the range corresponds to a wealth goal and the lower end represents a desired conservative minimum. We give a brief overview of our results using this objective function in Appendix D. This objective function produced results similar to the quadratic shortfall criteria, but with noticeably worse CVAR. Hence, it appears that the quadratic shortfall (expected value constraint) objective function discussed in Section 9 gives somewhat better overall results.

[13] Recall that the optimal strategy for minimizing $Pr[W_T < 0]$ was not very robust in terms of bootstrap stress tests.

Strategy	$Median[W_T]$	$Pr[W_T < 0]$	5% CVAR
Const. equity fraction $p = .40$	961	.15	−461
Const. equity fraction $p = .60$	2931	.07	−389
Const. equity fraction $p = .80$	6151	.054	−411
Minimize probability of ruin (Section 7) $\min Pr\left[W_T < W^{min}\right]$; $W^{min} = 200$	456	.030	−183
Mean-CVAR (Section 8) $\max E[\text{CVAR}_\alpha + \kappa W_T]$; $\alpha = .05$; $\kappa = +10^{-8}$	320	.036	−184
Sharp target (Appendix D) $W_L = 100, W_U = 1178$	1138	.031	−204
Quadratic shortfall (Section 9) $\min E\left[(\min(W_T - W^*,0))^2\right]$: W^* : $E[W_T] = 1000$	1169	.026	−112

TABLE 11.1: *Optimal controls determined by solving for strategies in the synthetic market, assuming the scenario in Table 4.1. Reported results use these controls in the historical market and are based on* 10,000 *stationary block bootstrap resamples of the historical data from 1926:1 to 2016:12 with expected blocksize* $\hat{b} = 2$ *years.* W_T *denotes real terminal wealth after* 60 *years, measured in thousands of dollars. Surplus cash is included in the median terminal wealth, where applicable.*

11 Comparison of strategies

Table 11.1 compares the results for several of the strategies discussed earlier.[14] This comparison is in the historical market, with an expected blocksize of $\hat{b} = 2$ years. The focus is on median terminal wealth (since mean terminal wealth can be misleading due to a small number of simulated paths with extreme results), as well as the two risk measures which we view as most important in this context: ruin probability and 5% CVAR. Table 11.1 shows that in terms of minimizing risk, the quadratic shortfall objective function with an expected value constraint from Section 9 seems to be superior to the other objective functions. It also offers a relatively high median terminal wealth. It is outperformed significantly on this dimension by the constant equity fraction strategies with $p = 0.60$ and $p = 0.80$, but these constant weight strategies also have much higher risk exposures.

Figure 11.1 plots kernel-smoothed probability densities of terminal wealth W_T in the historical market for the three constant weight strategies and the quadratic shortfall strategy from Table 11.1. This figure highlights some of the differences between the simpler constant weight approaches and the quadratic shortfall strategy. This latter strategy clearly sacrifices a lot of upside potential in exchange for downside protection, concentrating the wealth distribution in a narrow range, compared to the constant weight cases.

If we are concerned that too much upside is sacrificed for the quadratic shortfall method, we can try using a higher expected value constraint. Suppose, for example, that we target $E[W_T] = 2500$ in the synthetic market. Then in the historical bootstrap market ($\hat{b} = 2$ years), we obtain $Median[W_T] = 2961$, which is approximately the median obtained for the constant weight $p = .6$ case in Table 11.1. The quadratic shortfall risk measures in this case are $Pr[W_T < 0] = .04$, and 5% CVAR$= -331$. These results are still superior to the constant weight $p = .6$ case, but the

[14]Table 11.1 excludes some strategies which performed relatively poorly, such as minimizing the probability of ruin with $W^{min} = 0$ and the mean-CVAR strategy with $\kappa = -10^{-8}$.

