

# Robust Numerical Methods for Contingent Claims under Jump Diffusion Processes

Y. d'Halluin\*, P.A. Forsyth†, and K.R. Vetzal‡

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## Abstract

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An implicit method is developed for the numerical solution of option pricing models where it is assumed that the underlying process is a jump diffusion. This method can be applied to a variety of contingent claim valuations, including American options, various kinds of exotic options, and models with uncertain volatility or transaction costs. Proofs of timestepping stability and convergence of a fixed point iteration scheme are presented. For typical model parameters, it is shown that the fixed point iteration reduces the error by two orders of magnitude at each iteration. The correlation integral is computed using a fast Fourier transform (FFT) method. Techniques are developed for avoiding wrap-around effects. Numerical tests of convergence for a variety of options are presented.

**Keywords:** Jump diffusion, implicit discretization, iterative solution

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## 1 Introduction

It is well known that the constant volatility Black-Scholes model cannot account for the “volatility smile” which is observed in market prices for contingent claims. One widely used method to account for the smile effect is to assume that the volatility is deterministic, but a function of asset price and time. This local volatility function is then determined by calibration to market prices (Dupire, 1994; Coleman et al., 1999; Andersen and Brotherton-Ratcliffe, 1998). This approach has been criticized because of overfitting and non-stationarity of parameters.

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\*Y. d'Halluin, School of Computer Science, University of Waterloo, Waterloo ON, Canada N2L 3G1 (e-mail: ydhallui@elora.uwaterloo.ca).

†P. A. Forsyth, School of Computer Science, University of Waterloo, Waterloo ON, Canada N2L 3G1 (e-mail: paforsyt@elora.uwaterloo.ca).

‡K. R. Vetzal, Centre for Advanced Studies in Finance, University of Waterloo, Waterloo ON, Canada N2L 3G1 (e-mail: kvetzal@watarts.uwaterloo.ca).

There is increasing empirical evidence that the usual assumption of geometric Brownian motion should be augmented by discontinuous jump processes (see Eraker et al., 2003, for a review). Such models were originally introduced in the option valuation context by Merton (1976). It is also possible to develop more complex valuation models which include stochastic volatility as well as jumps (Bates, 1996; Scott, 1997; Bakshi et al., 1997).

Andersen and Andreasen (2000) have recently staked out a middle ground by combining the deterministic volatility approach with lognormally distributed Poisson jumps with constant parameters. They argue that this alleviates many of the concerns, noting that “by letting the jump-part of the process dynamics explain a significant part of the volatility smile/skew, we generally obtain a ‘reasonable’, stable [deterministic volatility] function, without the extreme short-term variation typical of the pure diffusion approach” (Andersen and Andreasen, 2000, p. 233).

However, most of the existing methods for pricing options under jump processes are confined to vanilla European options. There has been very little work on numerical methods for pricing exotic or path-dependent options of practical significance (e.g. discretely observed barrier, lookback, and Asian options). Numerical techniques are also required when jumps are combined with non-constant local volatilities to calibrate models to observed prices of European options, as in the model of Andersen and Andreasen (2000).

In general, the valuation of a contingent claim under a jump diffusion process requires solving a partial integro-differential equation (PIDE). The method suggested by Amin (1993) is an explicit type approach based on multinomial trees. As is well-known, such methods have timestep limitations due to stability considerations, and are generally only first order correct. Zhang (1997) develops a method which treats the jump integral term in explicit fashion, and the remaining terms in the PIDE implicitly. Unfortunately, rather restrictive stability conditions are required. Meyer (1998) uses the method of lines to value American options where the underlying asset can jump to a finite number of states. More recently, a method based on use of a wavelet transform has been suggested by Matache et al. (2002). The basic idea is to use a wavelet transform to approximate the dense matrix discrete integral operator by dropping small terms. Andersen and Andreasen (2000) use an operator splitting type of approach combined with a fast Fourier transform (FFT) evaluation of a convolution integral to price European options with jump diffusion, where the diffusion terms involve non-constant local volatilities. However, an operator splitting approach cannot easily handle American options or nonlinear option valuation models (e.g. transaction costs or uncertain parameters, as discussed in Wilmott (1998) and references provided there).

The objective of this paper is to develop robust numerical methods for solving the option pricing PIDE which results from a jump diffusion model. Our technique is similar in some respects to Zhang (1997), though less constrained in terms of stability restrictions. Our method also offers a higher rate of convergence than Zhang’s. Similar comments apply if we compare our approach to that of Andersen and Andreasen (2000). For some simple cases, their approach might be slightly more efficient than ours, but we offer a more general purpose method which is capable of handling a much wider array of contracts. We confine our attention to the relatively simple case of compound Poisson jump diffusion processes for a single underlying stochastic variable, deferring the treatment of the more complicated cases of general Lévy processes and multiple state variables to future work. As in Andersen and Andreasen (2000), we do not assume constant coefficients for the diffusion part of the process. Note that we have recently become aware of a paper (Vazquez and Oosterlee, 2003) which is similar in spirit to the methods suggested in this work, although the details of the implementation differ.

The main results of this paper are as follows.

- We prove that the jump diffusion term can be discretized explicitly, and, when coupled with

a fully implicit treatment of the usual PDE, the resulting timestepping method is unconditionally stable.

- We prove that a simple fixed point iteration scheme can be used to solve the discretized algebraic equations, and that this iteration is globally convergent. In fact, for typical values of the timestep size and Poisson arrival intensity, the  $l_\infty$  error is reduced by two orders of magnitude at each iteration.
- We also develop a method for efficiently computing the jump integral term. We make no assumptions about the probability density for the jump term. This general approach requires the evaluation of correlation type integrals, as in Zhang (1997). We also show how to eliminate the wrap-around effects which often plague FFT methods. The correlation integral term can be rapidly computed using FFT methods.
- In contrast with previous work, we do not assume that the grid is equally spaced in either the underlying asset price or its logarithm. This is a major advantage for the pricing of contracts with barrier provisions, which typically require a fine grid spacing near barriers in order to achieve sufficient accuracy.

A major advantage of the method developed here is that it is straightforward to add a jump process to existing option pricing software. In particular, existing software that uses an implicit approach for valuing American options can be simply modified to price American options with jump diffusion. A variety of exotic and path-dependent contracts can be handled in a straightforward way, and nonlinear models such as transaction costs or uncertain parameters (Pooley et al., 2003) can also be easily extended to the jump diffusion case. In this paper, we include numerical examples for pricing European, American and Parisian options.

## 2 The Basic Model

This section presents the model for the evolution of the price of the underlying asset and the general form of the PIDE to be solved for option valuation. Let  $S$  represent the underlying stock price. Movements in this variable over time are assumed to be described by a jump diffusion process of the form

$$\frac{dS}{S} = \nu dt + \sigma dz + (\eta - 1)dq, \quad (2.1)$$

where  $\nu$  is the drift rate,  $\sigma$  is the volatility associated with the continuous (Brownian) component of the process,  $dz$  is the increment of a Gauss-Wiener process,  $dq$  is a Poisson process which is assumed to be independent of the Brownian part (note that  $dq = 0$  with probability  $1 - \lambda dt$  and  $dq = 1$  with probability  $\lambda dt$ , where  $\lambda$  is the Poisson arrival intensity), and  $\eta - 1$  is an impulse function producing a jump from  $S$  to  $S\eta$ . We denote the expected relative jump size by  $\kappa = E(\eta - 1)$ .

Under equation (2.1), the stock price  $S$  has two sources of uncertainty. The term  $\sigma dz$  corresponds to normal levels of uncertainty while the term  $dq$  describes exceptional events. If the Poisson event does not occur ( $dq = 0$ ), then equation (2.1) is equivalent to the usual stochastic process of geometric Brownian motion assumed in the Black-Scholes model (with the additional assumption that  $\sigma$  is constant). If, on the other hand, the Poisson event occurs, then equation (2.1) can be written as

$$\frac{dS}{S} \simeq (\eta - 1), \quad (2.2)$$

where  $\eta - 1$  is an impulse function producing a jump from  $S$  to  $S\eta$ . Consequently, the resulting sample path for the stock  $S$  will be continuous most of the time with finite negative or positive jumps with various amplitudes occurring at discrete points in time.

Let  $V(S, t)$  be the value of a contingent claim that depends on the underlying stock price  $S$  and time  $t$ . As is well-known, the following backward PIDE may be solved to determine  $V$ :

$$V_\tau = \frac{1}{2}\sigma^2 S^2 V_{SS} + (r - \lambda\kappa)SV_S - rV + \left( \lambda \int_0^\infty V(S\eta)g(\eta)d\eta - \lambda V \right), \quad (2.3)$$

where  $\tau = T - t$  is the time until expiry at date  $T$ ,  $r$  is the continuously compounded risk free interest rate, and  $g(\eta)$  is the probability density function of the jump amplitude  $\eta$  with the obvious properties that  $\forall \eta, g(\eta) \geq 0$  and  $\int_0^\infty g(\eta)d\eta = 1$ . An important special case is where  $\sigma$  is constant and the jump size distribution is lognormal, this being the well-known model of Merton (1976). For brevity, the details of the derivation of equation (2.3) have been omitted (for further details, see Merton, 1976; Wilmott, 1998; Andersen and Andreasen, 2000, among others). For future convenience, note that equation (2.3) can be rewritten in slightly different form as

$$V_\tau = \frac{1}{2}\sigma^2 S^2 V_{SS} + (r - \lambda\kappa)SV_S - (r + \lambda)V + \lambda \int_0^\infty V(S\eta)g(\eta)d\eta. \quad (2.4)$$

**Remark 2.1 (Viscosity solution).** *In what follows it will be understood that we are seeking weak viscosity solutions (Crandall et al., 1992) to equation (2.4). More details concerning the existence and uniqueness of solutions to (2.4) are discussed in Pham (1998), Briani et al. (2003), and Amadori (2000). A proof of the convergence of an explicit method to the viscosity solution is given in Briani et al. (2003).*

## 2.1 Boundary Conditions

As  $S \rightarrow 0$ , equation (2.3) reduces to

$$V_\tau = -rV. \quad (2.5)$$

As  $S \rightarrow \infty$ , we make the common assumption that

$$V_{SS} \simeq 0 \quad ; \quad S \rightarrow \infty \quad (2.6)$$

which means that

$$V \simeq A(\tau)S + B(\tau) \quad ; \quad S \rightarrow \infty. \quad (2.7)$$

Assuming equation (2.7) holds, then equation (2.3) reduces to

$$V_\tau = \frac{1}{2}\sigma^2 S^2 V_{SS} + rSV_S - rV \quad ; \quad S \rightarrow \infty. \quad (2.8)$$

Consequently, at both  $S = 0, S \rightarrow \infty$ , the PIDE (2.4) reduces to the Black-Scholes PDE, and the usual boundary conditions can be imposed. For example, if a finite computational domain  $[0, S_{max}]$  is used, then Dirichlet conditions can be imposed at  $S = S_{max}$ . The Dirichlet condition is determined by substituting equation (2.7) into equation (2.8), giving ordinary differential equations for  $A(\tau), B(\tau)$ . The initial conditions for  $A(\tau), B(\tau)$  are given from the option payoffs.

### 3 Implicit Discretization Methods

This section explores discretization methods for the PIDE, where the terms not involving the jump integral are handled implicitly. A straightforward approach to the numerical solution of equation (2.4) would be to use standard numerical discretization methods for the non-integral terms (as described, for example in Tavella and Randall, 2000), in combination with numerical integration methods such as Simpson's rule or Gaussian quadrature. However, such an approach is computationally expensive, as noted by Tavella and Randall. It is more efficient to transform the integral in equation (2.4) into a correlation integral. This allows efficient FFT methods to be used to evaluate the integral for all values of  $S$ .

Let

$$I(S) = \int_0^\infty V(S\eta)g(\eta)d\eta. \quad (3.1)$$

Setting  $x = \log(S)$  and using the change of variable  $y = \log(\eta)$  gives

$$I = \int_{-\infty}^\infty \bar{V}(x+y)\bar{f}(y)dy. \quad (3.2)$$

where  $\bar{f}(y) = g(e^y)e^y$  and  $\bar{V}(y) = V(e^y)$ . Note that  $\bar{f}(y)$  is the probability density of a jump of size  $y = \log(\eta)$ . Equation (3.2) corresponds to the correlation product  $\otimes$  of  $\bar{V}(y)$  and  $\bar{f}(y)$ , so we can write (3.2) more succinctly as

$$I = \bar{V} \otimes \bar{f}. \quad (3.3)$$

We can write the correlation integral (3.2) in discrete form as

$$I_i = \sum_{j=-N/2+1}^{j=N/2} \bar{V}_{i+j}\bar{f}_j\Delta y + O((\Delta y)^2), \quad (3.4)$$

where  $I_i = I(i\Delta x)$ ,  $\bar{V}_j = \bar{V}(j\Delta x)$ ,

$$\bar{f}_j = \frac{1}{\Delta x} \int_{x_j-\Delta x/2}^{x_j+\Delta x/2} \bar{f}(x)dx, \quad (3.5)$$

and  $x_j = j\Delta x$ . Note that we have assumed that  $\Delta y = \Delta x$ , and that in (3.4)  $N$  is selected sufficiently large so that the solution in areas of interest is unaffected by the application of an asymptotic boundary condition for large values of  $S$ . In particular, we assume that  $\bar{V}_{N/2+j}, j > 0$  can be approximated by an asymptotic boundary condition. In practice, since  $\bar{f}_j$  decays rapidly for  $|j| > 0$ , this should not cause any difficulty. Also note that  $\bar{V}_{-N/2+j}, j < 0$ , can be interpolated from known values  $V_k$  since these points represent values near  $S = 0$ . An important property to note is that

$$\begin{aligned} \bar{f}_j &\geq 0, \quad \forall j \\ \sum_{j=-N/2+1}^{j=N/2} \bar{f}_j\Delta y &\leq 1. \end{aligned} \quad (3.6)$$

This follows because  $\bar{f}(y)$  is a probability density function and  $\bar{f}_j$  is defined by equation (3.5).

