Computational Finance: Hedging Your Bets

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Introduction

- Derivative contracts are used by firms and individuals as protection against risk—they act as financial insurance.

- For example, a call option gives its holder the right, but not the obligation, to buy some underlying asset at a future time T for a fixed price K, regardless of its price S_T at that time. Payoff: $\max(S_T - K, 0)$.

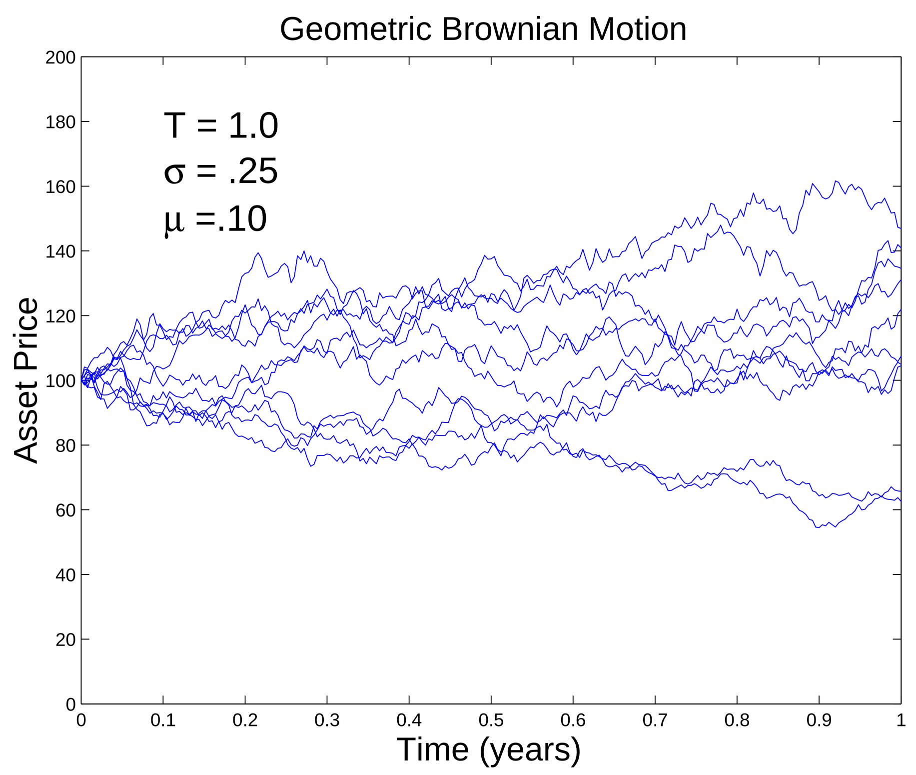
- The issuer (e.g. a bank) needs (1) to determine the price at which to sell the option; and (2) a way to *hedge* their risk associated with the contract.

The Black–Scholes Model

- A basic model for the evolution of the asset price S through time is geometric Brownian motion (GBM):

$$\frac{dS}{S} = \mu\, dt \;+\; \sigma\, \phi \sqrt{dt}$$

$\mu =$ drift rate
$\sigma =$ volatility
$\phi =$ random draw from N(0,1)



- The price $V(S,t)$ of an option based on the underlying asset S is found by solving a partial differential equation (PDE):

$$\frac{\partial V}{\partial t} \;+\; \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \;+\; rS\frac{\partial V}{\partial S} \;-\; rV = 0$$

- In the Black–Scholes model, the bank can perfectly hedge their position by owning $\frac{\partial V}{\partial S}$ units of the stock: this is known as delta hedging.

What's Wrong with GBM?

- Equity return data suggests the market has *jumps* (i.e. sudden discontinuous changes in the asset price) in addition to GBM.

- Example: A Drug Company

This is not GBM!

