A Numerical Scheme for the Impulse Control Formulation for Pricing Variable Annuities with a Guaranteed Minimum Withdrawal Benefit (GMWB)

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Abstract

In this paper, we outline an impulse stochastic control formulation for pricing variable annuities with a Guaranteed Minimum Withdrawal Benefit (GMWB) assuming the policyholder is allowed to withdraw funds continuously. We develop a single numerical scheme for solving the Hamilton-Jacobi-Bellman (HJB) variational inequality corresponding to the impulse control problem, and for pricing realistic discrete withdrawal contracts. We prove the convergence of our scheme to the viscosity solution of the continuous withdrawal problem, provided a strong comparison result holds. The convergence to the viscosity solution is also proved for the discrete withdrawal case. Numerical experiments are conducted, which show a region where the optimal control appears to be non-unique.

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1 Introduction

Variable annuities with a Guaranteed Minimum Withdrawal Benefit (GMWB) are extremely popular since these contracts provide investors with the tax-deferred feature of variable annuities as well as the additional benefit of the guaranteed minimum payment. In 2004, sixty-nine percent of all variable annuity contracts sold in the US included a GMWB option [4].

A GMWB contract involves payment of a lump sum to an insurance company. This lump sum is then invested in risky assets. The holder of this contract may withdraw up to a specified amount in each year for the life of the contract, regardless of the performance of the risky asset. The holder may also withdraw more than the specified amount, subject to certain penalties and conditions on

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the remaining guaranteed withdrawals. Upon contract expiry, the holder can convert the remaining investment in the risky assets to cash, and withdraw this amount.

The insurance company is in effect providing a cash flow guarantee, even though the investment return is uncertain. In return for this guarantee, the insurance company receives a proportional fee based on the value of the risky asset.

This contract is quite complex, since the holder has many options, including the timing and amount of withdrawals. In the continuous withdrawal limit, the no-arbitrage value of the guarantee can be determined as the solution to a singular stochastic control problem [12, 6].

However, as pointed out in [19], it is often advantageous to formulate these contracts as impulse control problems[1]. This formulation has the advantage that it is easier to incorporate the complex features of real contracts, such as various reset provision features.

Another criticism of the singular stochastic control formulation arises from the fact that real contracts typically only allow withdrawals at discrete intervals (e.g. the anniversary of contract inception). Consequently, it might be supposed that a discrete withdrawal formulation would be preferred [4, 6]. In [6], a penalty approach is used to solve the singular control problem. Some experimental computations were also included in [6] which indicated that, in the limit that withdrawals are allowed at infinitesimal intervals, the value of the discrete withdrawal contract converged to the solution of the singular control problem. No proof of this convergence was given in [6]. It is also pointed out in [6], that some complex features are straightforward to include in a discrete withdrawal formulation, but are very difficult to include in a continuous withdrawal, singular control formulation.

The objective of this paper is to provide a single numerical method which can be used to price both discrete withdrawal contracts and continuous withdrawal contracts. We consider as a starting point the impulse control formulation of the continuous withdrawal contract by introducing a strictly positive fixed cost. Then, we formulate a numerical method for pricing the discrete withdrawal contract. Our main results are the following

- We prove that the discrete withdrawal scheme converges to the viscosity solution of the discrete withdrawal contract.

- We generalize the scheme for the discrete withdrawal case to the continuous withdrawal case by setting each discretization timestep as a possible withdrawal time. Provided a strong comparison result holds, we prove that the scheme converges to the unique viscosity solution of the HJB variational inequality corresponding to the impulse control problem by verifying the l_∞ stability, monotonicity and consistency of the scheme and using the basic results in [3, 2].

- In the continuous withdrawal limit, the numerical results demonstrate that our scheme can solve the impulse control problem with a nonzero fixed cost as well as the singular control problem by setting the fixed cost to be zero, although the convergence is proved only for the former case.

- We provide some numerical tests which indicate that the no-arbitrage fee for the discrete withdrawal contract is very close to the continuous contract fee (i.e. to within a few basis points) even for fairly infrequent withdrawal intervals (e.g. once every half a year).

[1] Refer to [11, 13, 16, 14] for some applications of impulse control problems.

- Our numerical results appear to show that the optimal control strategy may not be unique. That is, there exists a region where different control strategies can result in the same guarantee value.

The advantage of this approach is that we now have a single scheme which can price realistic discrete withdrawal contracts, as well as the limiting case of allowing continuous withdrawals. In both cases, the numerical technique can be shown to converge to the viscosity solution.

2 Contract Description

There exist many variations of GMWB variable annuity contracts. In the following, we briefly describe a typical contract that we consider in this paper. The contract consists of a so called personal sub-account and a virtual guarantee account. The funds in the sub-account are managed by the insurance company investing in a diversified reference portfolio of a specific class of assets. At the inception of the policy, the policyholder pays a lump-sum premium to the insurer. This premium forms the initial balance of the sub-account and that of the guarantee account. Prior to the contract maturity, the policyholder is also committed to pay an annual insurance fee proportional to the sub-account balance.

A GMWB option allows the policyholder to withdraw funds from the sub-account at prespecified times (e.g., on a annual or semi-annual basis). Each withdrawal reduces the balance of the guarantee account by the corresponding amount. The policyholder can keep withdrawing as long as the balance of the guarantee account is above zero, even when the sub-account balance falls to zero prior to the policy maturity.

Following [12, 6], we assume the net amount received by the policyholder after a withdrawal is subject to a withdrawal level specified in the contract. If the withdrawal amount does not exceed the contract withdrawal level, then the policyholder receives the complete withdrawal amount. Otherwise, if the withdrawal amount is above the contract level, then the investor receives the remaining amount after a proportional penalty charge is imposed. At the maturity of the policy, the policyholder can choose to receive either the remaining balance of the sub-account if it is positive or the remaining balance of the guarantee account subject to a penalty charge.

As discussed in [4], for some variations of GMWB contracts, the balance of the guarantee account can increase at certain points in time if no withdrawals have been made so far. In [12, 6] another possibility is discussed whereby an excessive withdrawal may result in a decrease greater than the withdrawal amount in the guarantee account. In this paper, we do not consider these contractual complications and leave them for future research.

Prior to presenting the pricing equations, we first introduce the following notation.

2.1 Problem notation

Let S denote the value of the reference portfolio of assets underlying the variable annuity policy. Following [6], we assume that the risk adjusted process of S is modeled by a stochastic differential equation (SDE) given by

$$dS = rSdt + \sigma SdZ, \tag{2.1}$$

where $r \geq 0$ is the riskless interest rate, σ is the volatility, dZ is an increment of the standard Gauss-Wiener process.

Let W denote the balance of the personal variable annuity sub-account. Let A denote the current balance of the guarantee account. Let w_0 be the initial sub-account balance and guarantee account balance which is the same as the premium paid upfront. Then A can be any value lying in $[0, w_0]$. Let $\alpha \geq 0$ denote the proportional annual insurance rate paid by the policyholder. Then from (2.1) the risk adjusted dynamics of W follows an SDE given by

$$\begin{aligned} dW &= (r-\alpha)Wdt + \sigma W dZ + dA, \quad \text{if } W > 0 && (2.2)\\ W &= 0, \quad \text{if } W = 0. && (2.3) \end{aligned}$$

Note that the above equations indicate that W will stay at zero from the time it reaches zero.

Let T denote the maturity of the policy. Let $V(W, A, \tau)$ denote the no-arbitrage value of the variable annuity with GMWB at time $t = T - \tau$ when the value of the sub-account is W and the balance of the guarantee account is A. Here we use τ to represent the time to maturity of the contract.

3 Continuous Withdrawal Model

Under the continuous withdrawal scenario, we denote by $\hat{\gamma}$ the control variable representing the continuous withdrawal rate. Following [6], we assume $0 \leq \hat{\gamma} \leq \lambda$, where λ is the upper bound of $\hat{\gamma}$. As shown in [6], the dynamics of A is determined by that of $\hat{\gamma}$ as follows:

$$A(t) = A(0) - \int_0^t \hat{\gamma}(s)ds. \tag{3.1}$$

3.1 Singular control formulation

In this subsection, we recall the singular stochastic control formulation presented in [6]. Let $\hat{f}(\hat{\gamma})$ be a function of $\hat{\gamma}$ denoting the rate of cash flow received by the policyholder due to the continuous withdrawal. According to [6], a penalty is charged if the withdrawal rate exceeds the contract withdrawal rate, denoted G_r. Specifically, we assume that if $\hat{\gamma} \leq G_r$, there is no penalty imposed; if $\hat{\gamma} > G_r$, then there is a proportional penalty charge $\kappa(\hat{\gamma} - G_r)$, that is, the net revenue rate received by the policyholder is $\hat{\gamma} - \kappa(\hat{\gamma} - G_r)$ if $\hat{\gamma} > G_r$, where κ is a positive constant. Consequently, we can write $\hat{f}(\hat{\gamma})$ as a piecewise linear function

$$\hat{f}(\hat{\gamma}) = \begin{cases} \hat{\gamma} & \text{if } 0 \leq \hat{\gamma} \leq G_r, \\ \hat{\gamma} - \kappa(\hat{\gamma} - G_r) & \text{if } \hat{\gamma} > G_r. \end{cases} \tag{3.2}$$

As shown in [6], the annuity value $V(W, A, \tau)$, assuming equations (2.1-3.2), is given by the solution of the following Hamilton-Jacobi-Bellman (HJB) equation

$$V_\tau - \mathcal{L}V - \sup_{\hat{\gamma} \in [0,\lambda]} \left[\hat{f}(\hat{\gamma}) - \hat{\gamma}V_W - \hat{\gamma}V_A\right] = 0 \tag{3.3}$$

where the operator $\mathcal{L}$ is

$$\mathcal{L}V = \frac{1}{2}\sigma^2 W^2 V_{WW} + (r-\alpha)WV_W - rV. \tag{3.4}$$

Since the function $\hat{f}(\hat{\gamma})$ is piecewise linear, the maximum in (3.3) is achieved at $\hat{\gamma} = 0$, $\hat{\gamma} = G_r$, or $\hat{\gamma} = \lambda$. Thus, equation (3.3) is identical to the following free boundary value problem resulting from evaluating the objective function of the maximization problem at $\hat{\gamma} = 0, G_r, \lambda$, respectively

$$
\begin{aligned}
&V_\tau - \mathcal{L}V \geq 0, && (3.5)\\
&V_\tau - \mathcal{L}V - G_r(1 - V_W - V_A) \geq 0, && (3.6)\\
&V_\tau - \mathcal{L}V - \kappa G_r - \lambda\big[(1-\kappa) - V_W - V_A\big] \geq 0, && (3.7)
\end{aligned}
$$

where the equality holds in at least one of the three cases above. Since $\hat{f}(\hat{\gamma}) = \hat{\gamma}$ for $\hat{\gamma} \in [0, G_r]$, inequalities (3.5-3.6) are identical to

$$V_\tau - \mathcal{L}V - \sup_{\hat{\gamma}\in[0,G_r]} \big[\hat{\gamma}(1 - V_W - V_A)\big] \geq 0. \tag{3.8}$$

Taking the limit $\lambda \to \infty$ (corresponding to an infinite withdrawal rate, or a finite withdrawal amount), inequality (3.7) is equivalent to

$$V_W + V_A - (1-\kappa) \geq 0, \tag{3.9}$$

where the expression $V_\tau - \mathcal{L}V - \kappa G_r$ in (3.7) becomes negligible as $\lambda \to \infty$.