The discrete form of the correlation integral (3.4) uses an equally spaced grid in  $\log S$  coordinates. While this is convenient for a FFT evaluation of the correlation integral, it is not particularly

suitable for discretizing the PDE. We will use an unequally spaced grid in  $S$  coordinates for the PDE discretization  $[S_0, \dots, S_p]$ . Let

$$V_i^n = V(S_i, \tau_n). \quad (3.7)$$

Now,  $\bar{V}_j$  will not necessarily coincide with any of the discrete values  $V_k$  in equation (3.7). Consequently, we will linearly interpolate (using Lagrange basis functions defined on the  $S$  grid) to determine the appropriate values, i.e. if

$$S_{\Upsilon(j)} \leq e^{j\Delta x} \leq S_{\Upsilon(j)+1}, \quad (3.8)$$

then

$$\bar{V}_j = \psi_{\Upsilon(j)} V_{\Upsilon(j)} + (1 - \psi_{\Upsilon(j)}) V_{\Upsilon(j)+1} + O((\Delta S_{\Upsilon(j)+1/2})^2), \quad (3.9)$$

where  $\psi_{\Upsilon(j)}$  is an interpolation weight, and  $\Delta S_{i+1/2} = S_{i+1} - S_i$ . We are now faced with the problem that the integral  $I_i$  is evaluated at a point  $S = e^{x_i}$  which does not coincide with a grid point  $S_k$ . We simply linearly interpolate the  $I_i$  to get the desired value. If  $e^{x_{\Pi(k)}} \leq S_k \leq e^{x_{\Pi(k)+1}}$ , then

$$I(S_k) = \phi_{\Pi(k)} I_{\Pi(k)} + (1 - \phi_{\Pi(k)}) I_{\Pi(k)+1} + O((e^{x_{\Pi(k)}} - e^{x_{\Pi(k)+1}})^2), \quad (3.10)$$

where  $\phi_{\Pi(k)}$  is an interpolation weight. Note that

$$\begin{aligned} 0 &\leq \phi_i \leq 1 \\ 0 &\leq \psi_i \leq 1. \end{aligned} \quad (3.11)$$

Combining equations (3.4), (3.9), and (3.10) gives

$$I(S_k) = \sum_{j=-N/2+1}^{j=N/2} \chi(V, k, j) \bar{f}_j \Delta y, \quad (3.12)$$

where  $V = [V_0, V_1, \dots, V_p]'$  and

$$\begin{aligned} \chi(V, k, j) &= \phi_{\Pi(k)} [\psi_{\Upsilon(\Pi(k)+j)} V_{\Upsilon(\Pi(k)+j)} + (1 - \psi_{\Upsilon(\Pi(k)+j)}) V_{\Upsilon(\Pi(k)+j)+1}] \\ &+ (1 - \phi_{\Pi(k)}) [\psi_{\Upsilon(\Pi(k)+1+j)} V_{\Upsilon(\Pi(k)+1+j)} + (1 - \psi_{\Upsilon(\Pi(k)+1+j)}) V_{\Upsilon(\Pi(k)+1+j)+1}]. \end{aligned} \quad (3.13)$$

For future reference, note that  $\chi(V, k, j)$  is linear in  $V$ , and that if  $\iota = [1, 1, \dots, 1]'$ , then it follows from properties (3.11) that

$$\chi(\iota, k, j) = 1 \quad \forall k, j. \quad (3.14)$$

We can now consider the complete discretization of equation (2.4). The integral term is approximated using equation (3.12). We use a fully implicit method for the usual PDE, and then use a weighted timestepping method for the jump integral term. Letting  $V_i^n$  denote the solution at node  $i$  and time level  $n$ , the discrete equations can be written as

$$\begin{aligned} V_i^{n+1} [1 + (\alpha_i + \beta_i + r + \lambda) \Delta \tau] - \Delta \tau \beta_i V_{i+1}^{n+1} - \Delta \tau \alpha_i V_{i-1}^{n+1} \\ = V_i^n + (1 - \theta_J) \Delta \tau \lambda \sum_{j=-N/2+1}^{j=N/2} \chi(V^{n+1}, i, j) \bar{f}_j \Delta y + \theta_J \Delta \tau \lambda \sum_{j=-N/2+1}^{j=N/2} \chi(V^n, i, j) \bar{f}_j \Delta y. \end{aligned} \quad (3.15)$$

Note that  $\theta_J = 0$  corresponds to an implicit handling of the jump integral, whereas  $\theta_J = 1$  indicates an explicit treatment of this term.

Discretizing the first derivative term of equation (2.4) with central differences leads to

$$\begin{aligned}\alpha_{i,central} &= \frac{\sigma_i^2 S_i^2}{(S_i - S_{i-1})(S_{i+1} - S_{i-1})} - \frac{(r - \lambda\kappa)S_i}{S_{i+1} - S_{i-1}} \\ \beta_{i,central} &= \frac{\sigma_i^2 S_i^2}{(S_{i+1} - S_i)(S_{i+1} - S_{i-1})} + \frac{(r - \lambda\kappa)S_i}{S_{i+1} - S_{i-1}}.\end{aligned}\quad (3.16)$$

If  $\alpha_{i,central}$  or  $\beta_{i,central}$  is negative, oscillations may appear in the numerical solution. These can be avoided by using forward or backward differences at the problem nodes, leading to (forward difference)

$$\begin{aligned}\alpha_{i,forward} &= \frac{\sigma_i^2 S_i^2}{(S_i - S_{i-1})(S_{i+1} - S_{i-1})} \\ \beta_{i,forward} &= \frac{\sigma_i^2 S_i^2}{(S_{i+1} - S_i)(S_{i+1} - S_{i-1})} + \frac{(r - \lambda\kappa)S_i}{S_{i+1} - S_i},\end{aligned}\quad (3.17)$$

or (backward difference)

$$\begin{aligned}\alpha_{i,backward} &= \frac{\sigma_i^2 S_i^2}{(S_i - S_{i-1})(S_{i+1} - S_{i-1})} - \frac{(r - \lambda\kappa)S_i}{S_{i+1} - S_i} \\ \beta_{i,backward} &= \frac{\sigma_i^2 S_i^2}{(S_{i+1} - S_i)(S_{i+1} - S_{i-1})}.\end{aligned}\quad (3.18)$$

Algorithmically, we decide between a central or forward discretization at each node for equation (3.15) as follows:

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If [ $\alpha_{i,central} \geq 0$  and  $\beta_{i,central} \geq 0$ ] then
     $\alpha_i = \alpha_{i,central}$ 
     $\beta_i = \beta_{i,central}$ 
ElseIf [ $\beta_{i,forward} \geq 0$ ] then
     $\alpha_i = \alpha_{i,forward}$ 
     $\beta_i = \beta_{i,forward}$ 
Else
     $\alpha_i = \alpha_{i,backward}$ 
     $\beta_i = \beta_{i,backward}$ 
EndIf

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(3.19)

Note that the test condition (3.19) guarantees that  $\alpha_i$  and  $\beta_i$  are non-negative. For typical parameter values and grid spacing, forward or backward differencing is rarely required for single factor options. In practice, since this occurs at only a small number of nodes remote from the region of interest, the limited use of a low order scheme does not result in poor convergence as the mesh is refined. As we shall see, requiring that all  $\alpha_i$  and  $\beta_i$  are non-negative has important theoretical ramifications.

As  $S \rightarrow 0$ , equation (2.3) reduces to  $V_\tau = -rV$ , which is simply incorporated into the discrete equations (3.15) by setting  $\alpha_i = \beta_i = \lambda = 0$  at  $S_i = 0$ . In practice we truncate the  $S$  grid

at some large value  $S_p = S_{\max}$ , where we impose Dirichlet conditions. This is done by replacing equation (3.15) at  $S = S_{\max} = S_p$  with the specification that  $V_p^{n+1}$  is equal to the relevant Dirichlet condition.

We now proceed to consider the stability of the discretization (3.15). In particular, we have the following result:

**Theorem 3.1 (Stability of scheme (3.15)).** *The discretization method (3.15) is unconditionally stable for any choice of  $\theta_J, 0 \leq \theta_J \leq 1$ , provided that*

- $\alpha_i, \beta_i \geq 0$ ;
- the discrete probability density  $\bar{f}_j$  has the properties (3.6);
- the interpolation weights satisfy (3.11);
- $r, \lambda \geq 0$ .

*Proof.* Let  $V^n = [V_0^n, V_1^n, \dots, V_p^n]'$  be the discrete solution vector to equation (3.15). Suppose the initial solution vector is perturbed, i.e.

$$\hat{V}^0 = V^0 + E^0, \quad (3.20)$$

where  $E^n = [E_0^n, \dots, E_p^n]'$  is the perturbation vector. Note that  $E_p^n = 0$  since Dirichlet boundary conditions are imposed at this node. Then we obtain the following equation for the propagation of the perturbation (noting that  $\chi$  is a linear operator)

$$\begin{aligned} & E_i^{n+1} [1 + (\alpha_i + \beta_i + r + \lambda)\Delta\tau] - \Delta\tau\beta_i E_{i+1}^{n+1} - \Delta\tau\alpha_i E_{i-1}^{n+1} \\ &= E_i^n + (1 - \theta_J)\Delta\tau\lambda \sum_{j=-N/2+1}^{j=N/2} \chi(E^{n+1}, i, j)\bar{f}_j\Delta y + \theta_J\Delta\tau\lambda \sum_{j=-N/2+1}^{j=N/2} \chi(E^n, i, j)\bar{f}_j\Delta y. \end{aligned} \quad (3.21)$$

Defining

$$\|E\|^n = \max_i |E_i|^n, \quad (3.22)$$

it follows from properties (3.6), (3.11), and (3.14) and  $\alpha_i, \beta_i \geq 0$  that

$$\begin{aligned} |E_i^{n+1}| [1 + (\alpha_i + \beta_i + r + \lambda)\Delta\tau] &\leq \|E\|^n + (1 - \theta_J)\Delta\tau\lambda\|E\|^{n+1} + \theta_J\Delta\tau\lambda\|E\|^n \\ &\quad + \Delta\tau\beta_i|E_{i+1}^{n+1}| + \Delta\tau\alpha_i|E_{i-1}^{n+1}|. \end{aligned} \quad (3.23)$$

This implies

$$|E_i^{n+1}| [1 + (\alpha_i + \beta_i + r + \lambda)\Delta\tau] \leq (\Delta\tau\beta_i + \Delta\tau\alpha_i)\|E\|^{n+1} \quad (3.24)$$

$$+ \|E\|^n + (1 - \theta_J)\Delta\tau\lambda\|E\|^{n+1} + \theta_J\Delta\tau\lambda\|E\|^n. \quad (3.25)$$

Now, equation (3.25) is valid for all  $i < p$ . In particular, it is true for node  $i^*$ , where

$$\max_i |E_i^{n+1}| = |E_{i^*}^{n+1}|. \quad (3.26)$$

Writing equation (3.25) for  $i = i^*$  gives

$$\|E\|^{n+1} [1 + (r + \theta_J\lambda)\Delta\tau] = \|E\|^n (1 + \theta_J\Delta\tau\lambda), \quad (3.27)$$



and thus

$$\begin{aligned} \|E\|^{n+1} &\leq \|E\|^n \frac{(1 + \theta_J \Delta\tau \lambda)}{(1 + (r + \theta_J \lambda) \Delta\tau)} \\ &\leq 1. \quad \blacksquare \end{aligned} \tag{3.28}$$

**Remark 3.1 (Unconditional stability with explicit evaluation of the integral).** *This result is somewhat surprising, since we can discretize the correlation integral term explicitly ( $\theta_J = 1$ ), yet scheme (3.15) remains unconditionally stable. Note that Zhang (1997) derived a conditionally stable method. The conditional stability was a result of a slightly different timestepping approach compared to that in equation (3.15).*

**Remark 3.2 (Convergence to the viscosity solution).** *Following along the lines of Briani et al. (2003), it is straightforward to show that the discretization (3.15) is monotone and consistent. Since it is also unconditionally stable, the results of Barles (1997) then imply that our discretized solution converges to the viscosity solution.*

**Remark 3.3 (Extension to uncertain volatility/transaction costs).** *It is simple to extend scheme (3.15) to the case of a nonlinear model with uncertain volatility or transaction costs. Based on Remark 3.2 and the results in Pooley et al. (2003), it is then straightforward to show convergence to the viscosity solution.*

## 4 Crank-Nicolson Discretization

The discretization method used in the previous section is only first order correct in the time direction. In order to improve the timestepping error, we can use a Crank-Nicolson method. Such an approach results in the following set of discrete equations

$$\begin{aligned} V_i^{n+1} &\left[ 1 + (\alpha_i + \beta_i + r + \lambda) \frac{\Delta\tau}{2} \right] - \frac{\Delta\tau}{2} \beta_i V_{i+1}^{n+1} - \frac{\Delta\tau}{2} \alpha_i V_{i-1}^{n+1} \\ &= V_i^n \left[ 1 - (\alpha_i + \beta_i + r + \lambda) \frac{\Delta\tau}{2} \right] + \frac{\Delta\tau}{2} \alpha_i V_{i-1}^n + \frac{\Delta\tau}{2} \beta_i V_{i+1}^n \\ &\quad + (1 - \theta_J) \lambda \Delta\tau \sum_{j=-N/2+1}^{j=N/2} \chi(V^{n+1}, i, j) \bar{f}_j \Delta y + \theta_J \lambda \Delta\tau \sum_{j=-N/2+1}^{j=N/2} \chi(V^n, i, j) \bar{f}_j \Delta y. \end{aligned} \tag{4.1}$$

A full Crank-Nicolson method is obtained by setting  $\theta_J = 1/2$  in equation (4.1). If we define the matrix  $M$  such that

$$-[MV^n]_i = V_i^n (\alpha_i + \beta_i + r + \lambda) \frac{\Delta\tau}{2} - \frac{\Delta\tau}{2} \beta_i V_{i+1}^n - \frac{\Delta\tau}{2} \alpha_i V_{i-1}^n - \frac{\Delta\tau}{2} \lambda \sum_{j=-N/2+1}^{j=N/2} \chi(V^n, i, j) \bar{f}_j \Delta y, \tag{4.2}$$

then we can write equation (4.1) as

$$[I - M] V^{n+1} = [I + M] V^n. \tag{4.3}$$

Alternatively, we can define  $B = [I - M]^{-1} [I + M]$ , so that equation (4.3) can be written as

$$V^n = B^n V^0. \tag{4.4}$$

Consequently, an initial perturbation vector  $E^0$  will generate a perturbation at the  $n$ th step,  $E^n$ , given by  $E^n = B^n E^0$ .