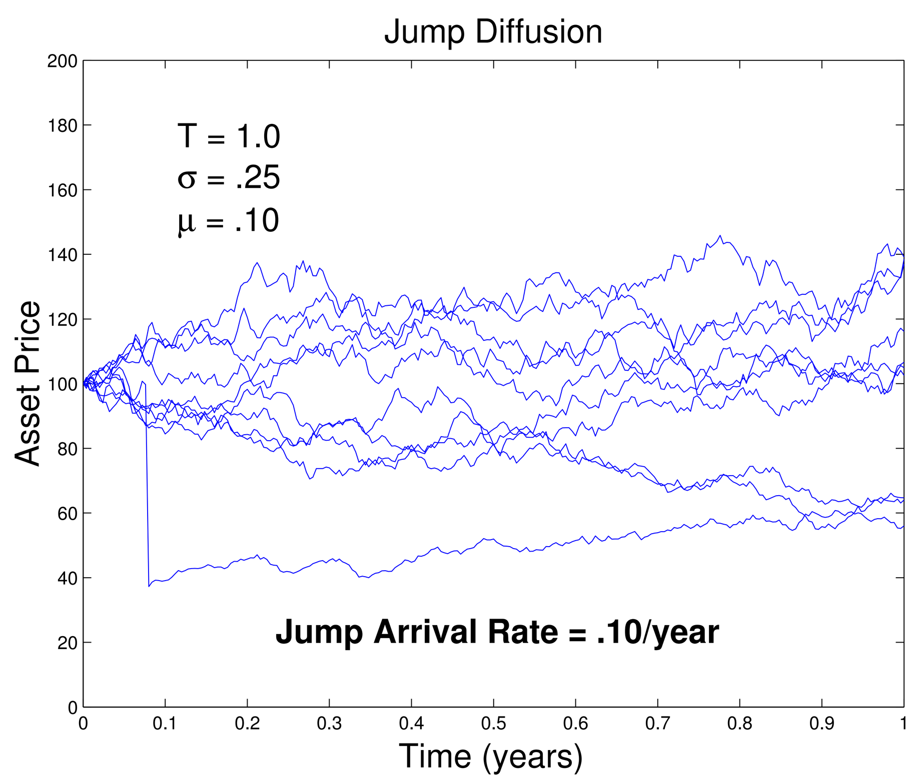
The Jump-Diffusion Model

- A new process to model jumps:

$$\frac{dS}{S} = \mu\, dt \;+\; \sigma\, \phi \sqrt{dt} \;+\; (J-1)\, dq$$

$J =$ random jump size
$\lambda =$ mean arrival rate of Poisson jumps: $S \rightarrow JS$

$$dq = \begin{cases} 0 & \text{with probability } 1 - \lambda dt \\ 1 & \text{with probability } \lambda dt \end{cases}$$



- The jump-diffusion model more accurately captures the properties of the options market.

- The option value is obtained by solving a partial integro-differential equation (PIDE). In addition, perfect hedging can only be accomplished by using an infinite number of hedging instruments.

A New Hedging Strategy

- Hedge using a portfolio that contains the underlying asset and a practical number of calls/puts with different characteristics.

- At each hedge rebalance date, choose portfolio weights to minimize an objective function of the form

$$\xi\Big(\text{Money Lost Due to Transaction Costs}\Big)^2 + (1-\xi)\Big(\text{Random Change Due to Jumps}\Big)^2.$$

- Choose $\xi \in [0,1]$ to minimize the standard deviation of the profit and loss (P&L) of the overall position at expiry.

- Carry out Monte Carlo simulations of the hedging strategy, assuming the underlying asset follows a jump-diffusion process and the option prices are given by the solution of the PIDE.

- Simulation results for hedging a long-term American put using short-term puts and calls:

Profit & Loss / Initial Option Price

Hedging Strategy	Mean	Standard Deviation	0.2% Quantile
Delta Hedge	0.05	1.05	-8.94
New Strategy (Eight Options)	-0.06	0.02	-0.14

- Conclusions for the new hedging strategy:
  - 99.8% of the time, we lose no more than 14% of initial option premium.
  - need to mark up the *perfect market* price by $\simeq 6\%$ to break even.

Hedge Fund Strategy

- Sell options and employ a delta hedge.

- In a real market with jumps, hedge fund managers make money most of the time and get big bonuses.

- Investors left with large unhedged jump losses!