Consequently, combining inequalities (3.8-3.9) and using the fact that the equality holds in one of the two cases results in the following HJB variational inequality, as proposed in [6]:

$$\min\left\{ V_\tau - \mathcal{L}V - \sup_{\hat{\gamma}\in[0,G_r]} \big(\hat{\gamma} - \hat{\gamma}V_W - \hat{\gamma}V_A\big), V_W + V_A - (1-\kappa) \right\} = 0. \tag{3.10}$$

3.2 Impulse control formulation

As discussed in [19], it is advantageous to reformulate the pricing equation (3.10) with a similar HJB variational inequality based on an impulse control argument. Roughly speaking, the policyholder can choose to either withdraw continuously at a rate no greater than G_r or withdraw a finite amount instantaneously; withdrawing a finite amount is subject to a penalty charge proportional to the amount of the withdrawal as well as subject to a strictly positive fixed cost, denoted by c. Due to the associated penalty, the withdrawal of a finite amount is optimal only at some discrete stopping times t_s^n.

Since the amount of a finite withdrawal can be infinitesimally small, it is difficult to distinguish the two cases: withdrawing at a finite rate or withdrawing an infinitesimal amount. This results in non-uniqueness of the solution to the impulse control formulation. As a result, the nonzero fixed cost c is introduced as a technical tool to distinguish these two cases and resolve the non-uniqueness problem. The nonzero fixed cost is commonly assumed in the impulse control literature [1, 11, 13, 16, 14]. Note that the discrete withdrawal model proposed in Section 4 allows the fixed cost to be zero.

Next we outline the impulse control formulation. In regions where it is optimal to withdraw continuously we must have

$$
\begin{aligned}
&V_\tau - \mathcal{L}V - \sup_{\hat{\gamma}\in[0,G_r]} \big(\hat{\gamma} - \hat{\gamma}V_W - \hat{\gamma}V_A\big) = 0 && (3.11)\\
&V - \sup_{\gamma\in(0,A]} \big[V(\max(W-\gamma, 0), A-\gamma, \tau) + (1-\kappa)\gamma - c\big] \geq 0\ , && (3.12)
\end{aligned}
$$

where the operator $\mathcal{L}$ is given in (3.4) and $c > 0$ is the fixed cost. Equation (3.11) represents the case of continuous withdrawal with the withdrawal rate residing between 0 and G_r. Since $\hat{\gamma} \leq G_r$, there is no penalty applied and the complete withdrawal rate $\hat{\gamma}$ is received by the policyholder. Equation (3.12) indicates that it is not optimal to withdraw a finite amount.

In regions where it is optimal to withdraw a finite amount, we have

$$V_\tau - \mathcal{L}V - \sup_{\hat{\gamma} \in [0, G_r]} \big(\hat{\gamma} - \hat{\gamma} V_W - \hat{\gamma} V_A \big) \geq 0 \tag{3.13}$$

$$V - \sup_{\gamma \in (0, A]} \big[V(\max(W - \gamma, 0), A - \gamma, \tau) + (1 - \kappa)\gamma - c \big] = 0 \ . \tag{3.14}$$

Equation (3.14) represents the case with a finite nonzero withdrawal amount $\gamma \in (0, A]$. Note that γ represents the actual withdrawal amount, as opposed to the withdrawal rate $\hat{\gamma}$. After an instantaneous withdrawal of γ, the balance of the sub-account and the guarantee account decreases from W and A to $\max(W - \gamma, 0)$ and $A - \gamma$, respectively; at the same time, a penalized cash flow $(1 - \kappa)\gamma - c$ is provided to the policyholder.

Combining equations (3.11-3.12) and (3.13-3.14) gives the following HJB variational inequality

$$\min \left\{ V_\tau - \mathcal{L}V - \sup_{\hat{\gamma} \in [0, G_r]} \big(\hat{\gamma} - \hat{\gamma} V_W - \hat{\gamma} V_A \big), V - \sup_{\gamma \in (0, A]} \big[V(\max(W - \gamma, 0), A - \gamma, \tau) + (1 - \kappa)\gamma - c \big] \right\} = 0, \tag{3.15}$$

Remark 3.1. *In [6], a reset provision feature is incorporated into the discrete withdrawal formulation (4.3). Following the method in [6], we can easily incorporate this feature into our impulse control formulation (3.15) under the continuous withdrawal scenario. However, as pointed in [6], it is not straightforward to incorporate the reset provision feature into the singular control formulation (3.10).*

In the following, we will consider only the impulse control formulation (3.15) with $c > 0$. Although $c > 0$ is required in our theoretical formulation, our numerical scheme proposed in later sections accepts both $c > 0$ and $c = 0$. However, convergence is proved only for the $c > 0$ case. In practice, of course, we would expect that a very small $c > 0$ will have very little effect on the computed solution, and we verify this in our numerical experiments. Indeed, our results for small $c > 0$ are the same (to within discretization errors) as those reported in [6] based on the singular control formulation.

Remark 3.2 (Hedging Fees)**.** *Equation (3.15) is derived making specific assumptions concerning the allocation of fees [12, 6]. In some cases, the asset manager may charge fees separately from the guarantee provider, in which case the the guarantee equation will be slightly different [18, 17]. However, all the numerical methods developed for equation (3.15) can be easily generalized to cases where different fee allocation is modelled.*

3.3 Boundary conditions for the impulse control problem

In order to completely specify the GMWB variable annuity pricing problem, we need to provide boundary conditions for equation (3.15). Following [6], the terminal boundary condition is

$$V(W, A, \tau = 0) = \max(W, (1 - \kappa)A - c). \tag{3.16}$$

This means the policyholder obtains the maximum of the remaining guarantee withdrawal net after the penalty charge ($(1-\kappa)A - c$) or the remaining sub-account balance (W).

The domain for equation (3.15) is $(W, A) \in [0, \infty] \times [0, w_0]$. For computational purposes, we need to solve the equation in a finite computational domain $[0, W_{\max}] \times [0, w_0]$.

As $A \to 0$, that is, the guarantee account balance approaches zero, the withdrawal rate $\hat{\gamma}$ must approach zero. Thus by taking $\hat{\gamma} \to 0$ and $A \to 0$ in equation (3.15), we obtain a linear PDE

$$V_\tau - \mathcal{L}V = 0 \tag{3.17}$$

at $A = 0$. Note that this is essentially a Dirichlet boundary condition at $A = 0$ because we can simply solve equation (3.17) independently without using any information other than at $A = 0$.

As $A \to w_0$, since $\hat{\gamma} \geq 0$, the characteristics of the PDE in (3.15) are outgoing in the A direction at $A = w_0$. As a result, we can directly solve equation (3.15) along the $A = w_0$ boundary, no further information is needed.

As $W \to 0$, following [6], we assume $V_W = 0$. Taking the limit $W \to 0$ in (3.15) and applying $V_W = 0$, we obtain

$$\min\left\{ V_\tau - rV - \sup_{\hat{\gamma} \in [0, G_r]} \big(\hat{\gamma} - \hat{\gamma} V_A\big), V - \sup_{\gamma \in (0, A]} \big[V(0, A - \gamma, \tau) + (1-\kappa)\gamma - c\big] \right\} = 0 \tag{3.18}$$

at $W = 0$. Thus, similar to equation (3.17), equation (3.18) is essentially a Dirichlet boundary condition since we can solve the equation without requiring any information other than at $W = 0$.

As $W \to \infty$, according to [6], the value function satisfies $V(W, A, \tau) \to e^{-\alpha\tau} W$. As a result, we impose the Dirichlet boundary condition

$$V(W, A, \tau) = e^{-\alpha\tau} W, \quad \text{if } W = W_{\max}. \tag{3.19}$$

Note that since we will choose $W_{\max} \gg w_0$, evaluating V at $W = W_{\max}$ using equation (3.16) gives $V = W_{\max}$, which is the same as evaluating V at $\tau = 0$ using equation (3.19).

4 Discrete Withdrawal Model

In practice, GMWB variable annuities allow withdrawals only at discrete observation times [4, 6], denoted by t_O^i, $i = 1, 2, \ldots, K$, where $t_O^K = T$, and we denote by $t_O^0 = 0$ the inception time of the policy. The pricing model under the discrete withdrawal scenario was suggested in [6].

Following [6], we assume there is no withdrawal allowed at $t = 0$. Let $\tau_O^k = T - t_O^i$ be the time to maturity at the ith withdrawal time with $\tau_O^0 = T$ and $\tau_O^K = 0$, where $k = K - i$. Let $\Delta\tau_O^{k+1} = \tau_O^{k+1} - \tau_O^k$. We denote by γ^k the control variable representing the discrete withdrawal amount at $\tau = \tau_O^k$; γ^k can take any value in $\gamma^k \in [0, A]$.