The stability of the operator  $B$  is defined in terms of the power boundedness of  $B$ . If  $n$  is the number of timesteps and  $p$  is the number of grid nodes, then given some matrix norm  $\|\cdot\|$ , we say that  $B$  is *strictly stable* if

$$\|B^n\| \leq 1 \quad \forall n, p. \quad (4.5)$$

Following Giles (1997), *strong stability* is defined as

$$\|B^n\| \leq C \quad \forall n, p, \quad (4.6)$$

and *algebraic stability* is defined as

$$\|B^n\| \leq C n^s p^q \quad \forall n, m. \quad (4.7)$$

where  $C, s, q \geq 0$  are constants independent of  $n$  and  $p$ .

Algebraic stability is obviously a weaker condition than either strict or strong stability. Note that the Lax Equivalence Theorem states that strong stability is a necessary and sufficient condition for convergence for all initial data. Weaker algebraic stability yields convergence only for certain initial data. For a more detailed discussion of this, see Giles (1997).

If  $\mu_i$  are the eigenvalues of  $B$ , then a necessary condition for strong stability is that  $|\mu_i| \leq 1$ , and that any  $|\mu_i| = 1$  has multiplicity one. From equation (4.2) and properties (3.6), we have that

- The off-diagonals of  $M$  are all non-negative.
- The diagonals of  $M$  (excluding the last row) are strictly negative.
- Assuming that  $r > 0$ ,  $\sum_{j=0}^{j=p} M_{ij} < 0$  for  $i = 0, \dots, p-1$ .
- The last row of  $M$  is identically zero due to the Dirichlet boundary condition.

It then follows that all the Gerschgorin disks of  $M$  are strictly contained in the left half of the complex plane, with one eigenvalue identically zero. Hence all the eigenvalues of  $B$  are strictly less than one in magnitude, with one eigenvalue having modulus one. As a result,  $B$  satisfies the necessary conditions for strict stability.

However, since  $B$  is non-symmetric, this is not sufficient for power boundedness of  $B$  (see Borovikh and Spijker, 2000, for a counterexample). As discussed in Kraaijevanger et al. (1987) and Lenferink and Spijker (1991), we can guarantee algebraic stability by examining the  $\gamma$  numerical range of the matrix  $M$ . In the case  $\gamma = 1$ , the numerical range of  $M$  coincides with the convex hull of the Gerschgorin disks of  $M$  when the maximum norm is used in equation (4.7). These results can be summarized in the following theorem:

**Theorem 4.1 (Algebraic Stability of Crank-Nicolson Timestepping).** *The Crank-Nicolson discretization (4.1) is algebraically stable in the sense that*

$$\|B^n\|_\infty \leq C n^{1/2} \quad \forall n, p,$$

where  $C$  is independent of  $n, p$ .

*Proof.* Since all the Gerschgorin disks of  $M$  are in the left half of the complex plane, this follows from the results in Lenferink and Spijker (1991). ■

In fact, we believe that the algebraic stability estimate is overly pessimistic. For the case of constant coefficients with a log-spaced grid, in Appendix A we show using Von Neumann analysis that Crank-Nicolson timestepping with the correlation product is unconditionally strictly stable. However, it is interesting to note that if we use Crank-Nicolson weighting for the PDE terms and an explicit method for the jump diffusion term ( $\theta_J = 1$  in equation (4.1)), then a Von Neumann analysis shows that this method is only conditionally stable ( $\lambda\Delta\tau$  must be sufficiently small).

## 5 Fixed Point Iteration Method

When using an implicit discretization, it is computationally inefficient to solve the full linear system because the correlation product term makes the system dense. Consequently, in this section we will explore the use of a fixed point iteration to solve the linear system which results from an implicit discretization of the correlation product term. This idea was suggested in Tavella and Randall (2000), but no convergence analysis was given.

Define the matrix  $\hat{M}$  such that

$$-[\hat{M}V^n]_i = V_i^n(\alpha_i + \beta_i + r + \lambda)\Delta\tau - \Delta\tau\beta_i V_{i+1}^n - \Delta\tau\alpha_i V_{i-1}^n, \quad (5.1)$$

and the vector  $\Omega(V^n)$  such that

$$[\Omega(V^n)]_i = \sum_{j=-N/2+1}^{j=N/2} \chi(V^n, i, j) \bar{f}_j \Delta y. \quad (5.2)$$

Note that  $\Omega(V^n)$  is a linear function of  $V^n$ . Thus we can write a fully implicit ( $\theta = 0$ ) or Crank Nicolson ( $\theta = 1/2$ ) discretization as

$$[I - (1 - \theta)\hat{M}]V^{n+1} = [I + \theta\hat{M}]V^n + (1 - \theta)\lambda\Delta\tau\Omega(V^{n+1}) + \theta\lambda\Delta\tau\Omega(V^n). \quad (5.3)$$

We can then derive the fixed point iteration method as follows.

**Fixed Point Iteration**

Let  $(V^{n+1})^0 = V^n$   
 Let  $\hat{V}^k = (V^{n+1})^k$   
 For  $k = 0, 1, 2, \dots$  until convergence  
     Solve  
         
$$\begin{aligned} & [I - (1 - \theta)\hat{M}] \hat{V}^{k+1} \\ & = [I + \theta\hat{M}] V^n \\ & + (1 - \theta)\lambda\Delta\tau\Omega(\hat{V}^k) + \theta\lambda\Delta\tau\Omega(V^n) \end{aligned} \quad (5.4)$$
  
     If  $\max_i \frac{|\hat{V}_i^{k+1} - \hat{V}_i^k|}{\max(1, |\hat{V}_i^{k+1}|)} < \textit{tolerance}$  then quit  
     EndFor

Letting  $e^k = V^{n+1} - \hat{V}^k$ , the convergence of the fixed point scheme can be summarized in the following theorem:

**Theorem 5.1 (Convergence of the fixed point iteration).** *Provided that*

- $\alpha_i, \beta_i \geq 0$  (see Section 3);
- the discrete probability density  $\bar{f}_j$  has the properties (3.6);
- the interpolation weights satisfy (3.11);
- $r > 0, \lambda \geq 0$ ;

then the fixed point iteration (5.4) is globally convergent, and the maximum error at each iteration satisfies

$$\|e^{k+1}\|_\infty \leq \|e^k\|_\infty \frac{(1-\theta)\lambda\Delta\tau}{1+(1-\theta)(r+\lambda)\Delta\tau}.$$

*Proof.* It is easily seen from iteration (5.4) that  $e^k$  satisfies

$$\left[ I - (1-\theta)\hat{M} \right] e^{k+1} = (1-\theta)\lambda\Delta\tau\Omega(e^k). \quad (5.5)$$

Following the same steps used to prove Theorem 3.1, we therefore obtain

$$\begin{aligned} \|e^{k+1}\|_\infty &\leq \|e^k\|_\infty \frac{(1-\theta)\lambda\Delta\tau}{1+(1-\theta)(r+\lambda)\Delta\tau} \\ &< 1. \quad \blacksquare \end{aligned} \quad (5.6)$$

Note that typically  $\lambda\Delta\tau \ll 1$ , so that

$$\|e^{k+1}\|_\infty \simeq \|e^k\|_\infty (1-\theta)\lambda\Delta\tau, \quad (5.7)$$

which will result in rapid convergence. It is also interesting to observe that the number of iterations required for convergence is independent of the number of nodes in the  $S$  grid.

## 6 Details Regarding Evaluation of the Correlation Integral

To complete the discussion of our numerical algorithm, we need to consider issues such as evaluating the jump integral term, interpolation, and wrap-around effects. Note that each iteration of the scheme (5.4) requires evaluation of a correlation integral for all points on the PDE grid.

Fast evaluation of this integral using FFT methods necessitates transformation to an equally spaced grid in  $x = \log(S)$  coordinates. If the original PDE grid is equally spaced in  $\log(S)$ , then there is clearly no difficulty. However, this type of grid spacing is highly inefficient for cases involving discontinuous payoffs or barriers. We therefore prefer not to restrict the type of grid used for the original PDE. Recall that the correlation integral is

$$I(x) = \int_{-\infty}^{\infty} \bar{V}(x+y)\bar{f}(y)dy,$$

or, in discrete form

$$I_i = \sum_{j=-N/2+1}^{j=N/2} \bar{V}_{i+j}\bar{f}_j\Delta y + O((\Delta y)^2),$$

where  $I_i = I(i\Delta x)$ ,  $\bar{V}_j = \bar{V}(j\Delta x)$ ,  $\bar{f}_j = \bar{f}(j\Delta y)$ . We have also assumed that  $\Delta y = \Delta x$ , and that  $\bar{V}(\log S) = V(S)$ .

Now,  $\bar{V}_j$  will not necessarily coincide with any of the discrete values  $V_k$  in equation (3.15). Consequently, we will linearly interpolate to determine the appropriate values, as in equation (3.9). Since equation (3.4) has the form of a discrete correlation, FFT methods are an obvious choice to compute this efficiently. Assuming that  $\bar{f}$  is real, then

$$\text{FFT}(I)_k = (\text{FFT}(\bar{V}))_k (\text{FFT}(\bar{f}))_k^*, \quad (6.1)$$

where  $(\cdot)^*$  denotes the complex conjugate. Since  $\bar{f}(z)$  is the probability density of  $z = \log \eta$ , which is a specified function, we can simply precompute  $\text{FFT}(\bar{f})$  on the required equally spaced grid in  $z$  coordinates. We can then carry out an inverse FFT to obtain the values of the correlation integral on the equally spaced  $x = \log S$  grid. A further interpolation step is required to obtain the value of the correlation integral on the original  $S$  grid (equation (3.10)).

We can summarize the steps needed to generate the required values  $I(S_k)$ ,  $k = 0, \dots, p$  as follows.

- Interpolate the discrete values of  $V$  onto an equally spaced  $\log(S)$  grid. This generates the required values of  $\bar{V}_j$ .
  - Carry out the FFT on this data.
  - Compute the correlation in the frequency domain (with precomputed  $\text{FFT}(\bar{f})$ ), using equation (6.1).
  - Invert the FFT of the correlation.
  - Interpolate the discrete values of  $I(x_i)$  onto the original  $S$  grid.
- (6.2)

Note that as long as linear or higher order interpolation is used, this procedure is second order correct, which is consistent with the discretization error in the PDE and the midpoint rule used to evaluate the integral (3.4).

In principle, we can avoid the interpolation steps in the above procedure if we use special techniques for computing the FFT for unequally spaced data. There are several methods for computing the inverse FFT problem (i.e. given unequally spaced data, determine the Fourier coefficients), as well as the forward FFT problem (given the Fourier coefficients, determine the inverse transform values on an unequally spaced grid) (see, for example Ware, 1998; Duijndam and Schonewille, 1999; Potts et al., 2001). However, it should be noted that we are not particularly interested in obtaining highly accurate estimates of the discrete Fourier coefficients, as we simply need to evaluate the correlation integral correct to second order. Consequently, for our purposes there is no particular benefit in terms of accuracy in using these methods. We will use the interpolation method (6.2) followed by the standard FFT to calculate our illustrative results below in Section 8.

An alternative approach for evaluation of the correlation integral could involve a fast Gauss transform (FGT) (see Greengard and Strain, 1991). This method has complexity of  $O(N)$ . The use of this method has been explored in the general option pricing context by Broadie and Yamamoto (2002). In the particular case of jump diffusions, this approach would work for the case where the jump size is lognormally distributed. It is not clear if it could be applied for other jump distributions. We have carried out some numerical experiments using public domain FGT software, and it appears

that the FFT approach used in here is superior to use of an FGT, at least for grid sizes of practical interest. Note that all of the theoretical results in terms of convergence rates given previously would be unchanged if the FGT were used instead of the FFT.

Another issue requiring attention is that the FFT algorithm effectively assumes that the input functions are periodic. This may cause wrap-around pollution unless special care is taken when implementing the algorithm. The integral (3.2) is approximated on the finite domain

$$I(x) = \int_{y_{\min}}^{y_{\max}} \bar{V}(x+y)\bar{f}(y)dy. \quad (6.3)$$

The PDE part of the PIDE (2.4) is computed using the finite computational domain  $[0, S_{\max}]$ , using the discrete grid  $S_0, S_1, \dots, S_p$ . Initially, we chose

$$\begin{aligned} y_{\max} &= \log(S_{\max}) \\ y_{\min} &= \log(S_1), \end{aligned} \quad (6.4)$$

assuming  $S_1 > 0$ . Note that  $y_{\min} = \log(S_1)$  since normally  $S_0 = 0$ , so that  $\log(S_0) = -\infty$ .