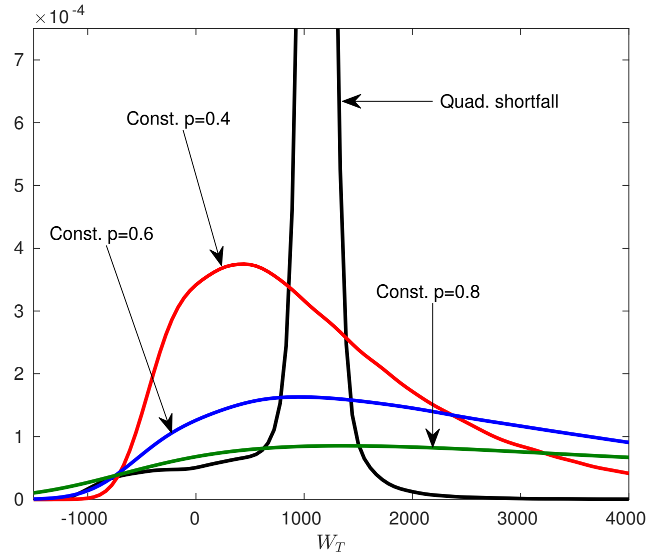


FIGURE 11.1: *Kernel smoothed probability densities for three constant weight strategies and the quadratic shortfall strategy, assuming the scenario in Table 4.1. Densities based on stationary block bootstrap resampling of the historical data from 1926:1 to 2016:12 with an expected blocksize of* $\hat{b} = 2.0$ *years.* W_T *denotes real terminal wealth after* 60 *years, measured in thousands of dollars. The quadratic shortfall method enforces the constraint that* $E[W_T] = 1000$ *in the synthetic market used to determine the optimal control for that strategy.*

quadratic shortfall strategy has to maintain a relatively high allocation to equities in order to hit the expected value target, so that there is less freedom to reduce risk.

As an additional stress test, we consider a case where the optimal strategy was computed with the historical parameters, but, going forward, the stock returns are reduced by 200 basis points per year relative to the historical average. Obviously all strategies in this case are adversely affected, but the quadratic shortfall strategy computed using incorrect parameter estimates is still superior to the constant weight strategies.

12 Conclusion

DC pension plan holders generally have no choice but to invest in risky assets in order to achieve even minimal salary replacement levels. We make the conservative assumption that the DC plan holder requires fixed cash flows for 30 years after retirement, after an accumulation period of 30 years. We also assume that the holder does not choose to annuitize, which is consistent with observed behaviour.

Our main result is that an objective function which focuses purely on a risk measure such as minimizing the probability of ruin or maximizing CVAR[15] performs well in a synthetic market, but poorly in bootstrap backtests (the historical market). The main problem here seems to be that these strategies are not robust due to real interest rate shocks introduced by the resampling process.

[15] Our definition of CVAR is mean of the worst α fraction of terminal wealth, not the losses, so we want to maximize CVAR to minimize risk.

In addition, even in the synthetic market, we observe that the small decreases in the probability of ruin come at the cost of drastically reducing the median terminal wealth (i.e. a bequest or an additional longevity safety valve). Greater robustness is achieved by targeting a final wealth greater than zero, which acts as a buffer against uncertainties in market parameters.

Minimizing the quadratic shortfall with an expected terminal wealth constraint appears to be a good strategy in general, as long as the expected terminal wealth constraint is sufficiently large to buffer the real interest rate shocks. This method results in an acceptable probability of ruin, and a significant median terminal wealth. This strategy is also robust to the misspecification of the drift of the risky asset, and is superior (by most measures) to standard constant weight strategies. However, this approach requires some experimentation in order to set the expected terminal wealth constraint appropriately.

It is interesting to observe that a robust strategy involves aiming for a significant size of terminal wealth (which may turn out to be a bequest) in order to have a small probability of ruin. In this instance, the investor and her heirs are likely to agree on the strategy.

We would like to emphasize that it is important to stress test any strategy, e.g. by bootstrapping the historical data. Some strategies which appear to work very well in the synthetic market fail in the bootstrap stress tests. However, we believe that our tests point the way to some promising choices of objective function for full life cycle DC plan asset allocation.