Let $f(\gamma^k)$ be a function of γ^k representing the cash flow received by the policyholder at the observation time $\tau = \tau_O^k$. Let $\bar{G}(t_O^i)$ denote the contract withdrawal level at $t = t_O^i$. A penalty charge will be imposed if $\gamma^k > \bar{G}(t_O^i)$. In this paper, we assume $\bar{G}(t_O^i) = G_r \cdot (t_O^i - t_O^{i-1})$, where G_r is the contract withdrawal rate as introduced in the continuous withdrawal model and $t_O^i - t_O^{i-1}$ denotes the interval between two consecutive withdrawal times. Let $G(\tau_O^k)$ denote the corresponding contract withdrawal level in terms of backward times $\tau = T - t$. Then we have

$$G(\tau_O^k) = G_r(\tau_O^{k+1} - \tau_O^k) = G_r \Delta\tau_O^{k+1}, \quad k = 0, \ldots, K-1. \tag{4.1}$$

Let $G^k = G(\tau_O^k)$. Then if $\gamma^k \le G^k$, there is no penalty imposed; if $\gamma^k > G^k$, then there is a proportional penalty charge $\kappa(\gamma^k - G^k)$ and a fixed cost c associated with the excessive withdrawal, that is, the net amounted received by the policyholder is $\gamma^k - \kappa(\gamma^k - G^k) - c$ if $\gamma^k > G^k$, where κ is a positive constant and c is a non-negative constant. Note that in the discrete withdrawal model, we allow the fixed cost to be zero. Consequently, we can write f as the piecewise function

$$f(\gamma^k) = \begin{cases} \gamma^k & \text{if } 0 \le \gamma^k \le G^k, \\ \gamma^k - \kappa(\gamma^k - G^k) - c & \text{if } \gamma^k > G^k. \end{cases} \tag{4.2}$$

According to equation (4.2), if $c > 0$, then $f(\gamma^k)$ is uniformly continuous for $\gamma^k \in [0, G^k] \cup (G^k, A]$ and exhibits a discontinuity at $\gamma^k = G^k$.

4.1 Pricing equation for the discrete withdrawal problem

As shown in [6], at the withdrawal time $\tau = \tau_O^k$, V satisfies the following no-arbitrage condition

$$V(W, A, \tau_O^{k+}) = \sup_{\gamma^k \in [0,A]} \Big[V\big(\max(W - \gamma^k, 0), A - \gamma^k, \tau_O^k\big) + f(\gamma^k) \Big], \quad k = 0, \ldots, K-1, \tag{4.3}$$

where τ_O^{k+} denotes the time infinitesimally after τ_O^k.

Within each time interval $[\tau_O^{k+}, \tau_O^{k+1}]$, $k = 0, \ldots, K-1$, the annuity value function $V(W, A, \tau)$, assuming equations (2.1-2.3), solves the following linear PDE which has A dependence only through equation (4.3):

$$V_\tau - \mathcal{L}V = 0, \quad \tau \in [\tau_O^{k+}, \tau_O^{k+1}], \quad k = 0, \ldots, K-1. \tag{4.4}$$

where the operator $\mathcal{L}$ is given in (3.4).

4.2 Boundary conditions for the discrete withdrawal problem

We next determine the boundary conditions for equation (4.4). Similar to the condition (3.16) in the continuous withdrawal case, we use the following terminal boundary conditions from [6]:

$$V(W, A, \tau = 0) = \max(W, (1-\kappa)A - c). \tag{4.5}$$

Real contracts can contain a variety of payoffs. We use equation (4.5) since it is the same as the payoff in the continuous case (3.16).

As $A \to 0$, the withdrawal amount γ^k approaches zero. Hence the no-arbitrage condition (4.3) reduces to

$$V\big(W, A, \tau_O^{k+}\big) = V\big(W, A, \tau_O^k\big), \quad k = 0, \ldots, K-1, \tag{4.6}$$

which means that at the boundary $A = 0$, we only solve the linear PDE (4.4) for all $\tau \in [0, T]$.

At $A = w_0$, we simply solve the equations (4.3-4.4).

At $W = 0$, the no-arbitrage condition (4.3) becomes

$$V(0, A, \tau_O^{k+}) = \sup_{\gamma^k \in [0,A]} \Big[V\big(0, A - \gamma^k, \tau_O^k\big) + f(\gamma^k) \Big], \quad k = 0, \ldots, K-1. \tag{4.7}$$

By taking the limit $W \to 0$, equation (4.4) reduces to

$$V_\tau - rV = 0. \tag{4.8}$$

We solve equations (4.7-4.8) at the boundary $W = 0$.

At $W = W_{\max}$, we apply the Dirichlet condition as condition (3.19):

$$V(W, A, \tau) = e^{-\alpha\tau} W, \quad \text{if } W = W_{\max} \tag{4.9}$$

Let us define solution domains

$$\begin{aligned} \bar{\Omega}_k &= [0, W_{\max}] \times [0, w_0] \times [\tau_O^{k+}, \tau_O^{k+1}] \\ \bar{\Omega} &= \bigcup_k \bar{\Omega}_k = [0, W_{\max}] \times [0, w_0] \times \bigcup_k [\tau_O^{k+}, \tau_O^{k+1}], \quad k = 0, \ldots, K-1. \end{aligned} \tag{4.10}$$

Definition 4.1 (Pricing problem under the discrete withdrawal scenario)**.** *The pricing problem for GMWB variable annuities under the discrete withdrawal scenario is defined in $\bar{\Omega}$ as follows: within each domain $\bar{\Omega}_k$, $k = 0, \ldots, K-1$, the solution to the problem is the viscosity solution of a decoupled set of linear PDEs (4.4) along the A direction with boundary conditions (4.8-4.9) and initial condition $V(W, A, \tau_O^{k+})$ computed from the nonlinear algebraic equation (4.3).*

We next give an auxiliary result and then show that the pricing problem described in Definition 4.1 is well defined in the sense that the solution to the problem is unique.

Lemma 4.2. *If $V(W, A, \tau_O^k)$ is uniformly continuous on $(W, A) \in [0, W_{\max}] \times [0, w_0]$, then $V(W, A, \tau_O^{k+})$ given by equation (4.3) is uniformly continuous on $(W, A) \in [0, W_{\max}] \times [0, w_0]$.*

Proof. See Appendix A. □

Proposition 4.3. *There exists a unique viscosity solution to the GMWB variable annuity pricing problem described in Definition 4.1. In particular, the solution is continuous on (W, A, τ) within each domain $\bar{\Omega}_k$, $k = 0, \ldots, K-1$.*

Proof. The terminal boundary condition (4.5) implies that $V(W, A, \tau_O^0)$ is uniformly continuous on $(W, A) \in [0, W_{\max}] \times [0, w_0]$. Then according to Lemma 4.2, $V(W, A, \tau_O^{0+})$ is a uniformly continuous function of (W, A). Since in addition the boundary equation (4.8) at $W = 0$ is the limit of equation (4.4) towards the boundary and boundary equation (4.9) at $W = W_{\max}$ is a standard Dirichlet condition, then there exists a unique continuous viscosity solution to equation (4.4) in the domain $\bar{\Omega}_0$ with initial condition $V(W, A, \tau_O^{0+})$ and boundary conditions (4.8-4.9). Consequently, the proposition holds by applying the above arguments to each interval $[\tau_O^{k+}, \tau_O^{k+1}]$, $k = 1, \ldots, K-1$. □

Remark 4.4. *We do not define the problem on the continuous region $\tau \in [0, T]$ since the solution can be discontinuous (and hence not well defined) across the observation times τ_O^k, $k = 0, \ldots, K-1$ in the τ direction for fixed (W, A) due to the no-arbitrage condition (4.3).*

5 Numerical Scheme for the Discrete Withdrawal Model

We use an unequally spaced grid in the W direction for the PDE discretization, represented by $[W_0, W_1, \ldots, W_{i_{\max}}]$ with $W_{i_{\max}} = W_{\max}$. Similarly, we use an unequally spaced grid in the A direction denoted by $[A_0, A_1, \ldots, A_{j_{\max}}]$ with $A_{j_{\max}} = w_0$. We denote by $0 = \Delta\tau < \ldots < N\Delta\tau = T$ the discrete timesteps. Let $\tau^n = n\Delta\tau$ denote the nth timestep. We assume each discrete withdrawal time τ_O^k coincides with a discrete timestep, denoted by τ^{n_k} with $\tau^{n_0} = \tau^0 = 0$. Let $V(W_i, A_j, \tau^n)$

denote the exact solution of equations (4.3-4.4) when the value of the variable annuity sub-account is W_i, the guarantee account balance is A_j and discrete time is τ^n. Let $V_{i,j}^n$ denote an approximation of the exact solution $V(W_i, A_j, \tau^n)$.

It will be convenient to define $\Delta W_{\max} = \max_i\big(W_{i+1} - W_i\big)$, $\Delta W_{\min} = \min_i\big(W_{i+1} - W_i\big)$, $\Delta A_{\max} = \max_j\big(A_{j+1} - A_j\big)$, $\Delta A_{\min} = \min_j\big(A_{j+1} - A_j\big)$. We assume that there is a mesh size/timestep parameter h such that

$$\Delta W_{\max} = C_1 h \ \ ; \ \ \Delta A_{\max} = C_2 h \ \ ; \ \ \Delta\tau = C_3 h \ \ ; \ \ \Delta W_{\min} = C_1' h \ \ ; \ \ \Delta A_{\min} = C_2' h. \tag{5.1}$$

where $C_1, C_1', C_2, C_2', C_3$ are constants independent of h.