Generally,  $\bar{f}(y)$  (which represents the probability density of a jump of  $S \rightarrow S\eta$  (where  $y = \log \eta$ )) is rapidly decaying for  $|y| \gg 0$ . However,  $\bar{V}(y)$  does not decay to zero near  $y = y_{\min}, y_{\max}$ . Typically,  $V(S) \leq \text{Const. } S$  as  $S \rightarrow \infty$ , and  $V(S) \simeq \text{Const.}$  as  $S \rightarrow 0$ , or in  $y = \log S$  coordinates,

$$\bar{V}(y) \leq \text{Const. } e^y, \quad y \rightarrow \infty \quad (6.5)$$

$$\simeq \text{Const.}, \quad y \rightarrow -\infty. \quad (6.6)$$

This will cause undesirable wrap-around effects if we use an FFT approach to evaluate the integral (6.3), since the discrete Fourier transform (DFT) is effectively applied to the periodic extension of the input functions. To avoid these problems, we extend the domain of the integral to the left and right by a size which reflects the width of the probability density. In other words, we use the values  $\bar{V}(y), y \in [y_{\min} - \Delta y^-, y_{\max} + \Delta y^+]$  as input to the correlation evaluation (6.1).

In order to determine values in the extended region, we solve the following PDE-PIDE in the region  $[0, S_{\max}e^{\Delta y^+}]$ .

$$\begin{aligned} V_\tau &= -rV \\ &S = 0 \\ V_\tau &= \frac{1}{2}\sigma^2 S^2 V_{SS} + (r - \lambda\kappa)SV_S - (r + \lambda)V + \lambda \int_0^\infty V(S\eta)g(\eta)d\eta \\ &0 < S < S_{\max} \\ V_\tau &= \frac{1}{2}\sigma^2 S^2 V_{SS} + rSV_S - rV \\ &S_{\max} \leq S \leq S_{\max}e^{\Delta y^+} . \end{aligned} \quad (6.7)$$

The extended region  $[S_{\max}, S_{\max}e^{\Delta y^+}]$  can be regarded as a buffer zone which reduces the effect of FFT wrap-around. Note that we have assumed that  $S_{\max}$  is sufficiently large so that it is valid to assume that approximation (2.8) holds.

The values of  $\bar{V}(u)$  for  $u \in [y_{\max}, y_{\max} + \Delta y^+]$  are estimated using simple linear interpolation. The values in the left extension can be determined from interpolation on the original  $S$  grid.

This extended domain is then used as input to the forward DFT, the correlation computation (in the spectral domain), and the inverse DFT. The values in the domain extensions are affected

by wrap-around and are discarded, since they are not needed. (Recall from equation (6.7) that we solve the PDE at  $S = 0$ , and in the right domain extension  $[S_{max}, S_{max}e^{\Delta y^+}]$ , so that the values of the correlation integrals are not needed in these regions). In Appendix B, we show how to estimate  $\Delta y^+, \Delta y^-$  so that the errors due to wrap-around are within a user specified tolerance.

## 7 American Options

Next, we briefly describe how to extend the ideas presented thus far to the case of American options. Suppose that we have to value an American style option where the holder of the contract can exercise at any time and receive a payoff of  $V^*(S, \tau)$ . This pricing problem can be written as the differential linear complementarity problem

$$V_\tau - \left( \frac{1}{2} \sigma^2 S^2 V_{SS} + (r - \lambda \kappa) S V_S - (r + \lambda) V + \lambda \int_0^\infty V(S\eta) g(\eta) d\eta \right) \geq 0 \quad (7.1)$$

$$V - V^* \geq 0, \quad (7.2)$$

where at least one of equations (7.1) and (7.2) must hold with equality. As mentioned in the introduction, we are seeking viscosity solutions to equations (7.1)-eq7.2. This is formulated more precisely in Pham (1998). We can easily combine the fixed point iteration with the penalty method described in Forsyth and Vetzal (2002) to solve this complementarity problem. For a detailed analysis of the convergence of iteration for the discretized algebraic equations, see d'Halluin et al. (2003).

## 8 Results

This section presents numerical results for various options, including vanilla European and American options, digital options, and options with barrier features. Unless stated otherwise, we use the Crank-Nicolson discretization scheme (4.1). The discrete system of equations is solved using the fixed point iteration method (5.4) with a convergence tolerance of  $10^{-6}$ .

We begin by considering European options under the assumptions that the continuous part of the underlying stock price process follows geometric Brownian motion and that the proportional jump size is lognormally distributed, where the jump size distribution  $g(\eta)$  is given by

$$g(\eta) = \frac{e^{\left( -\frac{(\log(\eta) - \mu)^2}{2\gamma^2} \right)}}{\sqrt{2\pi}\gamma\eta}. \quad (8.1)$$

This allows us to check the accuracy of our algorithm against the analytic solution of Merton (1976). Table 1 contains the input parameters. These are roughly the same as those estimated by Andersen and Andreasen (2000) using European call options on the S&P 500 stock index in April of 1999.

We are particularly interested in the convergence properties of the algorithm as the grid is refined. For each test, as we double the number of grid points we cut the timestep size ( $\Delta\tau = .01$  on the coarsest grid) in half. The convergence ratio presented in the tables below is defined in the following way. Let

$$\Delta\tau = \max_n (\tau^{n+1} - \tau^n),$$

$$\Delta S = \max_i (S_{i+1} - S_i).$$

Parameter Values	
$\sigma$	0.15
$r$	0.05
$\gamma$	0.45
$\mu$	-0.90
$\lambda$	0.10
$T$	0.25
$K$	100.00

TABLE 1: *Input data used to value European options under the lognormal jump diffusion process. These parameters are approximately the same as those reported in Andersen and Andreasen (2000) using European call options on the S&P 500 stock index in April of 1999.*

Size of $S$ grid	No. of Timesteps	Interpolation Scheme					
		Linear		Quadratic		Cubic	
		Value	$R$	Value	$R$	Value	$R$
128	25	3.146361	n.a.	3.145896	n.a.	3.146361	n.a.
255	50	3.148354	n.a.	3.148249	n.a.	3.148354	n.a.
509	100	3.148856	3.973	3.148831	4.039	3.148832	4.175
1017	200	3.148983	3.949	3.148977	3.990	3.148977	3.287
2033	400	3.149015	4.001	3.149014	4.007	3.149014	3.997
4065	800	3.149023	3.997	3.149023	4.002	3.149023	3.997

TABLE 2: *Value of a European put option at  $S = 100$  using Crank-Nicolson timestepping for linear, quadratic and cubic interpolation. The interpolation schemes are used to transfer data between the non-uniform  $S$  grid and the uniform log-spaced FFT grid. The input parameters are provided in Table 1. The convergence ratio  $R$  is defined in equation (8.2). The exact solution is 3.149026. The number of points used for the FFT grid is  $2^\alpha$ , where  $\alpha$  is the smallest integer such that the number of nodes in the non-uniform  $S$  grid  $p \leq 2^\alpha$ .*

Note that we are allowing here for the possibility of using variable timestep sizes (to be explained below), although most of our tests will simply use a constant timestep size. If we then carry out a convergence study letting  $h \rightarrow 0$  where  $\Delta S = \text{Const. } h$ ,  $\Delta\tau = \text{Const. } h$ , then we can assume that the error in the solution (at a given node) is  $V_{\text{approx}}(h) = V_{\text{exact}} + \text{Const. } h^\xi$ , and the convergence ratio is defined as

$$R = \frac{V_{\text{approx}}(h/2) - V_{\text{approx}}(h)}{V_{\text{approx}}(h/4) - V_{\text{approx}}(h/2)}. \quad (8.2)$$

In the case of quadratic convergence ( $\xi = 2$ ), then  $R = 4$ , while for linear convergence ( $\xi = 1$ ),  $R = 2$ .

Recall that interpolation is required to transform data from the clustered PDE grid to the equally spaced log  $S$  grid, and vice versa. In Table 2, we compare linear interpolation (see equations (3.9)-(3.10)) with quadratic and cubic Lagrange interpolation for a vanilla European put option with different numbers of points on the FFT grid.

In Table 2 we observe quadratic convergence to the exact solution for all three interpolation schemes. Note that our earlier theoretical analysis for stability and convergence of the fixed point iteration was based on linear interpolation. This was required because linear interpolation is the only Lagrange interpolation method which has non-negative weights. Although it is not the case for these particular parameter values, our numerical experiments indicate that quadratic interpolation



Size of $S$ grid	No. of Timesteps	$S = 90$		$S = 100$		$S = 110$	
		Value	$R$	Value	$R$	Value	$R$
128	25	0.526562	n.a.	4.388091	n.a.	12.641501	n.a.
255	50	0.527379	n.a.	4.390462	n.a.	12.642942	n.a.
509	100	0.527574	4.186	4.391050	4.039	12.643290	4.125
1017	200	0.527622	4.042	4.391197	3.991	12.643377	4.008
2033	400	0.527634	4.046	4.391233	4.005	12.643399	4.059
4065	800	0.527637	4.023	4.391243	4.002	12.643404	4.049

TABLE 3: Value of a European call option using Crank-Nicolson timestepping. The input parameters are provided in Table 1. The convergence ratio  $R$  is defined in equation (8.2). The exact solution is 0.527638 at  $S = 90$ , 4.391246 at  $S = 100$ , and 12.643406 at  $S = 110$ . The number of points used for the FFT grid is  $2^\alpha$ , where  $\alpha$  is the smallest integer such that the number of nodes in the non-uniform  $S$  grid  $p \leq 2^\alpha$ . Quadratic interpolation is used.

is often more efficient than linear interpolation (although the rate of convergence rate is theoretically the same for both methods). Consequently, in all subsequent examples we will use quadratic interpolation. In Table 3 we show the convergence rate for a call option using the data in Table 1. For each value of  $S$  in the table, we observe quite smooth second order convergence.

We now consider the issues raised by the presence of a discontinuity in the payoff. Oscillations are more likely to be a problem in this context if we use Crank-Nicolson timestepping, and, unless care is taken, rates of convergence can be reduced. A detailed discussion of this can be found in Pooley et al. (2003) for the case without jumps. Following Rannacher (1984), it is possible to restore quadratic convergence if any discontinuities in the payoff (arising either due to the payoff function itself in the case of a digital option, or from the application of a discretely observed barrier) are  $l_2$  projected onto the space of linear Lagrange basis functions, and a fully implicit method is used for a small number of timesteps after any discontinuities arise. We will refer to this technique as Rannacher timestepping. While this method does ensure quadratic convergence, it does not guarantee the absence of oscillations. Typically, however, the use of the fully implicit timesteps smooths out the function enough that oscillations are not a problem.

We will investigate the application of Rannacher timestepping in the jump diffusion context for a digital put option which pays \$1 at maturity if the underlying stock price is below the strike price, and zero otherwise. Table 4 gives a convergence study for the digital put with jumps, using Rannacher timestepping (with two fully implicit steps) and  $l_2$  projection. As shown in this table, quadratic convergence is generally achieved, though perhaps a bit more erratically than for the vanilla payoff as shown in Table 3. Figure 1 provides plots of the solution value for a digital put along with the hedging parameters delta ( $V_S$ ) and gamma ( $V_{SS}$ ).

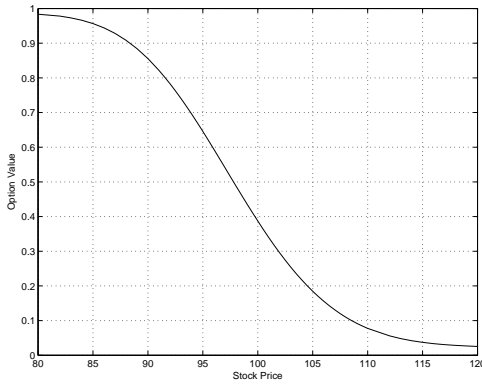
Our next two numerical tests incorporate the use of an automatic timestep size selector as described in Johnson (1987). It is not generally possible to achieve second order convergence for American options using constant timesteps (Forsyth and Vetzal, 2002). An initial timestep is given and the next timestep is computed according to

$$\frac{\tau^{n+2} - \tau^{n+1}}{\tau^{n+1} - \tau^n} = \frac{d}{\max_i \frac{|V_i^{n+1} - V_i^n|}{\max(1, |V_i^n|)}}, \quad (8.3)$$

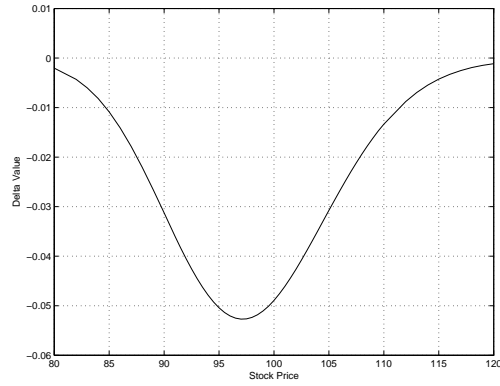
where  $d$  specifies the maximum relative change allowed. Initially we set  $d = .1$ , and we divide this value by two at each grid refinement. An initial timestep of .01 on the coarsest grid is used, and this initial timestep is reduced by a factor of four at each refinement.

Size of $S$ grid	No. of Timesteps	$S = 90$		$S = 100$		$S = 110$	
		Value	$R$	Value	$R$	Value	$R$
128	25	0.855482	n.a.	0.387139	n.a.	0.077539	n.a.
255	50	0.855034	n.a.	0.387151	n.a.	0.077830	n.a.
509	100	0.854935	4.577	0.387152	7.157	0.077899	4.160
1017	200	0.854910	3.924	0.387153	2.219	0.077917	3.896
2033	400	0.854902	2.824	0.387153	3.590	0.077922	3.970
4065	800	0.854899	3.327	0.387153	4.124	0.077923	4.005

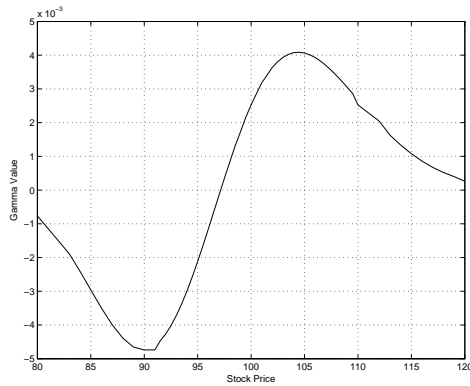
TABLE 4: Value of a European digital put option using Rannacher timestepping and  $l_2$  projection. The input parameters are provided in Table 1. The convergence ratio  $R$  is defined in equation (8.2). The exact solution is 0.854898 at  $S = 90$ , 0.387153 at  $S = 100$ , and 0.077923 at  $S = 110$ . The number of points used for the FFT grid is  $2^\alpha$ , where  $\alpha$  is the smallest integer such that the number of nodes in the non-uniform  $S$  grid  $p \leq 2^\alpha$ . Quadratic interpolation is used.