Any strategy which involves investing in risky assets to meet fixed cash flows has a non-zero probability of portfolio depletion before the horizon date. The best that can be done is to make this probability acceptably small. Nevertheless, failure can occur, which begs the question of what happens then. A possible backup in many cases would be the use of the retiree's other assets, such as real estate. For example, it may be possible to use a reverse mortgage to monetize the retiree's home. As long as the value of any real estate asset is larger than (the negative of) the 5% CVAR, then we can regard the real estate asset as at least a partial hedge against the tail risk.

We have restricted attention in this paper to requiring a fixed (real) withdrawal during the decumulation phase. Another alternative is to allow the withdrawal to vary in response to the then current portfolio value, based on an estimate of remaining lifetime. This shifts the risk from portfolio depletion to volatile decumulation cash flows (Waring and Siegel, 2015; Westmacott and Daley, 2015). In this case, the control problem objective function would be to minimize the withdrawal volatility and maximize the cumulative withdrawals. We intend to study this approach in the future.

Acknowledgements

P. A. Forsyth's work was supported by the Natural Sciences and Engineering Research Council of Canada (NSERC).

Conflicts of interest

The authors have no conflicts of interest to report.

μ	σ	λ	p_{up}	η_1	η_2
.08753	.14801	.34065	.25806	4.67877	5.60389

TABLE A.1: *Estimated annualized parameters for the double exponential jump diffusion model given in equation* (2.1) *applied to the value-weighted CRSP Deciles (1-10) index, deflated by the CPI. Sample period 1926:1 to 2016:12. CRSP and CPI data from Dimensional Returns 2.0 under licence from Dimensional Fund Advisors Canada.*

Appendices

A Calibration of model parameters

To estimate the jump diffusion model parameters, we use the thresholding technique described in Mancini (2009) and Cont and Mancini (2011). This procedure is considered to be relatively efficient for fairly low frequency data, such as the monthly frequency used here. For details, see Dang and Forsyth (2016) and Forsyth and Vetzal (2017a). We use a threshold parameter $\alpha = 3$ in our estimates.[16]

Table A.1 provides the resulting annualized parameter estimates for the double exponential jump diffusion given in equation (2.1). The drift rate μ corresponds to an expected annual return of almost 9%. The diffusive volatility σ might seem slightly low at less than 15%, but recall that the overall effective volatility includes this amount plus the contribution to volatility from jumps. The jump intensity λ implies that jumps can be expected to occur approximately every 3 years. When a jump happens, it is about 3 times more likely to be a move down than a move up. Upward jumps are a little larger on average than downward jumps.

Figure A.1 shows the normalized histogram of real CRSP Deciles (1-10) index log returns for the period 1926:1-2016:12. The standard normal density and scaled jump diffusion density are also shown. The improved fit from the jump diffusion model is readily apparent.

The historical average annualized real interest rate for one-month US T-bills from 1926:1 to2016:12 was $r = 0.004835$. The volatility of the one-month T-bill return was about .018, which justifies ignoring the randomness of short term interest rates, at least as a first approximation. We test the effect of this assumption on optimal strategies by applying the computed strategies to the historical market, which is constructed using bootstrap resamples of the data series and so includes the effect of stochastic real interest rates.

B Bootstrap resampling

We use bootstrap resampling to study how the various strategies would have performed on actual historical data. A single bootstrap resampled path can be constructed as follows. Divide the total investment horizon of T years into k blocks of size b years, so that $T = kb$. We then select k blocks at random (with replacement) from the historical data (from both the deflated equity and T-bill indexes). Each block starts at a random month. A single path is formed by concatenating these blocks. The historical data is wrapped around to avoid end effects, as in the circular block bootstrap (Politis and White, 2004; Patton et al., 2009). This procedure is then repeated for many paths.

[16]This parameter has the intuitive interpretation that if the absolute value of the log return in a period is larger an α standard deviation Brownian motion return, then it is identified as a jump.