We use standard finite difference methods to discretize the operator $\mathcal{L}V$ as given in (3.4). Let $(\mathcal{L}_h V)_{i,j}^n$ denote the discrete value of the differential operator (3.4) at node (W_i, A_j, τ^n). The operator (3.4) can be discretized using central, forward, or backward differencing in the W, A directions to give

$$(\mathcal{L}_h V)_{i,j}^n = \alpha_i V_{i-1,j}^n + \beta_i V_{i+1,j}^n - (\alpha_i + \beta_i + r) V_{i,j}^n \ , \quad i < i_{\max}, \tag{5.2}$$

where α_i and β_i are determined using an algorithm in [8]. The algorithm guarantees α_i and β_i satisfy the following positive coefficient condition:

$$\alpha_i \geq 0 \quad ; \quad \beta_i \geq 0 \ , \quad i = 0, \ldots, i_{\max} - 1. \tag{5.3}$$

At time $\tau = 0$, we apply terminal boundary condition (4.5) by

$$V_{i,j}^0 = \max(W_i, (1-\kappa)A_j - c), \quad i = 0, \ldots, i_{\max}, \ j = 0, \ldots, j_{\max}. \tag{5.4}$$

At a withdrawal time $\tau^{n_k} = \tau_O^k$, $k = 0, \ldots, K-1$, we apply the no-arbitrage condition (4.3) in the following manner. Let $V_{\hat{i},\hat{j}}^n$ be an approximation of $V\big(\max(W_i - \gamma_{i,j}^n, 0), A_j - \gamma_{i,j}^n, \tau^n\big)$ obtained by linear interpolation; in other words, if $\phi(W, A, \tau)$ is a smooth function on (W, A, τ) with $\phi_{i,j}^n = \phi(W_i, A_j, \tau^n)$, then we have

$$\phi_{\hat{i},\hat{j}}^n = \phi\big(\max(W_i - \gamma_{i,j}^n, 0), A_j - \gamma_{i,j}^n, \tau^n\big) + O\big((\Delta W_{\max} + \Delta A_{\max})^2\big). \tag{5.5}$$

Then at $\tau = \tau_O^k = \tau^{n_k}$, we solve the local optimization problem

$$V_{i,j}^{n+} = \sup_{\gamma_{i,j}^n \in [0, A_j]} \Big[V_{\hat{i},\hat{j}}^n + f\big(\gamma_{i,j}^n\big) \Big], \quad i = 0, \ldots, i_{\max} - 1, \ j = 0, \ldots, j_{\max}, \ n = n_k, \tag{5.6}$$

where τ^{n_k+} denotes the time infinitesimally after τ^{n_k}. We describe in Section 7 the method used to solve the optimization problem (5.6).

Within the interval $\tau \in [\tau_O^{k+}, \tau_O^{k+1}]$, $k = 0, \ldots, K-1$, we use a fully implicit timestepping scheme to discretize (4.4). Specifically, we compute $V_{i,j}^{n+1}$ by

$$\begin{aligned}
V_{i,j}^{n+1} &= V_{i,j}^{n+} + \Delta\tau\big(\mathcal{L}_h V\big)_{i,j}^{n+1}, \quad i = 0, \ldots, \ i_{\max} - 1, \ j = 0, \ldots, j_{\max}, \ n+1 = n_k + 1; \\
V_{i,j}^{n+1} &= V_{i,j}^{n} + \Delta\tau\big(\mathcal{L}_h V\big)_{i,j}^{n+1}, \quad i = 0, \ldots, \ i_{\max} - 1, \ j = 0, \ldots, j_{\max}, \ n+1 = n_k + 2, \ldots, n_{k+1}; \\
V_{i,j}^{n+1} &= e^{-\alpha\tau^{n+1}} W_{\max}, \quad i = i_{\max}, \ j = 0, \ldots, j_{\max}, \ n+1 = n_k + 1, \ldots, n_{k+1}.
\end{aligned} \tag{5.7}$$

Remark 5.1. *Assuming that* $\max(W_i - \gamma_{i,j}^n, 0)$ *and* $A_j - \gamma_{i,j}^n$ *reside within an interval* $[W_l, W_{l+1}]$ *and* $[A_m, A_{m+1}]$*, respectively, where* $0 \le l < i_{\max}$*,* $0 \le m < j_{\max}$*, then* $V_{\hat{i},\hat{j}}^n$ *is linearly interpolated using grid nodes* $V_{l,m}^n$*,* $V_{l+1,m}^n$*,* $V_{l,m+1}^n$ *and* $V_{l+1,m+1}^n$*.*

In the discrete equation (5.6), $V_{\hat{i},\hat{j}}^n$ *is a function of* $\gamma_{i,j}^n$*, representing the continuous curve on the interpolated surface, constructed by linear interpolation using discrete values* $V_{i,j}^n$*,* $i = 0, \ldots, i_{\max}$*,* $j = 0, \ldots, j_{\max}$*, along the piecewise line segments* $(W, A)(\gamma_{i,j}^n) = (\max(W_i - \gamma_{i,j}^n, 0), A_j - \gamma_{i,j}^n)$*. Since the values of* $V_{i,j}^n$ *are bounded (see Lemma 5.3), then* $V_{\hat{i},\hat{j}}^n$ *is uniformly continuous on* $\gamma_{i,j}^n$*.*

According to (4.2), if the fixed cost $c = 0$*, then* $f(\gamma^k)$ *is continuous on the closed interval* $[0, A_j]$*. Thus the supremum in (5.6) is achieved by a control* $\gamma^k \in [0, A_j]$*. If, on the other hand,* $c > 0$*, then* $f(\gamma^k)$ *is discontinuous at* $\gamma^k = G^k$*. We can write (5.6) as*

$$V_{i,j}^{n+} = \max\left\{ \sup_{\gamma_{i,j}^n \in [0, \min(G^k, A_j)]} \left[V_{\hat{i},\hat{j}}^n + f\big(\gamma_{i,j}^n\big) \right], \sup_{\gamma_{i,j}^n \in (G^k, A_j]} \left[V_{\hat{i},\hat{j}}^n + f\big(\gamma_{i,j}^n\big) \right] \right\} \tag{5.8}$$

with the convention that $(G^k, A_j] = \emptyset$ *if* $G^k \ge A_j$*. Since* $V_{\hat{i},\hat{j}}^n$ *and* $f(\gamma_{i,j}^n)$ *are continuous on* $[0, \min(G^k, A_j)]$*, the first supremum in (5.8) can be achieved by a control* $\gamma^k \in [0, \min(G^k, A_j)]$

Equation (4.2) implies that (if $c > 0$*)*

$$f(\gamma^k = G^k) = \lim_{\gamma^k \to [G^k]^-} f(\gamma^k) > \lim_{\gamma^k \to [G^k]^+} f(\gamma^k), \quad \text{if } c > 0, \tag{5.9}$$

where $\lim_{\gamma^k \to [G^k]^-} f$ *and* $\lim_{\gamma^k \to [G^k]^+} f$ *represent the left and right limits of* f *at* $\gamma^k = G^k$*, respectively. Consequently, if the second supremum in (5.8) is achieved by the limiting point* $[G^k]^+$*, since* $f(G^k) > f([G^k]^+)$*, then the value of the first supremum in (5.8) will be greater than that of the second one. Thus, the supremum in (5.6) can be achieved by a control* $\gamma^k \in [0, A_j]$ *for the case when* $c > 0$*.*

5.1 Convergence of the numerical scheme

In this subsection, we prove the convergence of scheme (5.4-5.7) to the unique viscosity solution of the pricing problem defined in Definition 4.1 by showing that the scheme is l_∞ stable, pointwise consistence and monotone.

Definition 5.2 (l_∞ stability)**.** *Discretization (5.4-5.7) is* l_∞ *stable if*

$$\|V^{n+1}\|_\infty \quad \le \quad C_4 \quad , \tag{5.10}$$

for $0 \le n \le N-1$ *as* $\Delta\tau \to 0$*,* $\Delta W_{\min} \to 0$*,* $\Delta A_{\min} \to 0$*, where* C_4 *is a constant independent of* $\Delta\tau$*,* $\Delta W_{\min}$*,* $\Delta A_{\min}$*. Here* $\|V^{n+1}\|_\infty = \max_{i,j} |V_{i,j}^{n+1}|$*.*

Lemma 5.3 (l_∞ stability)**.** *If the discretization (5.2) satisfies the positive coefficient condition (5.3) and linear interpolation is used to compute* $V_{\hat{i},\hat{j}}^{n_k}$*, then the scheme is stable according to Definition 5.2.*

Proof. The Lemma directly follows from the stability proof of the corresponding scheme under the continuous withdrawal scenario in Lemma 6.1. ∎

We can write discrete equations (5.7) at a node (W_i, A_j, τ^{n+1}) for $\tau^{n_k+1} \le \tau^{n+1} \le \tau^{n_{k+1}}$ as

$$
\begin{aligned}
&\mathcal{G}_{i,j}^{n+1}\big(h, V_{i,j}^{n+1}, \{V_{l,m}^{n+1}\}_{\substack{l\neq i\\ m\neq j}}, V_{i,j}^{n+}, \{V_{i,j}^{n}\}\big)\\
&\equiv \begin{cases}
V_{i,j}^{n+1} - V_{i,j}^{n+} - \Delta\tau\big(\mathcal{L}_h V\big)_{i,j}^{n+1} & \text{if } 0 \le W_i < W_{i_{\max}}, \quad 0 \le A_j \le A_{j_{\max}}, \quad \tau^{n+1} = \tau^{n_k+1};\\
V_{i,j}^{n+1} - V_{i,j}^{n} - \Delta\tau\big(\mathcal{L}_h V\big)_{i,j}^{n+1} & \text{if } 0 \le W_i < W_{i_{\max}}, \quad 0 \le A_j \le A_{j_{\max}}, \quad \tau^{n_k+2} \le \tau^{n+1} \le \tau^{n_{k+1}};\\
V_{i,j}^{n+1} - e^{-\alpha\tau^{n+1}} W_{\max} & \text{if } W_i = W_{i_{\max}}, \quad\quad 0 \le A_j \le A_{j_{\max}}, \quad \tau^{n_k+1} \le \tau^{n+1} \le \tau^{n_{k+1}}
\end{cases}\\
&= 0,
\end{aligned}
\tag{5.11}
$$

where $\{V_{l,m}^{n+1}\}_{\substack{l\neq i\\ m\neq j}}$ is the set of values $V_{l,m}^{n+1}$, $l \neq i$, $l = 0, \ldots, i_{\max}$ and $m \neq j$, $m = 0, \ldots, j_{\max}$, and $\{V_{i,j}^{n}\}$ is the set of values $V_{i,j}^{n}$, $i = 0, \ldots, i_{\max}$, $j = 0, \ldots, j_{\max}$.