(a) Digital put value: Rannacher.



(b) Digital put delta: Rannacher.



(c) Digital put gamma: Rannacher.

FIGURE 1: Digital put option value, delta and gamma for Rannacher timestepping. The input parameters are provided in Table 1.

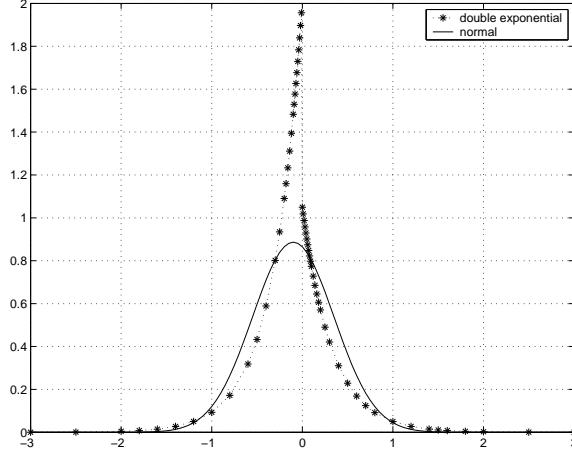


FIGURE 2: Overall comparison of the normal ( $\mu = -.10, \gamma = .45$ ) and double exponential probability density functions ( $p = 0.3445, \eta_1 = 3.0465, \eta_2 = 3.0775$ ).

The first test to incorporate variable timesteps involves an alternative distribution for the jump size. Kou (2002) suggests the double exponential distribution for the log jump size, observing that it has desirable analytical properties. In the model of Kou (2002),

$$\bar{f}(x) = p\eta_1 \exp(-\eta_1 x)H(x) + q\eta_2 \exp(\eta_2 x)H(-x), \quad (8.4)$$

where  $\eta_1 > 1, \eta_2 > 0, p > 0, q = 1 - p > 0$ , and  $H(\cdot)$  is the Heaviside function. As noted by Kou, the condition  $\eta_1 > 1$  is used to ensure that the proportional jump and stock price have finite expectation. In this model,  $\kappa = E[J - 1] = \frac{p\eta_1}{\eta_1 - 1} + \frac{q\eta_2}{\eta_2 + 1} - 1$ .

To provide a basis for comparison with the lognormal distribution, we attempted to find parameters for the double exponential distribution which match those used for the lognormal given in Table 1. This did not work well for those parameters, as the mean is too far below zero, resulting in only the left tail of the double exponential being used. To remedy this, we shifted the lognormal mean from its value of  $-.90$  in Table 1 to  $-.10$ . We then performed a numerical search to find parameters to match the first three central moments of the two distributions as closely as possible. We obtained values of  $p = 0.3445, \eta_1 = 3.0465$ , and  $\eta_2 = 3.0775$ . Figure 2 shows the double exponential probability density function and the normal probability density function for our parameter values. Note that the double exponential distribution has a discontinuity at zero. This can be expected to cause some problems for our numerical integration using an FFT method.

Table 5 presents numerical convergence tests for pricing a European call option. In an attempt to deal with the discontinuity at zero, the number of points used on the uniform-spaced  $x$  grid has been oversampled to a greater extent than in the lognormal case. In particular, the number of points on the FFT grid is  $8 \times 2^\alpha$ , where  $\alpha$  is the smallest integer such that  $2^\alpha$  is at least equal to the number of nodes in the  $S$  grid. Rannacher timestepping is used. In contrast to our earlier examples, we do not obtain second order convergence here. Instead the results indicate convergence at a linear (or perhaps slightly higher) rate to the exact solution. Despite the discontinuity, we observe smooth solution plots for the solution value, delta, and gamma in Figure 3.

Note that other numerical experiments indicate that we can achieve quadratic convergence in the double exponential case if we restrict the parameters so that the distribution is continuous at zero (i.e. set  $p = 0.50, \eta_1 = \eta_2$ ). This still requires a heavily oversampled FFT grid relative to the lognormal case in order to adequately capture the sharp peak of the distribution.

Size of $S$ grid	No. of Timesteps	$S = 90$		$S = 100$		$S = 110$	
		Value	$R$	Value	$R$	Value	$R$
128	34	0.671314	n.a.	3.969969	n.a.	11.78927	n.a.
255	65	0.672213	n.a.	3.972476	n.a.	11.79248	n.a.
509	132	0.672535	2.791	3.973107	3.972	11.79367	2.688
1017	266	0.672630	3.358	3.973322	2.936	11.79416	2.431
2033	533	0.672660	3.225	3.973407	2.511	11.79438	2.244
4065	1067	0.672670	2.917	3.973445	2.281	11.79448	2.130

TABLE 5: Value of a European vanilla call option using Rannacher timestepping with variable timestep sizes for the double exponential probability density function (8.4). The timesteps are selected using equation (8.3), with  $d = 0.1$  on the coarsest grid, and divided by two for each grid refinement. The input parameters are  $\sigma = 0.15$ ,  $r = 0.05$ ,  $\lambda = 0.1$ ,  $T = 0.25$ ,  $K = 100$ ,  $\eta_1 = 3.0465$ ,  $\eta_2 = 3.0775$ , and  $p = .3445$ . The convergence ratio  $R$  is defined in equation (8.2). The exact solution is 0.672677 at  $S = 90$ , 3.973479 at  $S = 100$ , and 11.794583 at  $S = 110$ . The number of points used for the FFT grid is  $8 \times 2^\alpha$ , where  $\alpha$  is the smallest integer such that the number of nodes in the non-uniform  $S$  grid  $p \leq 2^\alpha$ . Quadratic interpolation is used.

Size of $S$ grid	No. of Timesteps	$S = 90$		$S = 100$		$S = 110$	
		Value	$R$	Value	$R$	Value	$R$
128	32	10.000000	n.a.	3.236354	n.a.	1.417613	n.a.
255	58	10.002938	n.a.	3.240286	n.a.	1.419269	n.a.
509	117	10.003519	5.058	3.241045	5.182	1.419676	4.077
1017	235	10.003791	2.137	3.241207	4.699	1.419774	4.139
2033	470	10.003815	11.653	3.241243	4.463	1.419798	4.143
4065	940	10.003822	3.213	3.241251	4.331	1.419803	4.127

TABLE 6: Value of an American put option using Rannacher timestepping with variable timestep sizes. The timesteps are selected using equation (8.3), with  $d = 0.1$  on the coarsest grid, and divided by two for each grid refinement. The input parameters are provided in Table 1. The convergence ratio  $R$  is defined in equation (8.2). The approximate analytic values from Bates (1991) are 9.304946 at  $S = 90$ , 3.163112 at  $S = 100$ , and 1.411669 at  $S = 110$ . The number of points used for the FFT grid is  $2^\alpha$ , where  $\alpha$  is the smallest integer such that the number of nodes in the non-uniform  $S$  grid  $p \leq 2^\alpha$ . Quadratic interpolation is used.

Our second test involving variable timesteps is the valuation of an American put option. As mentioned in Section 6, this is easily handled in our framework by combining the fixed point iteration with the penalty method described in Forsyth and Vetzal (2002) and d’Halluin et al. (2003). As noted above, in this context it is also generally necessary to use variable timestepping to achieve faster than linear convergence. Table 6 presents the results for the case of lognormally distributed jumps. Once again, we generally observe second order convergence, although convergence is somewhat erratic near the exercise boundary. There is no evidence of oscillations in either the solution or its first two derivatives with respect to the underlying stock price (see Figure 4). Also note that in this case the analytic approximation of Bates (1991) is quite accurate for the out of the money case where  $S = 110$ , about eight cents too low when  $S = 100$ , and around 70 cents too low for the in the money case with  $S = 90$ . This suggests that (at least for our parameter values), Bates’s approximation is not very accurate (in terms of absolute pricing error), unless the option is deep out of the money.

The last set of results to be presented are for the case of a European call option with a Parisian

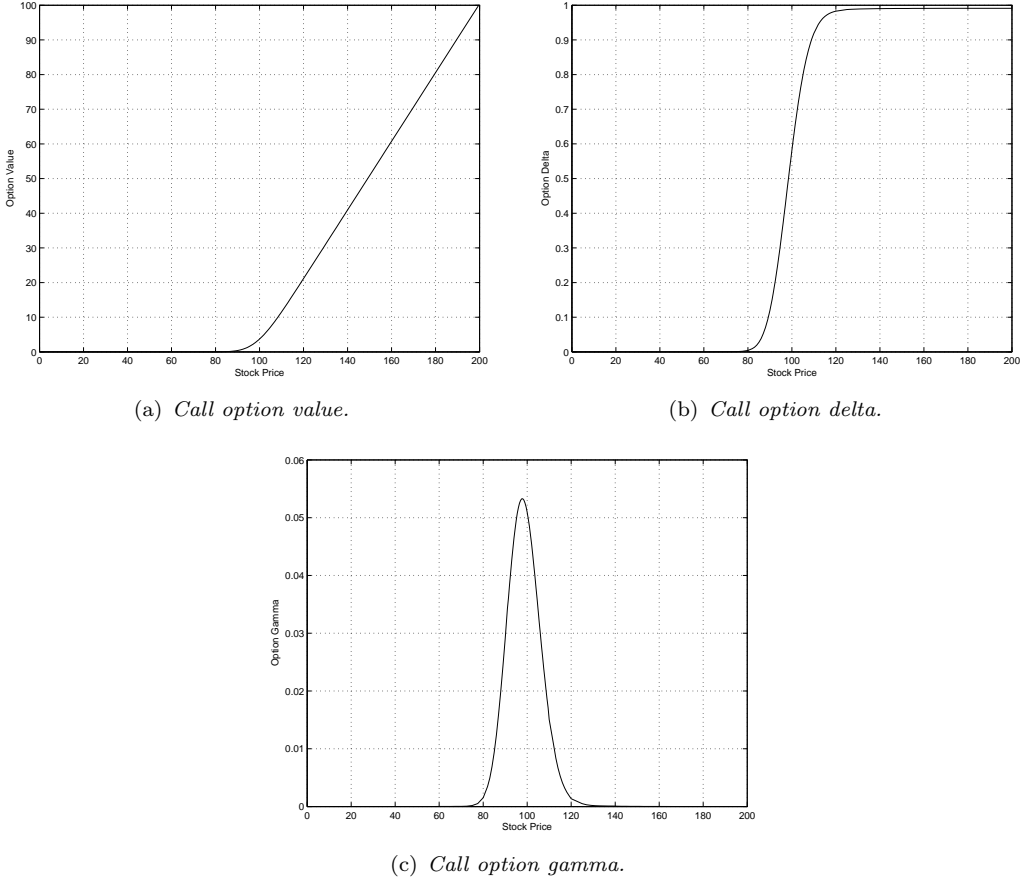
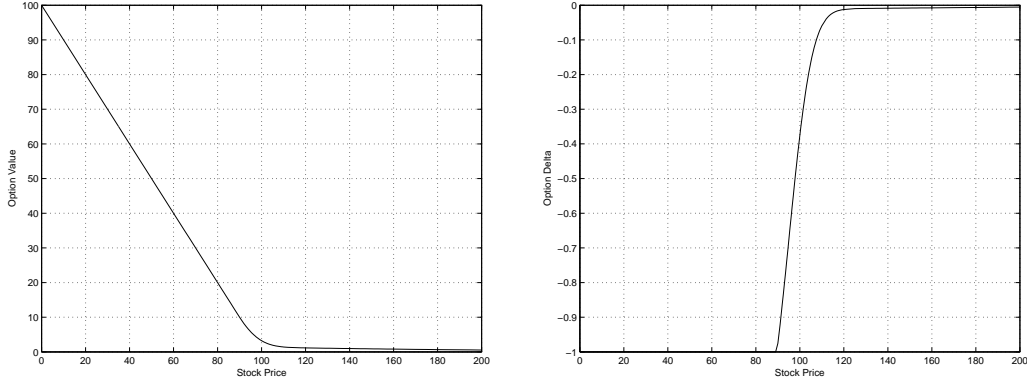


FIGURE 3: *Call option value, delta and gamma for Rannacher timestepping using the double exponential probability density function (8.4). The input parameters are provided in the caption to Table 5.*

knock-out feature. The particular case we consider here is an up-and-out call with daily discrete observation dates. This contract ceases to have value if  $S$  is above a specified barrier level for a specified number of consecutive monitoring dates. This can be valued by solving a set of one-dimensional problems which exchange information at monitoring dates (Vetzal and Forsyth, 1999). It is easy to incorporate jumps by simply adding a jump integral term to each of the one-dimensional problems. Other path-dependent contracts such as Asian options can also be handled using this approach of solving a set of one-dimensional problems (Zvan et al., 1999).

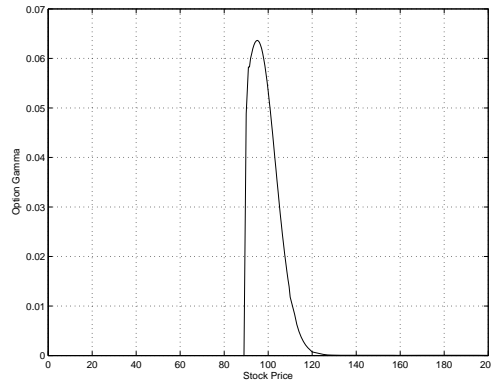
For our test, the barrier is set at  $S = 120$  and the required number of consecutive daily observations for knock-out is 10. We consider the lognormal jump distribution case with the same input parameters as in Table 1. Note that we specify the barrier observation interval as  $1/250$ , based on 250 trading days per year. In Table 7, we present our convergence results. We use constant timestepping ( $\Delta\tau = .002$  on the coarse grid) and the solution is  $l_2$  projected after each barrier observation date. Rannacher timestepping is used after each observation. As expected, quadratic convergence is obtained.