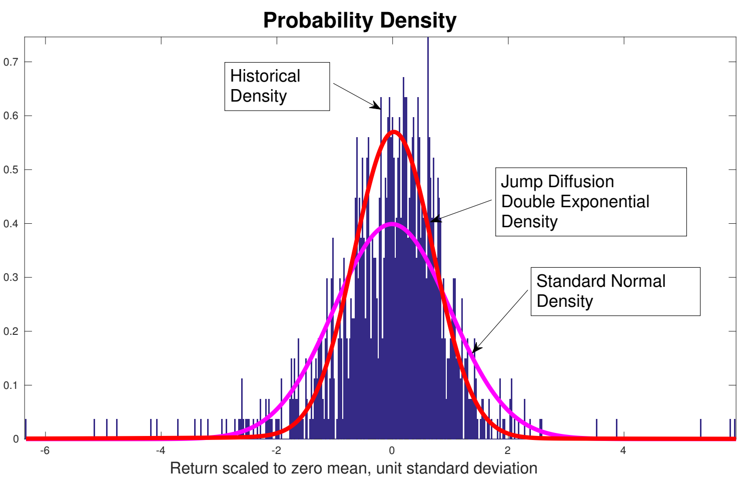


FIGURE A.1: *Histogram of real log returns of CRSP Deciles (1-10) index, scaled to zero mean and unit standard deviation. Standard normal density shown, as well as the fitted jump diffusion, double exponential distribution, also scaled to zero mean and unit standard deviation. Jump diffusion parameters: threshold ($\alpha = 3$) from Table A.1. Sample period 1926:1 to 2016:12. Source: CRSP and CPI data from Dimensional Returns 2.0 under licence from Dimensional Fund Advisors Canada.*

The sampling is done in blocks in order to account for possible serial dependence effects in the historical time series. The choice of blocksize is crucial and can have a large impact on the results (Cogneau and Zakalmouline, 2013). We simultaneously sample the real stock and bond returns from the historical data. This introduces random real interest rates in our samples, in contrast to the constant interest rates assumed in the synthetic market tests and in the determination of the optimal controls.

To reduce the impact of a fixed blocksize and to mitigate the edge effects at each block end, we use the stationary block bootstrap (Politis and White, 2004). The blocksize is randomly sampled from a geometric distribution with an expected blocksize $\hat{b}$. The optimal choice for $\hat{b}$ is determined using the algorithm described in Patton et al. (2009).[17] Calculated optimal values for $\hat{b}$ were 57 months for the T-bill index and 3.5 months for the real CRSP index. We adopt a paired sampling approach whereby we sample simultaneously from both stock and bond indexes, so we must use the same blocksize for both indexes. Since the recommended blocksizes are quite different for the two indexes, we sidestep this issue by presenting results for a range of blocksizes.

C Definition of CVAR

Let $p(W_T)$ be the probability density function of wealth at $t = T$. Let

$$\int_{-\infty}^{W_\alpha^*} p(W_T)\, dW_T = \alpha, \tag{C.1}$$

i.e. $Pr[W_T > W_\alpha^*] = 1 - \alpha$. We can interpret W_α^* as the Value at Risk (VAR) at level α. The Conditional Value at Risk (CVAR) at level α is then

$$\text{CVAR}_\alpha = \frac{\int_{-\infty}^{W_\alpha^*} W_T\, p(W_T)\, dW_T}{\alpha}, \tag{C.2}$$

[17] This approach has also been used in other tests of portfolio allocation problems recently (e.g. Dichtl et al., 2016).

which is the average of the worst α fraction of outcomes. Typically $\alpha = .01, .05$. Note that the definition of CVAR in equation (C.2) uses the probability density of the final wealth distribution, not the density of *loss*. Hence, in our case, a larger value of CVAR (i.e. a larger value of worst case terminal wealth) is desired. In our examples, we have both positive and negative values of CVAR.