Definition 5.4 (Pointwise consistency, discrete withdrawal)**.** *The scheme (5.11) is pointwise consistent with the PDE (4.4) and boundary conditions (4.8-4.9) if, for any smooth test function ϕ,*

$$
\lim_{h\to 0}\left|\mathcal{G}_{i,j}^{n+1}\big(h, \phi_{i,j}^{n+1}, \{\phi_{l,m}^{n+1}\}_{\substack{l\neq i\\ m\neq j}}, \phi_{i,j}^{n+}, \{\phi_{i,j}^{n}\}\big) - (\phi_\tau - \mathcal{L}\phi)_{i,j}^{n}\right| = 0\ , \tag{5.12}
$$

for any point in $\bar{\Omega}$.

With the above definition, it is straightforward to verify that scheme (5.11) is consistent using Taylor series.

Lemma 5.5 (Pointwise consistency)**.** *The discrete scheme (5.11) is pointwise consistent.*

The following result shows that scheme (5.11) is monotone according to the definition in [3, 2]:

Lemma 5.6 (Monotonicity)**.** *If discretization (5.2) satisfies the positive coefficient condition (5.3) then discretization (5.11) is monotone according to the definition in [3, 2], i.e.,*

$$
\begin{aligned}
&\mathcal{G}_{i,j}^{n+1}\big(h, V_{i,j}^{n+1}, \{X_{l,m}^{n+1}\}_{\substack{l\neq i\\ m\neq j}}, X_{i,j}^{n+}, \{X_{i,j}^{n}\}\big)\\
&\le \mathcal{G}_{i,j}^{n+1}\big(h, V_{i,j}^{n+1}, \{Y_{l,m}^{n+1}\}_{\substack{l\neq i\\ m\neq j}}, Y_{i,j}^{n+}, \{Y_{i,j}^{n}\}\big); \quad \textit{for all } X_{i,j}^{n} \ge Y_{i,j}^{n},\ \forall i, j, n.
\end{aligned}
\tag{5.13}
$$

Proof. It is straightforward to verify that the discretization (5.11) satisfies inequality (5.13) for all mesh nodes (W_i, A_j, τ^n). $\square$

Theorem 5.7 (Convergence to the viscosity solution)**.** *Assuming that scheme (5.4-5.7) satisfies all the conditions required for Lemmas 5.3 and 5.6, then as $h \to 0$, scheme (5.4-5.7) converges to the unique viscosity solution to the pricing problem defined in Definition 4.1 in the domain $\bar{\Omega}$.*

Proof. Let $V^h(\tau_O^{0+})$ denote the approximate solution computed by (5.6) at τ_O^{0+} with the mesh size/timestep parameter h. $V^h(\tau_O^{0+})$ is only defined at mesh nodes (W_i, A_j). Let $V_I^h(\tau_O^{0+})$ denote the value of the approximate solution which is interpolated using linear interpolation for any point (W, A). Let $V(\tau_O^{0+})$ be the exact solution to equation (4.3). Here we suppress the variables (W, A) in the above notation. Since (5.6) is a consistent discretization of equation (4.3), then $V_I^h(\tau_O^{0+})$ converges to $V(\tau_O^{0+})$ as $h \to 0$.

Let any $\mathbf{x} = (W, A, \tau) \in \bar{\Omega}_0$, where $\bar{\Omega}_0 = [0, W_{\max}] \times [0, w_0] \times [\tau_O^{0+}, \tau_O^1]$ as defined in (4.10). Let $V^h\big(V_I^h(\tau_O^{0+})\big)$ denote the approximate solution resulting from equation (5.7) with initial condition $V_I^h(\tau_O^{0+})$ at mesh nodes $(W_i, A_j, \tau^{n+1}) \in \bar{\Omega}_0$. Accordingly, let $V_I^h\big(\mathbf{x}, V_I^h(\tau_O^{0+})\big)$ be the value of the approximate solution at $\mathbf{x}$ obtained by linear interpolation using $V^h\big(V_I^h(\tau_O^{0+})\big)$ defined only at mesh nodes. Let $V\big(\mathbf{x}; V(\tau_O^{0+})\big)$ and $V\big(\mathbf{x}; V_I^h(\tau_O^{0+})\big)$ denote the unique viscosity solution to equation (4.4) and boundary conditions (4.8-4.9), with initial condition $V(\tau_O^{0+})$ and $V_I^h(\tau_O^{0+})$, respectively. Since $V_I^h(\tau_O^{0+}) \to V(\tau_O^{0+})$ as $h \to 0$, we have

$$V\Big(\mathbf{x}; V_I^h\big(\tau_O^{0+}\big)\Big) \to V\Big(\mathbf{x}; V\big(\tau_O^{0+}\big)\Big) \quad \text{as } h \to 0. \tag{5.14}$$

According to Lemmas 5.3, 5.5 and 5.6, scheme (5.7) is l_∞ stable and monotone, and pointwise consistent to PDE (4.4) and its boundary conditions (4.8-4.9). Thus, convergence results in [3, 2] imply that

$$V_I^h\Big(\mathbf{x}; V_I^h\big(\tau_O^{0+}\big)\Big) \to V\Big(\mathbf{x}; V_I^h\big(\tau_O^{0+}\big)\Big) \quad \text{as } h \to 0. \tag{5.15}$$

Using equation (5.14-5.15), we have

$$\begin{aligned}
&\Big|V_I^h\Big(\mathbf{x}; V_I^h\big(\tau_O^{0+}\big)\Big) - V\Big(\mathbf{x}; V\big(\tau_O^{0+}\big)\Big)\Big| \\
&\leq \Big|V_I^h\Big(\mathbf{x}; V_I^h\big(\tau_O^{0+}\big)\Big) - V\Big(\mathbf{x}; V_I^h\big(\tau_O^{0+}\big)\Big)\Big| + \Big|V\Big(\mathbf{x}; V_I^h\big(\tau_O^{0+}\big)\Big) - V\Big(\mathbf{x}; V\big(\tau_O^{0+}\big)\Big)\Big| \\
&\to 0 \qquad \text{as } h \to 0.
\end{aligned} \tag{5.16}$$

Thus we prove the Theorem in $\bar{\Omega}_0$. Equation (5.16) implies that $V_I^h(\tau_O^{1+}) \to V(\tau_O^{1+})$ as $h \to 0$. Consequently, the Theorem follows by sequentially applying the above argument to regions $\bar{\Omega}_k$, $k = 1, \ldots, K-1$. $\square$

6 Generalization of the Scheme to the Continuous Withdrawal Model

In this section, we consider the case when the discrete withdrawal interval approaches zero, i.e., $\Delta\tau_O^k \to 0$. We generalize our numerical scheme introduced in Section 5 to this case and prove the convergence of the scheme to the viscosity solution of the impulse control problem (3.15), provided a strong comparison result holds.

We assume $\Delta\tau_O^k = \Delta\tau$, $k = 0, \ldots, K-1$ and $K = N$. In other words, each discrete timestep τ^n corresponds to a withdrawal time τ_O^k. Then $\Delta\tau_O^k \to 0$ as we take $\Delta\tau \to 0$. In this case, according to (4.1) and the assumption $\Delta\tau_O^k = \Delta\tau$, the cash flow $f(\gamma_{i,j}^n)$ resulting from (4.2) becomes

$$f(\gamma_{i,j}^n) = \begin{cases} \gamma_{i,j}^n & \text{if } 0 \leq \gamma_{i,j}^n \leq G_r\Delta\tau, \\ \gamma_{i,j}^n - \kappa(\gamma_{i,j}^n - G_r\Delta\tau) - c & \text{if } \gamma_{i,j}^n > G_r\Delta\tau. \end{cases} \tag{6.1}$$

We impose condition (4.5) at $\tau = 0$

$$V_{i,j}^0 = \max(W_i, (1-\kappa)A_j - c)), \quad i = 0, \ldots i_{\max}, \ j = 0, \ldots, j_{\max}. \tag{6.2}$$

Meanwhile, discrete equations (5.6-5.7) turn into

$$V_{i,j}^{n+} = \sup_{\gamma_{i,j}^n \in [0, A_j]} \Big[V_{\hat{i},\hat{j}}^n + f\big(\gamma_{i,j}^n\big) \Big], \quad i = 0, \dots, i_{\max} - 1, \; j = 0, \dots, j_{\max}, \tag{6.3}$$

$$V_{i,j}^{n+1} = V_{i,j}^{n+} + \Delta\tau\big(\mathcal{L}_h V\big)_{i,j}^{n+1}, \quad i = 0, \dots, \; i_{\max} - 1, \; j = 0, \dots, j_{\max}, \tag{6.4}$$

$$V_{i,j}^{n+1} = e^{-\alpha\tau^{n+1}} W_{\max}, \quad i = i_{\max}, \; j = 0, \dots j_{\max} \tag{6.5}$$

for $n = 0, \dots, N-1$, where $\mathcal{L}_h$ is given in (5.2). Here $V_{\hat{i},\hat{j}}^n$ is the approximation of $V(\max(W_i - \gamma_{i,j}^n, 0), A_j - \gamma_{i,j}^n, \tau^n)$ by linear interpolation.

Substituting discrete equation (6.3) into (6.4) gives

$$V_{i,j}^{n+1} - \sup_{\gamma_{i,j}^n \in [0, A_j]} \Big[V_{\hat{i},\hat{j}}^n + f\big(\gamma_{i,j}^n\big) \Big] - \Delta\tau\big(\mathcal{L}_h V\big)_{i,j}^{n+1} = 0, \quad i = 0, \dots, i_{\max} - 1, \; j = 0, \dots, j_{\max}. \tag{6.6}$$

6.1 Convergence to the viscosity solution

Provided a strong comparison result for the PDE applies, [3, 2] demonstrate that a numerical scheme will converge to the viscosity solution of the equation if it is l_∞ stable, monotone, and pointwise consistent. In this subsection, we will prove the convergence of our numerical scheme (6.2-6.5) (or scheme (6.2), (6.5) and (6.6)) to the viscosity solution of problem (3.15) associated with boundary conditions (3.16-3.19) by verifying these three properties.