In Figure 5, we compare the solutions of a Parisian call knock-out option with discrete daily observation dates with and without jumps. To ensure a consistent basis for comparison, we use the following procedure:



(a) American put option value.

(b) American put option delta.



(c) American put option gamma.

FIGURE 4: American put option value, delta and gamma for Rannacher timestepping with 128 points on the non-uniform  $S$  grid and an initial timestep of 0.01.

1. Given some parameters (in this example we use the values provided in Table 1), compute the analytical solution  $V_{\text{jump}}$  at the strike  $K = 100$  of a vanilla European call option.
2. Use a constant volatility Black-Scholes model with no jumps to determine the implied volatility  $\sigma_{\text{implied}}$  which matches the option price to the jump diffusion value  $V_{\text{jump}}$  at the strike  $K$ .
3. Value the Parisian knock-out call option with discrete daily observation dates with jumps using the same parameters as in Step 1.
4. Value the Parisian knock-out call option with discrete daily observation dates using a constant volatility model (no jumps) but with the implied volatility  $\sigma_{\text{implied}}$  estimated in Step 2.

We observe in Figure 5 that the difference in pricing can be significant for these parameter values, depending on the underlying asset price. The largest differences are near  $S = 110$ , where the model with jumps produces values of about 8.78 (as shown in Table 7), but the values for the no-jump model are around 7.25. For  $S$  ranging between about 98 and around 119, the jump model produces higher option values, but outside this range (in either direction) the model without jumps produces higher values.

Size of $S$ grid	No. of Timesteps	$S = 90$		$S = 100$		$S = 110$	
		Value	$R$	Value	$R$	Value	$R$
101	125	0.524766	n.a.	4.193418	n.a.	8.762555	n.a.
201	250	0.523168	n.a.	4.212131	n.a.	8.779253	n.a.
401	500	0.522761	3.930	4.216747	4.053	8.782267	5.540
801	1000	0.522660	4.002	4.217902	3.997	8.783008	4.068
1601	2000	0.522634	4.015	4.218192	3.990	8.783199	3.875

TABLE 7: Value of an up-and-out Parisian call option using Rannacher timestepping with constant timesteps ( $\Delta\tau = .002$  on the coarsest grid) and  $l_2$  projection. The input parameters are given in Table 1. The barrier is set at  $S = 120$  and 10 consecutive daily observations are required to knock-out. The convergence ratio  $R$  is defined in equation (8.2). The number of points used for the FFT grid is  $2^\alpha$ , where  $\alpha$  is the smallest integer such that the number of nodes in the non-uniform  $S$  grid  $p \leq 2^\alpha$ . Quadratic interpolation is used.

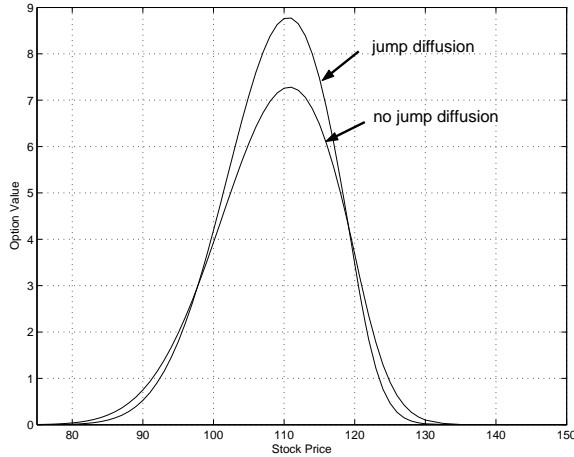


FIGURE 5: Parisian knock-out call option with discrete daily observation dates with and without jumps. The barrier is set at  $S = 120$  and the number of consecutive daily observations to knock-out is 10.

It is worth concluding this section by making some comparisons with other methods which have been proposed in the literature. When pricing options under the jump diffusion process, the main computational cost is the evaluation of the integral term of (2.4). The approach presented in Andersen and Andreasen (2000) is based on a FFT-ADI finite difference method. This method evaluates the convolution integral twice at each timestep, thus requiring a total of four FFT computations (two forward FFTs, and two reverse FFTs). Note that the method in Andersen and Andreasen (2000) is second order accurate. If  $N$  is the number of timesteps, and  $p$  the number of nodes in the  $S$  grid, then both Andersen and Andreasen’s method and the method in this work have complexity  $O(Np \log p)$ .

In Table 8, we see that the number of iterations required for convergence (at each timestep) depends on the convergence tolerance. For a typical convergence tolerance of  $10^{-6}$ , at most three iterations per step are required (on average). In this case, about six FFT computations are required per timestep. Consequently, for vanilla European options (with jumps), the method of Andersen and Andreasen (2000) may be more efficient than the pure Crank-Nicolson timestepping method developed here.

Number of points $N$	Timesteps	Iterations ( $tol = 10^{-6}$ )	Iterations ( $tol = 10^{-8}$ )
128	25	77	100
255	50	150	200
509	100	300	390
1017	200	600	600
2033	400	1091	1200
4065	800	1600	2400

TABLE 8: *Number of iterations for a European call option under jump diffusion using Crank-Nicolson timestepping. The input parameters are provided in Table 1. The convergence tolerance  $tol$  is defined in equation (5.4).*

However, in the case of American options, it is not clear how the approach in Andersen and Andreasen (2000) could be modified to handle the early exercise constraint implicitly, unless some form of iteration is used. In contrast, our technique can handle implicit treatment of the American constraint in a straightforward fashion.

The technique developed in Matache et al. (2002) uses a wavelet method for the evaluation of the jump integral term. This has complexity  $O(Np (\log p)^2)$ , in contrast to a complexity of  $O(Np \log p)$  (for one dimensional problems) for the method developed here. In addition, it is not obvious how to generalize the technique in Matache et al. (2002) to nonlinear cases such as uncertain volatility or transaction costs models, which can easily be handled using our method Pooley et al. (2003).

Finally, we note that the method in Zhang (1997) uses an explicit evaluation of the correlation integral term, and hence is only first order accurate. The approach of Meyer (1998) is restricted to cases where the underlying asset can only jump to a (small) finite number of states.

## 9 Conclusion

In this paper, we have shown that an explicit evaluation of the correlation integral in the jump diffusion PIDE, coupled with an implicit discretization of the usual PDE terms, is unconditionally stable. However, since this method is only first order correct, an implicit method is preferred. We show that Crank-Nicolson timestepping is algebraically stable, and in the special case of an equally spaced  $\log S$  grid with constant parameters, we can prove that Crank-Nicolson timestepping is strictly stable.

If implicit timestepping is used, then the direct evaluation of the correlation integral appearing in the PIDE would require a dense matrix solve. To avoid this computational complexity, a fixed point iteration method is developed. For typical parameter values, this fixed point iteration converges very quickly (the error is reduced by two orders of magnitude at each iteration).

Each fixed point iteration requires evaluation of the correlation integral. We use Lagrange interpolation to transfer the data on the clustered PDE grid to an equally spaced  $\log S$  grid. An FFT method is then used to evaluate the correlation integral, and Lagrange interpolation is used to transfer data back to the PDE grid. We demonstrate how to extend the  $\log S$  grid to avoid FFT wrap-around effects. This is done by taking into account the properties of the jump size probability density.

The methods developed in this paper can be applied to arbitrary jump size probability densities. Furthermore, we have demonstrated that this method can be used to obtain implicit solutions to American options with jump diffusion. Since the method used to handle the jump diffusion term



implicitly is a simple fixed point iteration, then it is a very simple matter to modify an existing exotic option pricing library to handle the jump diffusion case. All that is required is that a function be added to the library which, given the current vector of discrete option prices, returns the vector value of the correlation integral. This vector is then added to the right hand side of the fixed point iteration.

There are several obvious avenues for future research. One would be a detailed analysis of pricing and hedging various types of exotic options under a jump diffusion process. Similarly, it would be interesting to explore the effects of uncertain parameters or transactions costs, as described in Wilmott (1998) for the diffusion case. Another possibility would be to extend the analysis to more complex models for the evolution of the underlying state variable. Among the candidates here are more general Lévy processes than the jump diffusion case, or multifactor models such as those recently explored by Eraker et al. (2003), which feature stochastic volatility with Poisson jumps in both the state variable itself and its volatility.

## Appendices

### A Von Neumann Stability Analysis

In this Appendix, we will carry out a Von Neumann stability analysis for Crank-Nicolson timestepping in the special case of constant parameters and an equally spaced grid in  $\log S$  coordinates.

From equations (2.4) and (3.2),

$$V_\tau = \frac{1}{2}\sigma^2 S^2 V_{SS} + (r - \lambda\kappa)SV_S - (r + \lambda)V + \lambda \int_{-\infty}^{\infty} \bar{V}(y)f(y - \log S) dy. \quad (\text{A.1})$$

where  $\bar{V}(x, \tau) = V(\exp(x), \tau)$  and  $\bar{f}(y) = f(\exp(y))$ . Using the change of variable  $x = \log(S)$  and substituting into (A.1), we obtain

$$\bar{V}_\tau = \frac{1}{2}\sigma^2 \bar{V}_{xx} + (r - \lambda\kappa - \frac{1}{2}\sigma^2)\bar{V}_x - (r + \lambda)\bar{V} + \lambda \int_{-\infty}^{\infty} \bar{V}(y)\bar{f}(y - x)dy. \quad (\text{A.2})$$

From equation (A.2), it can be observed that the integral part of the PIDE is simply a correlation product. Using the correlation operator  $\otimes$  from equation (3.3), equation (A.2) can be written as

$$\bar{V}_\tau = \frac{1}{2}\sigma^2 \bar{V}_{xx} + (r - \lambda\kappa - \frac{1}{2}\sigma^2)\bar{V}_x - (r + \lambda)\bar{V} + \lambda \bar{V} \otimes \bar{f}. \quad (\text{A.3})$$

A Crank-Nicolson discretization of equation (A.3) is

$$\begin{aligned} \frac{\bar{V}_i^{n+1} - \bar{V}_i^n}{\Delta\tau} &= \frac{\lambda}{2} [(\bar{V} \otimes \bar{f})_i^n + (\bar{V} \otimes \bar{f})_i^{n+1}] + \\ &\frac{1}{2} \left[ \frac{1}{2}\sigma^2 \left( \frac{\bar{V}_{i+1}^{n+1} - 2\bar{V}_i^{n+1} + \bar{V}_{i-1}^{n+1}}{\Delta x^2} \right) + \left( r - \lambda\kappa - \frac{1}{2}\sigma^2 \right) \left( \frac{\bar{V}_{i+1}^{n+1} - \bar{V}_{i-1}^{n+1}}{2\Delta x} \right) - (r + \lambda)\bar{V}_i^{n+1} \right] + \\ &\frac{1}{2} \left[ \frac{1}{2}\sigma^2 \left( \frac{\bar{V}_{i+1}^n - 2\bar{V}_i^n + \bar{V}_{i-1}^n}{\Delta x^2} \right) + \left( r - \lambda\kappa - \frac{1}{2}\sigma^2 \right) \left( \frac{\bar{V}_{i+1}^n - \bar{V}_{i-1}^n}{2\Delta x} \right) - (r + \lambda)\bar{V}_i^n \right]. \quad (\text{A.4}) \end{aligned}$$

Equation (A.4) can be written as

$$\begin{aligned} & \bar{V}_i^{n+1} \left[ 1 + (\alpha + \beta + r + \lambda) \frac{\Delta\tau}{2} \right] - \frac{\Delta\tau}{2} \beta \bar{V}_{i+1}^{n+1} - \frac{\Delta\tau}{2} \alpha \bar{V}_{i-1}^{n+1} \\ &= \bar{V}_i^n \left[ 1 - (\alpha + \beta + r + \lambda) \frac{\Delta\tau}{2} \right] + \frac{\Delta\tau}{2} \beta \bar{V}_{i+1}^n + \frac{\Delta\tau}{2} \alpha \bar{V}_{i-1}^n + \frac{\Delta\tau}{2} \lambda [(\bar{V} \otimes \bar{f})_i^n + (\bar{V} \otimes \bar{f})_i^{n+1}], \end{aligned} \quad (\text{A.5})$$

where

$$\alpha = \frac{\sigma^2}{2\Delta x^2} - \frac{r - \lambda\kappa - \frac{\sigma^2}{2}}{2\Delta x} \quad (\text{A.6})$$

$$\beta = \frac{\sigma^2}{2\Delta x^2} + \frac{r - \lambda\kappa - \frac{\sigma^2}{2}}{2\Delta x}. \quad (\text{A.7})$$

Let  $\hat{V}^n = [\bar{V}_0^n, \bar{V}_1^n, \dots, \bar{V}_p^n]'$  be the discrete solution vector to equation (A.3). Suppose the initial solution vector is perturbed, i.e.  $\hat{V}^0 = \bar{V}^0 + E^0$ , where  $E^n = [E_0^n, \dots, E_p^n]'$  is the perturbation vector. Note that  $E_p^n = 0$  since Dirichlet boundary conditions are imposed at this node. Then, from equation (A.5), we obtain the following equation for the propagation of the perturbation

$$\begin{aligned} & E_i^{n+1} \left[ 1 + (\alpha + \beta + r + \lambda) \frac{\Delta\tau}{2} \right] - \frac{\Delta\tau}{2} \beta E_{i+1}^{n+1} - \frac{\Delta\tau}{2} \alpha E_{i-1}^{n+1} \\ &= E_i^n \left[ 1 - (\alpha + \beta + r + \lambda) \frac{\Delta\tau}{2} \right] + \frac{\Delta\tau}{2} \beta E_{i+1}^n + \frac{\Delta\tau}{2} \alpha E_{i-1}^n + \frac{\Delta\tau}{2} \lambda [(E \otimes \bar{f})_i^n + (E \otimes \bar{f})_i^{n+1}]. \end{aligned} \quad (\text{A.8})$$

In the following we determine the stability of our discretization scheme using the von Neumann approach (Richtmyer and Morton, 1967). In order to apply the Fourier transform method, we assume that the boundary conditions can be replaced by periodicity conditions. We define the inverse discrete Fourier transform (DFT) as follows (note that we have selected a particular scaling factor)

$$E_i^n = \frac{1}{X_N} \sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}} C_k^n \exp\left(\sqrt{-1} \frac{2\pi}{N} ik\right) \quad (\text{A.9})$$

$$f_i = \frac{1}{X_N} \sum_{l=-\frac{N}{2}+1}^{\frac{N}{2}} F_l \exp\left(\sqrt{-1} \frac{2\pi}{N} il\right), \quad (\text{A.10})$$

where  $C_k$  and  $F_l$  correspond respectively to the discrete Fourier coefficients of  $E$  and  $f$ , and  $X_N = x_{N/2} - x_{-N/2+1}$  is the width of the domain along the  $x$ -axis. Note that the notation  $C_k^n$  should be interpreted as  $(C_k)^n$ , i.e. in this case  $n$  is a power, not a superscript.