Given an expectation under control $\mathcal{P}$, $E^{\mathcal{P}}[\cdot]$, as noted by Rockafellar and Uryasev (2000) and Miller and Yang (2017), the mean-CVAR optimization can be expressed as

$$\max_{\mathcal{P}} \sup_{W^*} E^{\mathcal{P}}\bigg(W^* + \frac{1}{\alpha}\left[(W_T - W^*)^-\right] + \kappa W_T\bigg). \tag{C.3}$$

Following Miller and Yang (2017), we interchange the max and sup operations in equation (C.3), which allows us to rewrite the objective function (C.3) as

$$\sup_{W^*}\bigg\{\max_{\mathcal{P}} E^{\mathcal{P}}\bigg(W^* + \frac{1}{\alpha}\left[(W_T - W^*)^-\right] + \kappa W_T\bigg)\bigg\}, \tag{C.4}$$

and solve the inner optimization problem using an HJB equation (Dang and Forsyth, 2014; Forsyth and Labahn, 2017). Standard methods can then be used to solve the outer optimization problem.

D Sharp target

Another possible objective function is the *sharp target* suggested in Zhang et al. (2017):

$$\max_{\{(p_0,c_0),\ldots,(p_{M-1},c_{M-1})\}} E\left[(W_T - W_L)\mathbf{1}_{W_L \le W_T < W_U}\right]$$

$$\text{subject to } \begin{cases} (S_t, B_t) \text{ follow processes (2.3)-(2.4)}; \quad t \notin \mathcal{T} \\ W_i^+ = W_i^- + q_i - c_i\,;\ S_i^+ = p_i W_i^+\,;\ B_i^+ = W_i^+ - S_i^+\,; \quad t \in \mathcal{T} \\ p_i = p_i(W_i^+, t_i)\,;\ 0 \le p_i \le 1 \\ c_i = c_i(W_i^- + q_i, t_i)\,;\ c_i \ge 0 \end{cases}, \tag{D.1}$$

where W_L, W_U are parameters. We can think of W_L as a minimum required value of the final wealth and W_U as the desired value. We withdraw cash from the portfolio if investing the remaining amount in the risk-free asset (along any given path) ensures that $W_T > W_U$. The surplus (withdrawn amount) is also invested in the risk-free asset. Note that we have to specify what rule to use if a risk-free investment results in $W_T > W_U$, since otherwise the problem is not fully specified.

The idea of objective (D.1) is to reward outcomes between $W_L < W_T < W_U$, with higher reward for outcomes near W_U. There is no reward for outcomes $W_T > W_U$. A possible problem is that all outcomes $W_T < W_L$ are penalized equally. To be comparable with the results in Section 9 (quadratic shortfall with expected value constraint), we fix $W_L = 100$ and determine W_U so that $E[W_T] = 1000$ in the synthetic market. This gives $W_U = 1178$.

The results for the sharp target strategy are shown in Table D.1. Comparing the historical market results from this table with those for the quadratic shortfall strategy in Table 9.1, we see that the sharp target gives similar results, except that the 5% CVAR is notably worse. This can be traced the the fact that all shortfalls below W_L are weighted equally in the sharp target objective, while larger shortfalls are increasingly penalized with the quadratic shortfall objective function.

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$\hat{b}$	$Median[W_T]$	$Mean[W_T]$	$std[W_T]$	$Pr[W_T < 0]$	5% CVAR
		Synthetic market			
NA	1102	1023	372	.047	−507
		Historical market			
0.5	1108	1061	365	.045	−449
1.0	1120	1100	357	.042	−413
2.0	1138	1171	315	.031	−204
5.0	1177	1265	251	.018	+156

TABLE D.1: *Optimal control determined by solving problem* (D.1) *with* $W_L = 100$ *and* $W_U = 1178$, *so that* $E[W_T] = 1000$ *(excluding surplus cash) in the synthetic market, assuming the scenario in Table 4.1.* W_T *denotes real terminal wealth after* 60 *years, measured in thousands of dollars. Statistics for the synthetic market case are based on* 6.4×10^5 *Monte Carlo simulation runs. Statistics for the historical market cases are based on* 10,000 *stationary block bootstrap resamples of the historical data from 1926:1 to 2016:12.* $\hat{b}$ *is the expected blocksize, measured in years. Surplus cash is included in the mean, median, CVAR, and probability of ruin, but excluded from the standard deviation.*

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