Note that the authors of [7, 15] present numerical schemes for solving singular control problems arising in transaction cost models, and show the convergence of the scheme to the viscosity solution by following the framework of [3, 2].

6.1.1 Stability

At first we show the l_∞ stability of our scheme (6.2-6.5) by verifying Definition 5.2.

Lemma 6.1 (l_∞ stability)**.** *If the discretization (5.2) satisfies the positive coefficient condition (5.3) and linear interpolation is used to compute $V_{\hat{i},\hat{j}}^n$, then the scheme (6.2-6.5) satisfies*

$$\|V^{n+}\|_\infty \leq \|V^0\|_\infty + A_{j_{\max}} \quad \textit{and} \quad \|V^n\|_\infty \leq \|V^0\|_\infty + A_{j_{\max}} \tag{6.7}$$

for $0 \leq n \leq N$ as $\Delta\tau \to 0$, $\Delta W_{\min} \to 0$, $\Delta A_{\min} \to 0$, where $A_{j_{\max}} = w_0$.

The stability result (6.7) also holds for the discrete withdrawal case with $\Delta\tau_O^n > 0$.

Proof. Let us define $\|V_j^n\|_\infty = \max_i |V_{i,j}^n|$. As well, let $(V_j^n)_{max} = \max_i(V_{i,j}^n)$, $(V_j^{n+})_{max} = \max_i(V_{i,j}^{n+})$, $(V_j^n)_{min} = \min_i(V_{i,j}^n)$, and $(V_j^{n+})_{min} = \min_i(V_{i,j}^{n+})$. Here we only consider the continuous withdrawal case; the discrete withdrawal case follows from the same arguments. To prove the Lemma, it is sufficient to show

$$(V_j^n)_{max} \leq \|V^0\|_\infty + A_j \ , \tag{6.8}$$

$$(V_j^{n+})_{max} \leq \|V^0\|_\infty + A_j \tag{6.9}$$

$$(V_j^n)_{min} \geq 0 \tag{6.10}$$

$$(V_j^{n+})_{min} \geq 0 \tag{6.11}$$

for all $0 \le j \le j_{\max}$, $0 \le n \le N$. We will prove inequalities (6.8-6.11) using induction. From condition (6.2), it is obvious that inequalities (6.8), (6.10) hold when $n = 0$ and $0 \le j \le j_{\max}$.

Assume inequalities (6.8), (6.10) hold for $n \le n_*$ and $0 \le j \le j_{\max}$, where $n_* < N$. We next show inequalities (6.9), (6.11) hold for $n = n_*$, $0 \le j \le j_{\max}$ and then (6.8), (6.10) follow for $n = n_* + 1$, $0 \le j \le j_{\max}$.

We first consider discrete equation (6.3) at $n = n_*$. That is,

$$V_{i,j}^{n_*+} = \sup_{\gamma_{i,j}^{n_*} \in [0,A_j]} \left[V_{\hat{i},\hat{j}}^{n_*} + f\big(\gamma_{i,j}^{n_*}\big) \right], \quad i = 0,\dots,i_{\max}-1,\ j = 0,\dots,j_{\max}. \tag{6.12}$$

According to Remark 5.1, the supremum in the right hand side of (6.12) is achieved by a control, denoted by $\bar{\gamma}_{i,j}^{n_*}$. Assume that $\max(W_i - \bar{\gamma}_{i,j}^{n_*}, 0)$ and $A_j - \bar{\gamma}_{i,j}^{n_*}$ reside within an interval $[W_l, W_{l+1}]$ and $[A_m, A_{m+1}]$, respectively, where $0 \le l < i_{\max} - 1$, $0 \le m < j_{\max}$. Then computing $V_{\hat{i},\hat{j}}^{n_*}$ using linear interpolation results in

$$V_{\hat{i},\hat{j}}^{n_*} = x_A\big[x_W V_{l,m}^{n_*} + (1-x_W)V_{l+1,m}^{n_*}\big] + (1-x_A)\big[x_W V_{l,m+1}^{n_*} + (1-x_W)V_{l+1,m+1}^{n_*}\big], \tag{6.13}$$

where x_W and x_A are interpolation weights satisfying $0 \le x_W \le 1$ and $0 \le x_A \le 1$. Specifically, we have

$$x_A = \frac{A_{m+1} - (A_j - \bar{\gamma}_{i,j}^{n_*})}{A_{m+1} - A_m}. \tag{6.14}$$

Using equation (6.14) and the induction assumptions $V_{l,m}^{n_*} \le \|V^0\|_\infty + A_m$, $V_{l+1,m}^{n_*} \le \|V^0\|_\infty + A_m$, $V_{l,m+1}^{n_*} \le \|V^0\|_\infty + A_{m+1}$, $V_{l+1,m+1}^{n_*} \le \|V^0\|_\infty + A_{m+1}$, equation (6.13) leads to

$$V_{\hat{i},\hat{j}}^{n_*} \le \|V^0\|_\infty + A_j - \bar{\gamma}_{i,j}^{n_*}, \qquad \forall\ 0 \le i < i_{\max},\ 0 \le j \le j_{\max}. \tag{6.15}$$

Since $c, \kappa \ge 0$, equation (6.1) implies that

$$f\big(\bar{\gamma}_{i,j}^{n_*}\big) \le \bar{\gamma}_{i,j}^{n_*}, \qquad \forall\ 0 \le i < i_{\max},\ 0 \le j \le j_{\max}. \tag{6.16}$$

Equations (6.12) and (6.15-6.16) lead to (the max operator disappears since we have taken the optimal control $\bar{\gamma}_{i,j}^{n_*}$),

$$V_{i,j}^{n_*+} = V_{\hat{i},\hat{j}}^{n_*} + f\big(\bar{\gamma}_{i,j}^{n_*}\big) \le \|V^0\|_\infty + A_j \qquad \forall\ 0 \le i < i_{\max},\ 0 \le j \le j_{\max}. \tag{6.17}$$

This proves (6.9) at $n = n_*$, $0 \le j \le j_{\max}$. By the induction assumptions we have $V_{i,j}^{n_*} \ge 0$, hence from equations (6.12-6.13), we must have

$$V_{i,j}^{n_*+} \ge 0 \qquad \forall\ 0 \le i < i_{\max},\ 0 \le j \le j_{\max}. \tag{6.18}$$

hence equation (6.11) holds at $n = n_*$, $0 \le j \le j_{\max}$.

For any $i < i_{\max}$, $0 \le j \le j_{\max}$, and at $n = n_* + 1$, substituting (5.2) into (6.4) gives

$$V_{i,j}^{n_*+1}\big(1 + \Delta\tau(r + \alpha_i + \beta_i)\big) - \alpha_i \Delta\tau V_{i-1,j}^{n_*+1} - \beta_i \Delta\tau V_{i+1,j}^{n_*+1} = V_{i,j}^{n_*+} \tag{6.19}$$

Let i^* be the index such that $V_{i^*,j}^{n_*+1} = (V_j^{n_*+1})_{max}$. First consider the case when $i^* < i_{\max}$. Since $r \ge 0$, and $\alpha_i \ge 0$, $\beta_i \ge 0$, as indicated by the positive coefficient condition (5.3), equation (6.19) implies that

$$V_{i^*,j}^{n_*+1}\big(1 + \Delta\tau(r + \alpha_{i^*} + \beta_{i^*})\big) \le (V_j^{n_*+})_{max} + V_{i^*,j}^{n_*+1}\Delta\tau(\alpha_{i^*} + \beta_{i^*}). \tag{6.20}$$

Since we have just shown that $(V_j^{n_*+})_{max} \leq \|V^0\|_\infty + A_j$, inequality (6.20) results in

$$V_{i^*,j}^{n_*+1} \leq (V_j^{n_*+})_{max} \leq \|V^0\|_\infty + A_j. \tag{6.21}$$

Next consider the case when $i^* = i_{\max}$. Discrete equation (6.5) and $\|V^0\|_\infty \geq W_{\max}$ imply that

$$V_{i^*,j}^{n_*+1} = e^{-\alpha\tau^{n_*+1}} W_{\max} \leq \|V^0\|_\infty + A_j. \tag{6.22}$$

The inequality in (6.22) is due to $\alpha \geq 0$. Finally, inequalities (6.21-6.22) and the assumption $V_{i^*,j}^{n_*+1} = (V_j^{n_*+1})_{max}$ show that inequality (6.8) holds for all $0 \leq j \leq j_{\max}$ and $n = n_* + 1$. A similar argument shows equation (6.11) holds for all $0 \leq j \leq j_{\max}$ and $n = n_* + 1$. □

6.1.2 Consistency

It will be convenient to rewrite scheme (6.2), (6.5) and (6.6) using the following idea. If $A_j > G_r\Delta\tau$, we can separate the control region into two subregions: $[0, A_j] = [0, G_r\Delta\tau] \cup (G_r\Delta\tau, A_j]$. We will then write equation (6.6) in terms of these two subregions. Let us define

$$\mathcal{H}_{i,j}^{n+1}\big(h, V_{i,j}^{n+1}, \{V_{l,m}^{n+1}\}_{\substack{l\neq i\\ m\neq j}}, \{V_{i,j}^{n}\}\big) = \frac{1}{\Delta\tau}\Big[V_{i,j}^{n+1} - \sup_{\gamma_{i,j}^n \in [0, \min(A_j, G_r\Delta\tau)]} \big(V_{\hat{i},\hat{j}}^n + \gamma_{i,j}^n\big) - \Delta\tau\big(\mathcal{L}_h V\big)_{i,j}^{n+1}\Big] \tag{6.23}$$

and (assuming $A_j > G_r\Delta\tau$)

$$\mathcal{I}_{i,j}^{n+1}\big(h, V_{i,j}^{n+1}, \{V_{l,m}^{n+1}\}_{\substack{l\neq i\\ m\neq j}}, \{V_{i,j}^{n}\}\big) = V_{i,j}^{n+1} - \sup_{\gamma_{i,j}^n \in (G_r\Delta\tau, A_j]} \Big[V_{\hat{i},\hat{j}}^n + (1-\kappa)\gamma_{i,j}^n + \kappa G_r\Delta\tau - c\Big] - \Delta\tau\big(\mathcal{L}_h V\big)_{i,j}^{n+1}. \tag{6.24}$$