The forward transforms are

$$C_k^n = \frac{X_N}{N} \sum_{i=-\frac{N}{2}+1}^{\frac{N}{2}} E_i^n \exp\left(-\sqrt{-1} \frac{2\pi}{N} ik\right) \quad (\text{A.11})$$

$$F_l = \frac{X_N}{N} \sum_{i=-\frac{N}{2}+1}^{\frac{N}{2}} f_i \exp\left(-\sqrt{-1} \frac{2\pi}{N} il\right). \quad (\text{A.12})$$

The discrete correlation is given by

$$(E \otimes \bar{f})_i^n = \frac{X_N}{N} \sum_{j=-\frac{N}{2}+1}^{\frac{N}{2}} E_j^n f_{j-i}, \quad (\text{A.13})$$

which is second order accurate. Substituting (A.9) and (A.10) into (A.13), we obtain

$$\begin{aligned} (E \otimes \bar{f})_i^n &= \frac{X_N}{N} \sum_{j=-\frac{N}{2}+1}^{\frac{N}{2}} \frac{1}{X_N} \sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}} C_k^n \exp\left(\sqrt{-1} \frac{2\pi}{N} jk\right) \frac{1}{X_N} \sum_{l=-\frac{N}{2}+1}^{\frac{N}{2}} F_l \exp\left(\sqrt{-1} \frac{2\pi}{N} (j-i)l\right) \\ &= \frac{1}{X_N} \frac{1}{N} \sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}} \sum_{l=-\frac{N}{2}+1}^{\frac{N}{2}} C_k^n F_l \exp\left(-\sqrt{-1} \frac{2\pi}{N} il\right) \\ &\quad \times \sum_{j=-\frac{N}{2}+1}^{\frac{N}{2}} \exp\left(\sqrt{-1} \frac{2\pi}{N} jk\right) \exp\left(\sqrt{-1} \frac{2\pi}{N} jl\right). \end{aligned}$$

Using the orthogonality condition

$$\sum_{j=-\frac{N}{2}+1}^{\frac{N}{2}} \exp\left(\sqrt{-1} \frac{2\pi}{N} jk\right) \exp\left(\sqrt{-1} \frac{2\pi}{N} jl\right) = \begin{cases} N & \text{if } l = -k \\ 0 & \text{otherwise} \end{cases}, \quad (\text{A.14})$$

we find that

$$(E \otimes \bar{f})_i^n = \frac{1}{X_N} \sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}} C_k^n F_{-k} \exp\left(\sqrt{-1} \frac{2\pi}{N} ik\right). \quad (\text{A.15})$$

Substituting (A.9) and (A.15) into (A.8) gives

$$\begin{aligned} &\frac{1}{X_N} \sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}} C_k^{n+1} \exp\left(\sqrt{-1} \frac{2\pi}{N} ik\right) \left[1 + (\alpha + \beta + r + \lambda) \frac{\Delta\tau}{2}\right] \\ &- \frac{\Delta\tau}{2} \beta \frac{1}{X_N} \sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}} C_k^{n+1} \exp\left(\sqrt{-1} \frac{2\pi}{N} (i+1)k\right) - \frac{\Delta\tau}{2} \alpha \frac{1}{X_N} \sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}} C_k^{n+1} \exp\left(\sqrt{-1} \frac{2\pi}{N} (i-1)k\right) \\ &= \\ &\frac{1}{X_N} \sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}} C_k^n \exp\left(\sqrt{-1} \frac{2\pi}{N} ik\right) \left[1 - (\alpha + \beta + r + \lambda) \frac{\Delta\tau}{2}\right] \\ &+ \frac{\Delta\tau}{2} \beta \frac{1}{X_N} \sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}} C_k^n \exp\left(\sqrt{-1} \frac{2\pi}{N} (i+1)k\right) + \frac{\Delta\tau}{2} \alpha \frac{1}{X_N} \sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}} C_k^n \exp\left(\sqrt{-1} \frac{2\pi}{N} (i-1)k\right) \\ &+ \frac{\Delta\tau}{2} \lambda \left[ \frac{1}{X_N} \sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}} C_k^n F_{-k} \exp\left(\sqrt{-1} \frac{2\pi}{N} ik\right) + \frac{1}{X_N} \sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}} C_k^{n+1} F_{-k} \exp\left(\sqrt{-1} \frac{2\pi}{N} ik\right) \right] \quad (\text{A.16}) \end{aligned}$$

Because of linearity, each Fourier component can be treated separately. Equation (A.16) becomes

$$\begin{aligned}
& C_k^{n+1} \exp\left(\sqrt{-1} \frac{2\pi}{N} ik\right) \left[1 + (\alpha + \beta + r + \lambda) \frac{\Delta\tau}{2}\right] \\
& \quad - \frac{\Delta\tau}{2} \beta C_k^{n+1} \exp\left(\sqrt{-1} \frac{2\pi}{N} (i+1)k\right) - \frac{\Delta\tau}{2} \alpha C_k^{n+1} \exp\left(\sqrt{-1} \frac{2\pi}{N} (i-1)k\right) \\
& \quad = \\
& \quad C_k^n \exp\left(\sqrt{-1} \frac{2\pi}{N} ik\right) \left[1 - (\alpha + \beta + r + \lambda) \frac{\Delta\tau}{2}\right] \\
& \quad + \frac{\Delta\tau}{2} \beta C_k^n \exp\left(\sqrt{-1} \frac{2\pi}{N} (i+1)k\right) + \frac{\Delta\tau}{2} \alpha C_k^n \exp\left(\sqrt{-1} \frac{2\pi}{N} (i-1)k\right) \\
& \quad + \frac{\Delta\tau}{2} \lambda \left[ C_k^n F_{-k} \exp\left(\sqrt{-1} \frac{2\pi}{N} ik\right) + C_k^{n+1} F_{-k} \exp\left(\sqrt{-1} \frac{2\pi}{N} ik\right) \right]. \quad (\text{A.17})
\end{aligned}$$

Dividing equation (A.17) by  $C_k^n \exp\left(\sqrt{-1} \frac{2\pi}{N} ik\right)$ , we obtain

$$\begin{aligned}
& C_k \left[1 + (\alpha + \beta + r + \lambda) \frac{\Delta\tau}{2}\right] - \frac{\Delta\tau}{2} \beta C_k \exp\left(\sqrt{-1} \frac{2\pi}{N} k\right) - \\
& \quad \frac{\Delta\tau}{2} \alpha C_k \exp\left(-\sqrt{-1} \frac{2\pi}{N} k\right) - C_k \frac{\lambda \Delta\tau}{2} F_{-k} \\
& \quad = \\
& \quad \left[1 - (\alpha + \beta + r + \lambda) \frac{\Delta\tau}{2}\right] + \frac{\Delta\tau}{2} \beta \exp\left(\sqrt{-1} \frac{2\pi}{N} k\right) + \\
& \quad \quad \frac{\Delta\tau}{2} \alpha \exp\left(-\sqrt{-1} \frac{2\pi}{N} k\right) + \frac{\Delta\tau}{2} \lambda F_{-k}. \quad (\text{A.18})
\end{aligned}$$

Factoring the  $C_k$  term, equation A.18 becomes

$$C_k = \frac{\left[1 - (\alpha + \beta + r + \lambda) \frac{\Delta\tau}{2}\right] + \frac{\Delta\tau}{2} \beta \exp\left(\sqrt{-1} \frac{2\pi}{N} k\right) + \frac{\Delta\tau}{2} \alpha \exp\left(-\sqrt{-1} \frac{2\pi}{N} k\right) + \frac{\Delta\tau}{2} \lambda F_{-k}}{\left[1 + (\alpha + \beta + r + \lambda) \frac{\Delta\tau}{2}\right] - \frac{\Delta\tau}{2} \beta \exp\left(\sqrt{-1} \frac{2\pi}{N} k\right) - \frac{\Delta\tau}{2} \alpha \exp\left(-\sqrt{-1} \frac{2\pi}{N} k\right) - \frac{\Delta\tau}{2} \lambda F_{-k}}. \quad (\text{A.19})$$

Recalling (A.7), it follows that

$$\begin{aligned}
\alpha + \beta + r + \lambda &= \frac{\sigma^2}{2\Delta x^2} - \frac{(r - \lambda\kappa - \frac{1}{2}\sigma^2)}{2\Delta x} + \frac{\sigma^2}{2\Delta x^2} + \frac{(r - \lambda\kappa - \frac{1}{2}\sigma^2)}{2\Delta x} + r + \lambda \\
&= \frac{\sigma^2}{\Delta x^2} + r + \lambda,
\end{aligned}$$

and

$$\begin{aligned}
& \Delta\tau \beta \exp\left(\sqrt{-1} \frac{2\pi}{N} k\right) + \Delta\tau \alpha \exp\left(-\sqrt{-1} \frac{2\pi}{N} k\right) = \\
& \quad \frac{\sigma^2 \Delta\tau}{2\Delta x^2} \left[ \exp\left(\sqrt{-1} \frac{2\pi}{N} k\right) + \exp\left(-\sqrt{-1} \frac{2\pi}{N} k\right) \right] + \\
& \quad \frac{\Delta\tau (r - \lambda\kappa - \frac{1}{2}\sigma^2)}{2\Delta x} \times \left[ \exp\left(\sqrt{-1} \frac{2\pi}{N} k\right) + \exp\left(-\sqrt{-1} \frac{2\pi}{N} k\right) \right] \\
& \quad = \frac{\sigma^2 \Delta\tau}{\Delta x^2} \cos\left(\frac{2\pi}{N} k\right) + \frac{\sqrt{-1} \Delta\tau (r - \lambda\kappa - \frac{1}{2}\sigma^2)}{\Delta x} \sin\left(\frac{2\pi}{N} k\right).
\end{aligned}$$

Using the above results in (A.19), we find

$$C_k = \frac{\left[1 - \left(\frac{\sigma^2}{\Delta x^2} + r + \lambda\right) \frac{\Delta\tau}{2}\right] + \frac{1}{2} \left[\frac{\sigma^2 \Delta\tau}{\Delta x^2} \cos\left(\frac{2\pi}{N}k\right) + \sqrt{-1} \frac{\Delta\tau}{\Delta x} \left(r - \lambda\kappa - \frac{1}{2}\sigma^2\right) \sin\left(\frac{2\pi}{N}k\right) + \Delta\tau \lambda F_{-k}\right]}{\left[1 + \frac{1}{2} \left(\frac{\sigma^2}{\Delta x^2} + r + \lambda\right) \Delta\tau\right] - \frac{1}{2} \left[\frac{\sigma^2 \Delta\tau}{\Delta x^2} \cos\left(\frac{2\pi}{N}k\right) + \sqrt{-1} \frac{\Delta\tau}{\Delta x} \left(r - \lambda\kappa - \frac{1}{2}\sigma^2\right) \sin\left(\frac{2\pi}{N}k\right) + \Delta\tau \lambda F_{-k}\right]}. \quad (\text{A.20})$$

Letting

$$\begin{aligned} F_{-k}^R &= \text{Re}(F_{-k}) \\ F_{-k}^I &= \text{Im}(F_{-k}), \end{aligned}$$

equation (A.20) gives

$$|C_k|^2 = \frac{\left[1 - \left(\frac{\sigma^2}{\Delta x^2} + r + \lambda\right) \frac{\Delta\tau}{2} + \frac{\sigma^2 \Delta\tau}{2\Delta x^2} \cos\left(\frac{2\pi}{N}k\right) + \frac{\Delta\tau}{2} \lambda F_{-k}^R\right]^2 + \left[\frac{\Delta\tau}{2\Delta x} \left(r - \lambda\kappa - \frac{1}{2}\sigma^2\right) \sin\left(\frac{2\pi}{N}k\right) + \lambda \frac{\Delta\tau}{2} F_{-k}^I\right]^2}{\left[1 + \left(\frac{\sigma^2}{\Delta x^2} + r + \lambda\right) \frac{\Delta\tau}{2} - \frac{\sigma^2 \Delta\tau}{2\Delta x^2} \cos\left(\frac{2\pi}{N}k\right) - \frac{\Delta\tau}{2} \lambda F_{-k}^R\right]^2 + \left[\frac{\Delta\tau}{2\Delta x} \left(r - \lambda\kappa - \frac{1}{2}\sigma^2\right) \sin\left(\frac{2\pi}{N}k\right) + \lambda \frac{\Delta\tau}{2} F_{-k}^I\right]^2}, \quad (\text{A.21})$$

or

$$|C_k|^2 = \frac{\left[1 - \frac{r\Delta\tau}{2} - \frac{\sigma^2 \Delta\tau}{2\Delta x^2} \left(1 - \cos\left(\frac{2\pi}{N}k\right)\right) - \frac{\Delta\tau\lambda}{2} \left(1 - F_{-k}^R\right)\right]^2 + \left[\frac{\Delta\tau}{2\Delta x} \left(r - \lambda\kappa - \frac{1}{2}\sigma^2\right) \sin\left(\frac{2\pi}{N}k\right) + \lambda \frac{\Delta\tau}{2} F_{-k}^I\right]^2}{\left[1 + \frac{r\Delta\tau}{2} + \frac{\sigma^2 \Delta\tau}{2\Delta x^2} \left(1 - \cos\left(\frac{2\pi}{N}k\right)\right) + \frac{\Delta\tau\lambda}{2} \left(1 - F_{-k}^R\right)\right]^2 + \left[\frac{\Delta\tau}{2\Delta x} \left(r - \lambda\kappa - \frac{1}{2}\sigma^2\right) \sin\left(\frac{2\pi}{N}k\right) + \lambda \frac{\Delta\tau}{2} F_{-k}^I\right]^2}. \quad (\text{A.22})$$

Note that

$$F_{-k} = \frac{X_N}{N} \sum_{j=-\frac{N}{2}+1}^{\frac{N}{2}} \bar{f}_j \exp\left(\sqrt{-1} \frac{2\pi}{N}kj\right). \quad (\text{A.23})$$

Then, from (3.6), we have

$$\frac{X_N}{N} \sum_{j=-\frac{N}{2}+1}^{\frac{N}{2}} \bar{f}_j \leq 1, \quad (\text{A.24})$$

so that

$$|F_{-k}| \leq 1, \quad (\text{A.25})$$

and hence

$$-1 \leq F_{-k}^R \leq +1. \quad (\text{A.26})$$

It then follows that  $(\forall k \in -N/2 + 1, \dots, +N/2)$

$$\begin{aligned} \left|1 + \frac{r\Delta\tau}{2} + \frac{\sigma^2 \Delta\tau}{2\Delta x^2} \left(1 - \cos\left(\frac{2\pi}{N}k\right)\right) + \frac{\Delta\tau\lambda}{2} \left(1 - F_{-k}^R\right)\right| &\geq \\ \left|1 - \frac{r\Delta\tau}{2} - \frac{\sigma^2 \Delta\tau}{2\Delta x^2} \left(1 - \cos\left(\frac{2\pi}{N}k\right)\right) - \frac{\Delta\tau\lambda}{2} \left(1 - F_{-k}^R\right)\right|, & \end{aligned} \quad (\text{A.27})$$

and consequently  $|C_k| < 1, \forall k$ , so the scheme is unconditionally strictly stable.

## B Error Estimates for Correlation Integral

In this Appendix, we show how to extend the domain of integration of the integral (6.3) such that FFT wrap-around effects are less than a user specified tolerance. To avoid algebraic complication, we derive the results in an informal way. We focus on the error due to the FFT wrap-around. We assume that any other errors (interpolation, discretization of the integral, etc.) are second order in the asset grid spacing, and we ignore such errors in the following.

We make the assumptions

$$\begin{aligned}
y_{\min} &< 0 \\
y_{\max} &> 0 \\
\bar{V}(y) &\geq 0 \\
\bar{V}(y) &\leq \max(A_2, A_1 e^y) \\
\bar{f}(y) &\leq A_3 e^{-\gamma|y|}, \quad \forall y, \quad \gamma > 2 \\
\max \bar{V}(y) &\leq A_4 ; \quad y \in [y_{\min}, y_{\max}]
\end{aligned} \tag{B.1}$$

where  $A_1, A_2, A_3, A_4$  are constants independent of  $y$ . We assume that  $\bar{V}(y)$  is only given at discrete points on the interval  $y \in [y_{\min}, y_{\max}]$ .

Recall that we wish to compute an approximation to

$$I(x) = \int_{y_{\min}}^{y_{\max}} \bar{V}(x+y) \bar{f}(y) dy. \tag{B.2}$$

Considering the case where  $x = y_{\max}$ , equation (B.2) becomes

$$\begin{aligned}
I(y_{\max}) &= \int_{y_{\min}}^{y_{\max}} \bar{V}(y_{\max} + y) \bar{f}(y) dy \\
&= \int_{y_{\min}}^0 \bar{V}(y_{\max} + y) \bar{f}(y) dy + \int_0^{y_{\max}} \bar{V}(y_{\max} + y) \bar{f}(y) dy.
\end{aligned} \tag{B.3}$$

When using an FFT to evaluate the correlation integral, the term

$$\int_0^{y_{\max}} \bar{V}(y_{\max} + y) \bar{f}(y) dy$$

is actually evaluated using

$$\int_0^{y_{\max}} \bar{V}(y_{\min} + y) \bar{f}(y) dy \tag{B.4}$$

due to the wrap-around effect of the discrete FFT. The idea here is to extend the definition of  $\bar{V}$  to the interval  $y \in [y_{\min} - \Delta y^-, y_{\max} + \Delta y^+]$ . We make the assumption that the values of  $\bar{V}$  can be obtained in the extended regions correct to second order, and we ignore these errors in the following. Setting  $\Delta y^- = 0$  for the time being, equation (B.3) becomes

$$I(y_{\max}) = \int_{y_{\min}}^{\Delta y^+} \bar{V}(y_{\max} + y) \bar{f}(y) dy + \int_{\Delta y^+}^{y_{\max}} \bar{V}(y_{\max} + y) \bar{f}(y) dy. \tag{B.5}$$

Now, the wrap-around error  $E(y_{\max})$  which will occur using an FFT will be

$$\begin{aligned}
E(y_{\max}) &\simeq \int_{\Delta y^+}^{y_{\max}} |\bar{V}(y_{\max} + y) - \bar{V}(y_{\min} + \{y - \Delta y^+\})| \bar{f}(y) dy \\
&\leq \max \left[ \int_{\Delta y^+}^{y_{\max}} A_1 e^{(y_{\max} + y)} \bar{f}(y) dy, A_2 \int_{\Delta y^+}^{y_{\max}} \bar{f}(y) dy \right] \\
&\leq \max \left[ A_1 e^{y_{\max}} \int_{\Delta y^+}^{y_{\max}} e^y A_3 e^{-\gamma y} dy, A_2 \int_{\Delta y^+}^{y_{\max}} A_3 e^{-\gamma y} dy \right] \\
&\leq \max \left[ A_1 e^{y_{\max}} \frac{e^{\Delta y^+} A_3 e^{-\gamma \Delta y^+}}{\gamma - 1}, A_2 A_3 \frac{e^{-\gamma \Delta y^+}}{\gamma} \right] \\
&\leq A_3 e^{\Delta y^+} e^{-\gamma \Delta y^+} \max[A_1 e^{y_{\max}}, A_2] \\
&\leq A_3 e^{-\gamma \Delta y^+} e^{\Delta y^+} A_4
\end{aligned} \tag{B.6}$$

So, if we require that the relative error at  $x = y_{\max}$  be less than a given tolerance, then we select  $\Delta y^+$  such that

$$\frac{E(y_{\max})}{A_4} \leq A_3 e^{-\gamma \Delta y^+} e^{\Delta y^+} < tol_R. \tag{B.7}$$

For practical purposes, we assume that  $\bar{f}(\Delta y^+) \simeq A_3 e^{-\gamma \Delta y^+}$ , so that we can approximate equation (B.7) by

$$\frac{E(y_{\max})}{A_4} \simeq \bar{f}(\Delta y^+) e^{\Delta y^+} < tol_R. \tag{B.8}$$

Note that a relative error criteria is a reasonable choice at  $x = y_{\max}$  since  $\bar{V}(y_{\max})$  may be  $O(e^{y_{\max}})$ .

Following the same reasoning at  $x = y_{\min}$ , assuming that  $\Delta y^+ = 0$  for simplicity, we now extend the domain of  $\bar{V}$  to the left by  $\Delta y^-$ , and we assume that we can determine  $\bar{V}$  in  $[y_{\min} - \Delta y^-, y_{\min}]$  correct to second order. The error in  $I(y_{\min})$  due to wrap-around is given by

$$\begin{aligned}
E(y_{\min}) &\simeq \int_{y_{\min}}^{-\Delta y^-} |\bar{V}(y_{\min} + y) - \bar{V}(y_{\max} + y + \Delta y^-)| \bar{f}(y) dy \\
&\leq \max \left[ \int_{y_{\min}}^{-\Delta y^-} A_1 e^{(y_{\max} + y + \Delta y^-)} \bar{f}(y) dy, A_2 \int_{y_{\min}}^{-\Delta y^-} \bar{f}(y) dy \right] \\
&\leq \max \left[ A_1 e^{(y_{\max} + \Delta y^-)} \int_{y_{\min}}^{-\Delta y^-} e^y A_3 e^{\gamma y} dy, A_2 A_3 \frac{e^{-\gamma \Delta y^-}}{\gamma} \right] \\
&\leq \max \left[ A_1 e^{(y_{\max} + \Delta y^-)} A_3 e^{-\gamma \Delta y^-} \left( \frac{e^{-\Delta y^-}}{1 + \gamma} \right), A_2 A_3 \frac{e^{-\gamma \Delta y^-}}{\gamma} \right] \\
&\leq A_3 e^{-\gamma \Delta y^-} \max[A_2, A_1 e^{y_{\max}}] \\
&\leq A_3 e^{-\gamma \Delta y^-} A_4
\end{aligned} \tag{B.9}$$

Therefore we can require that the absolute error at  $x = y_{\min}$  be less than a specified tolerance if we select  $\Delta y^-$  such that

$$E(y_{\min}) \leq A_4 A_3 e^{-\gamma \Delta y^-} < tol_L. \tag{B.10}$$

Again, for practical purposes we assume that  $\bar{f}(-\Delta y^-) \simeq A_3 e^{-\gamma \Delta y^-}$  and so we approximate equation (B.7) to obtain

$$E(y_{\min}) \leq A_4 \bar{f}(-\Delta y^-) < tol_L. \quad (\text{B.11})$$

An estimate of  $A_4$  can be obtained from

$$A_4 \simeq \max_{0 \leq S \leq S_{\max}} V(S, \tau = 0). \quad (\text{B.12})$$

Note that an absolute error criteria is appropriate near  $x = y_{\min}$  since  $\bar{V}$  is bounded at  $y = y_{\min}$ .

Typically, we chose  $tol_L = tol_R = 10^{-6}$ . Since the wrap-around errors are largest at  $x = y_{\min}, x = y_{\max}$ , selecting the domain extensions which satisfy equations (B.7) and (B.10) will bound these errors at all other points. The domain extensions are illustrated in Figures 6-7.

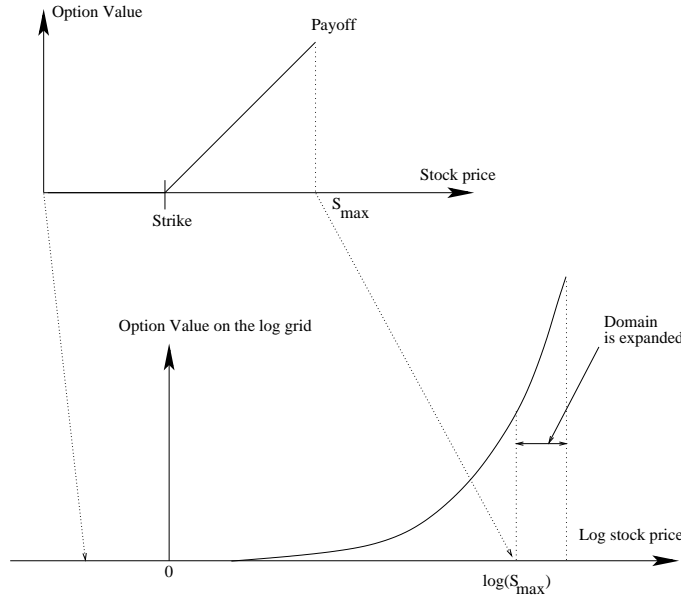


FIGURE 6: The value of the option is interpolated onto the log-spaced grid. The right hand side boundary of the log-spaced grid  $y_{\max} = \log(S_{\max})$  is expanded by  $\Delta y^+$ , where  $\Delta y^+$  is given by equation (B.7).

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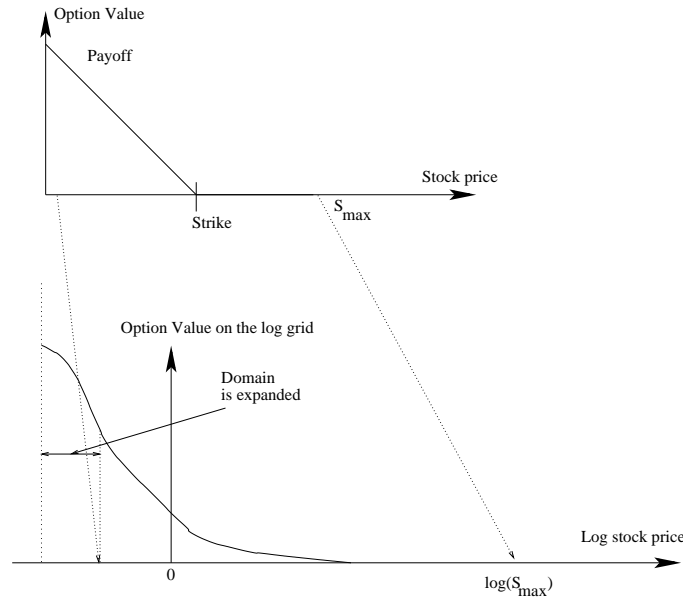


FIGURE 7: The value of the option is interpolated onto the log-spaced grid. However, the value of the option  $V(S, \tau)$  at  $S = 0$  is not used. The left hand side boundary grid point is chosen to be  $\log S_1$  where  $S_1$  is the grid point nearest to  $S = 0$ . This left boundary is then expanded by  $\Delta y^-$ , which is given by equation (B.10).

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