Note that within (6.23-6.24), the cash flow term $f(\gamma_{i,j}^n)$ in (6.6) is replaced by the piecewise representation given in (6.1) based on the subregion where the control $\gamma_{i,j}^n$ resides. Given the definitions of $\mathcal{H}$ and $\mathcal{I}$, we can write scheme (6.2), (6.5) and (6.6) in an equivalent way at a node (W_i, A_j, τ^{n+1}) as

$$\begin{aligned}
&\mathcal{G}_{i,j}^{n+1}\big(h, V_{i,j}^{n+1}, \{V_{l,m}^{n+1}\}_{\substack{l\neq i\\ m\neq j}}, \{V_{i,j}^{n}\}\big) \\
&\equiv \begin{cases}
\mathcal{H}_{i,j}^{n+1} & \text{if } 0 \leq W_i < W_{i_{\max}}, \quad 0 \leq A_j \leq G_r\Delta\tau, \quad 0 < \tau^{n+1} \leq T; \\
\min\Big\{\mathcal{H}_{i,j}^{n+1},\ \mathcal{I}_{i,j}^{n+1}\Big\} & \text{if } 0 \leq W_i < W_{i_{\max}}, \quad G_r\Delta\tau < A_j \leq A_{j_{\max}}, \quad 0 < \tau^{n+1} \leq T; \\
V_{i,j}^{n+1} - e^{-\alpha\tau^{n+1}} W_{\max} & \text{if } W_i = W_{i_{\max}}, \quad 0 \leq A_j \leq A_{j_{\max}}, \quad 0 < \tau^{n+1} \leq T; \\
V_{i,j}^{n+1} - \max(W_i, (1-\kappa)A_j - c) & \text{if } 0 \leq W_i \leq W_{i_{\max}}, \quad 0 \leq A_j \leq A_{j_{\max}}, \quad \tau^{n+1} = 0
\end{cases} \\
&= 0,
\end{aligned} \tag{6.25}$$

Let $\bar{\Omega} = [0, W_{\max}] \times [0, w_0] \times [0, T]$ be the closed domain in which our problem is defined. The domain $\bar{\Omega}$ can be divided into the following open regions:

$$\begin{aligned}
&\Omega_{in} = (0, W_{\max}) \times (0, w_0] \times (0, T] \ \ ; \ \ \Omega_{W_0} = \{0\} \times (0, w_0] \times (0, T] \ \ ; \\
&\Omega_{A_0} = [0, W_{\max}) \times \{0\} \times (0, T] \ \ ; \ \ \Omega_{W_m} = \{W_{\max}\} \times [0, w_0] \times (0, T] \ \ ; \\
&\Omega_{\tau^0} = [0, W_{\max}] \times [0, w_0] \times \{0\},
\end{aligned} \tag{6.26}$$

where Ω_{in} represents the interior region, and $\Omega_{W_0}, \Omega_{A_0}, \Omega_{W_m}, \Omega_{\tau^0}$ denote the boundary regions. Let us define vector $\mathbf{x} = (W, A, \tau)$, and let $DV(\mathbf{x})$ and $D^2V(\mathbf{x})$ be its first and second derivatives of $V(\mathbf{x})$, respectively. Let us define the following operators:

$$
\begin{aligned}
F_{in}\big(D^2V(\mathbf{x}), DV(\mathbf{x}), V(\mathbf{x}), \mathbf{x}\big) &= \min\Big\{V_\tau - \mathcal{L}V - \sup_{\hat{\gamma}\in[0,G_r]} \big(\hat{\gamma} - \hat{\gamma}V_W - \hat{\gamma}V_A\big), \\
&\qquad V - \sup_{\gamma\in(0,A]} \big[V(\max(W-\gamma,0), A-\gamma, \tau) + (1-\kappa)\gamma - c\big]\Big\}, \\
F_{W_0}\big(D^2V(\mathbf{x}), DV(\mathbf{x}), V(\mathbf{x}), \mathbf{x}\big) &= \min\Big\{V_\tau - rV - \sup_{\hat{\gamma}\in[0,G_r]} \big(\hat{\gamma} - \hat{\gamma}V_A\big), \\
&\qquad V - \sup_{\gamma\in(0,A]} \big[V(0, A-\gamma, \tau) + (1-\kappa)\gamma - c\big]\Big\}, \\
F_{A_0}\big(D^2V(\mathbf{x}), DV(\mathbf{x}), V(\mathbf{x}), \mathbf{x}\big) &= V_\tau - \mathcal{L}V, \\
F_{W_m}(V(\mathbf{x}), \mathbf{x}) &= V - e^{-\alpha\tau}W, \\
F_{\tau^0}(V(\mathbf{x}), \mathbf{x}) &= V - \max(W, (1-\kappa)A - c).
\end{aligned}
\tag{6.27}
$$

Then the pricing problem (3.15-3.19) can be combined into one equation as follows:

$$
F\big(D^2V(\mathbf{x}), DV(\mathbf{x}), V(\mathbf{x}), \mathbf{x}\big) = 0 \quad \text{for all } \mathbf{x} = (W, A, \tau) \in \bar{\Omega}\ , \tag{6.28}
$$

where F is defined by

$$
F = \begin{cases}
F_{in}\big(D^2V(\mathbf{x}), DV(\mathbf{x}), V(\mathbf{x}), \mathbf{x}\big) & \text{if } \mathbf{x} \in \Omega_{in}, \\
F_{W_0}\big(D^2V(\mathbf{x}), DV(\mathbf{x}), V(\mathbf{x}), \mathbf{x}\big) & \text{if } \mathbf{x} \in \Omega_{W_0}, \\
F_{A_0}\big(D^2V(\mathbf{x}), DV(\mathbf{x}), V(\mathbf{x}), \mathbf{x}\big) & \text{if } \mathbf{x} \in \Omega_{A_0}, \\
F_{W_m}\big(V(\mathbf{x}), \mathbf{x}\big) & \text{if } \mathbf{x} \in \Omega_{W_m}, \\
F_{\tau^0}\big(V(\mathbf{x}), \mathbf{x}\big) & \text{if } \mathbf{x} \in \Omega_{\tau_0}.
\end{cases} \tag{6.29}
$$

In order to demonstrate consistency as defined in [3, 2], we first need some intermediate results. We define operators

$$
\begin{aligned}
F_{A'}\big(D^2V(\mathbf{x}), DV(\mathbf{x}), V(\mathbf{x}), \mathbf{x}\big) &= V_\tau - \mathcal{L}V - \sup_{\hat{\gamma}\in[0,A/\Delta\tau]} \big(\hat{\gamma} - \hat{\gamma}V_W - \hat{\gamma}V_A\big), \text{ where } 0 \le A/\Delta\tau \le G_r, \\
F_{W'}\big(D^2V(\mathbf{x}), DV(\mathbf{x}), V(\mathbf{x}), \mathbf{x}\big) &= V_\tau - rV - \sup_{\hat{\gamma}\in[0,A/\Delta\tau]} \big(\hat{\gamma} - \hat{\gamma}V_A\big), \text{ where } 0 \le A/\Delta\tau \le G_r.
\end{aligned}
\tag{6.30}
$$

Lemma 6.2. *Let $\mathbf{x} = (W_i, A_j, \tau^{n+1})$. Suppose the mesh size and the timestep parameter satisfy conditions (5.1) and assume*

$$
\Delta W_{\min} \ge G_r \Delta\tau. \tag{6.31}
$$

Then for any smooth function $\phi(W, A, \tau)$ having bounded derivatives of all orders in $(W, A, \tau) \in \bar{\Omega}$, *with* $\phi_{i,j}^{n+1} = \phi(W_i, A_j, \tau^{n+1})$*, and for* h *sufficiently small, we have that*

$$
\begin{aligned}
&\mathcal{G}_{i,j}^{n+1}\big(h, \phi_{i,j}^{n+1} + \xi, \{\phi_{l,m}^{n+1} + \xi\}_{\substack{l\neq i\\ m\neq j}}, \{\phi_{i,j}^{n} + \xi\}\big) \\
&= \begin{cases}
F_{in} + O(h) + c(\mathbf{x})\xi & \text{if } 0 < W_i < W_{i_{\max}}, \quad G_r\Delta\tau < A_j \le A_{j_{\max}}, \quad 0 < \tau^{n+1} \le T; \\
F_{W_0} + O(h) + c(\mathbf{x})\xi & \text{if } W_i = 0, \quad G_r\Delta\tau < A_j \le A_{j_{\max}}, \quad 0 < \tau^{n+1} \le T; \\
F_{W'} + O(h) + c(\mathbf{x})\xi & \text{if } W_i = 0, \quad 0 < A_j \le G_r\Delta\tau, \quad 0 < \tau^{n+1} \le T; \\
F_{A_0} + O(h) + c(\mathbf{x})\xi & \text{if } 0 \le W_i < W_{i_{\max}}, \quad A_j = 0, \quad 0 < \tau^{n+1} \le T; \\
F_{A'} + O(h) + c(\mathbf{x})\xi & \text{if } 0 < W_i < W_{i_{\max}}, \quad 0 < A_j \le G_r\Delta\tau, \quad 0 < \tau^{n+1} \le T; \\
F_{W_m} + c(\mathbf{x})\xi & \text{if } W_i = W_{i_{\max}}, \quad 0 \le A_j \le A_{j_{\max}}, \quad 0 < \tau^{n+1} \le T; \\
F_{\tau^0} + c(\mathbf{x})\xi & \text{if } 0 \le W_i \le W_{i_{\max}}, \quad 0 \le A_j \le A_{j_{\max}}, \quad \tau^{n+1} = 0,
\end{cases}
\end{aligned} \tag{6.32}
$$

where ξ *is a constant,* $c(\mathbf{x})$ *is a bounded function of* $\mathbf{x}$ *satisfying* $|c(\mathbf{x})| \le \max(r, 1)$ *for all* $\mathbf{x} \in \bar{\Omega}$*, operators* $F_{in}, F_{W_0}, F_{A_0}, F_{A'}, F_{W'}$ *are functions of* $(D^2\phi(\mathbf{x}), D\phi(\mathbf{x}), \phi(\mathbf{x}), \mathbf{x})$*, and operators* F_{W_m}, F_{τ^0} *are functions of* $(\phi(\mathbf{x}), \mathbf{x})$*.*

Proof. We first consider the case when $0 < W_i < W_{i_{\max}}$, $G_r\Delta\tau < A_j \le A_{j_{\max}}$, and $0 < \tau^{n+1} \le T$. In this case, condition (6.31) implies that $W_i > \gamma_{i,j}^n$ for all $\gamma_{i,j}^n \in [0, G_r\Delta\tau]$. Thus, according to approximation (5.5), we have

$$
(\phi+\xi)_{\hat{i},\hat{j}}^n = \begin{cases}
\phi\big(W_i - \gamma_{i,j}^n, A_j - \gamma_{i,j}^n, \tau^n\big) + \xi + O\big((\Delta W_{\max} + \Delta A_{\max})^2\big), & \gamma_{i,j}^n \in [0, G_r\Delta\tau], \\
\phi\big(\max(W_i - \gamma_{i,j}^n, 0), A_j - \gamma_{i,j}^n, \tau^n\big) + \xi + O\big((\Delta W_{\max} + \Delta A_{\max})^2\big), & \gamma_{i,j}^n \in (G_r\Delta\tau, A_j].
\end{cases} \tag{6.33}
$$

Here we can take the term ξ out of the interpolation operation $(\cdot)_{\hat{i},\hat{j}}^n$ since it is a linear interpolation. Let us define a new variable $\hat{\gamma}_{i,j}^n = \gamma_{i,j}^n/(\Delta\tau)$. Then equation (6.33) becomes

$$
(\phi+\xi)_{\hat{i},\hat{j}}^n = \phi\big(W_i - \hat{\gamma}_{i,j}^n\Delta\tau, A_j - \hat{\gamma}_{i,j}^n\Delta\tau, \tau^n\big) + \xi + O\big((\Delta W_{\max} + \Delta A_{\max})^2\big), \quad \text{if } \hat{\gamma}_{i,j}^n \in [0, G_r]. \tag{6.34}
$$

Equation (6.34) implies

$$
\begin{aligned}
\frac{\big(\phi_{i,j}^{n+1} + \xi\big) - (\phi+\xi)_{\hat{i},\hat{j}}^n}{\Delta\tau} &= (\phi_\tau)_{i,j}^{n+1} + \hat{\gamma}_{i,j}^n(\phi_W)_{i,j}^n + \hat{\gamma}_{i,j}^n(\phi_A)_{i,j}^n \\
&\quad + O\Big(\Delta\tau + \big(\Delta W_{\max} + \Delta A_{\max}\big)^2/\Delta\tau\Big), \quad \text{if } \hat{\gamma}_{i,j}^n \in [0, G_r],
\end{aligned} \tag{6.35}
$$

where Taylor series is used to expand $\phi\big(W_i - \hat{\gamma}_{i,j}^n\Delta\tau, A_j - \hat{\gamma}_{i,j}^n\Delta\tau, \tau^n\big)$ at (W_i, A_j, τ^n). Note that the terms in the $O(\cdot)$ expressions are bounded functions of $\gamma_{i,j}^n$.

Assuming a discretization similar to that in [8] is used to discretize operator $\mathcal{L}\phi$, then from Taylor series expansions and equation (5.2), we obtain that

$$
\big(\mathcal{L}_h(\phi+\xi)\big)_{i,j}^{n+1} = (\mathcal{L}\phi)_{i,j}^{n+1} - r\xi + O(\Delta W_{\max}). \tag{6.36}
$$

Substituting equations (6.35), (6.36), $(\phi_W)_{i,j}^n = (\phi_W)_{i,j}^{n+1} + O(\Delta\tau)$, $(\phi_A)_{i,j}^n = (\phi_A)_{i,j}^{n+1} + O(\Delta\tau)$ into $\mathcal{H}_{i,j}^{n+1}$ given in (6.23), then unifying the mesh size/timestep parameter for the $O(\cdot)$ terms in terms of h in (5.1), leads to

$$
\begin{aligned}
&\mathcal{H}_{i,j}^{n+1}\big(h, \phi_{i,j}^{n+1}+\xi, \{\phi_{l,m}^{n+1}+\xi\}_{\substack{l\neq i\\ m\neq j}}, \{\phi_{i,j}^{n}+\xi\}\big)\\
&= \Big[\phi_\tau - \mathcal{L}\phi - \sup_{\hat{\gamma}_{i,j}^n\in[0,G_r]} \big[\hat{\gamma}_{i,j}^n - \hat{\gamma}_{i,j}^n\phi_W - \hat{\gamma}_{i,j}^n\phi_A + O(h)\big]\Big]_{i,j}^{n+1} + r\xi \\
&= \Big[\phi_\tau - \mathcal{L}\phi - \sup_{\hat{\gamma}_{i,j}^n\in[0,G_r]} \big(\hat{\gamma}_{i,j}^n - \hat{\gamma}_{i,j}^n\phi_W - \hat{\gamma}_{i,j}^n\phi_A\big)\Big]_{i,j}^{n+1} + O(h) + r\xi.
\end{aligned}
\tag{6.37}
$$

Here the constant for the $O(h)$ term in the first equality is a bounded function of $\hat{\gamma}_{i,j}^n$, that is, $O(h) = H(\hat{\gamma}_{i,j}^n)h$, where H is a bounded function of $\hat{\gamma}_{i,j}^n$. Since $\hat{\gamma}_{i,j}^n$ is bounded, we can move the $O(h)$ term out of the sup operator as shown in [5].

We next present an intermediate result. According to Remark 5.1, $\phi_{\hat{i},\hat{j}}^n$ is a uniformly continuous function of $\gamma_{i,j}^n$ on $[0, A_j]$. As a result, using (5.1) and (5.5) we have

$$
\begin{aligned}
&\sup_{\gamma_{i,j}^n\in(G_r\Delta\tau, A_j]} \Big[\phi_{\hat{i},\hat{j}}^n + (1-\kappa)\gamma_{i,j}^n - c\Big]\\
&= \max_{\gamma_{i,j}^n\in[G_r\Delta\tau, A_j]} \Big[\phi_{\hat{i},\hat{j}}^n + (1-\kappa)\gamma_{i,j}^n - c\Big]\\
&= \max_{\gamma_{i,j}^n\in[G_r\Delta\tau, A_j]} \big[\phi\big(\max(W_i-\gamma_{i,j}^n, 0), A_j - \gamma_{i,j}^n, \tau^n\big) + (1-\kappa)\gamma_{i,j}^n - c\big] + O(h^2).
\end{aligned}
\tag{6.38}
$$

It can be shown that $\phi\big(\max(W_i-\gamma_{i,j}^n, 0), A_j - \gamma_{i,j}^n, \tau^n\big) + (1-\kappa)\gamma_{i,j}^n - c$ is uniformly continuous on $\gamma_{i,j}^n \in [0, A_j]$. Consequently, from (6.38) we have

$$
\begin{aligned}
&\sup_{\gamma_{i,j}^n\in(G_r\Delta\tau, A_j]} \Big[\phi_{\hat{i},\hat{j}}^n + (1-\kappa)\gamma_{i,j}^n - c\Big] - \sup_{\gamma_{i,j}^n\in(0, A_j]} \big[\phi\big(\max(W_i-\gamma_{i,j}^n, 0), A_j - \gamma_{i,j}^n, \tau^n\big) + (1-\kappa)\gamma_{i,j}^n - c\big]\\
&= \max_{\gamma_{i,j}^n\in[G_r\Delta\tau, A_j]} \big[\phi\big(\max(W_i-\gamma_{i,j}^n, 0), A_j - \gamma_{i,j}^n, \tau^n\big) + (1-\kappa)\gamma_{i,j}^n\big] + O(h^2)\\
&\quad - \max_{\gamma_{i,j}^n\in[0, A_j]} \big[\phi\big(\max(W_i-\gamma_{i,j}^n, 0), A_j - \gamma_{i,j}^n, \tau^n\big) + (1-\kappa)\gamma_{i,j}^n\big]\\
&= O(h).
\end{aligned}
\tag{6.39}
$$

Note that the subtraction of two max expressions above produces an $O(h)$ error, since the function inside the max expressions is continuous on $\gamma_{i,j}^n \in [0, A_j]$ and the difference of the optimal values of $\gamma_{i,j}^n$ for two max expressions are bounded by $G_r\Delta\tau = O(h)$. Substituting (6.36) into $\mathcal{I}_{i,j}^{n+1}$ in (6.24), and using (6.39) and $\phi(\max(W_i-\gamma_{i,j}^n, 0), A_j-\gamma_{i,j}^n, \tau^n) = \phi(\max(W_i-\gamma_{i,j}^n, 0), A_j-\gamma_{i,j}^n, \tau^{n+1}) + O(h)$, gives

$$
\begin{aligned}
&\mathcal{I}_{i,j}^{n+1}\big(h, \phi_{i,j}^{n+1}+\xi, \{\phi_{l,m}^{n+1}+\xi\}_{\substack{l\neq i\\ m\neq j}}, \{\phi_{i,j}^{n}+\xi\}\big)\\
&= \phi_{i,j}^{n+1} - \sup_{\gamma_{i,j}^n\in(G_r\Delta\tau, A_j]} \Big[\phi_{\hat{i},\hat{j}}^n + (1-\kappa)\gamma_{i,j}^n - c\Big] - \kappa G_r\Delta\tau - \Delta\tau\big(\mathcal{L}\phi\big)_{i,j}^{n+1} + r\xi\Delta\tau + O(h^2)\\
&= \phi_{i,j}^{n+1} - \sup_{\gamma_{i,j}^n\in(0, A_j]} \big[\phi\big(\max(W_i-\gamma_{i,j}^n, 0), A_j - \gamma_{i,j}^n, \tau^{n+1}\big) + (1-\kappa)\gamma_{i,j}^n - c\big] + O(h),
\end{aligned}
\tag{6.40}
$$

where the last equality uses the fact that kG, $(\mathcal{L}\phi)_{i,j}^{n+1}$ and $r\xi$ are all bounded.

According to (6.25), (6.27), (6.37) and (6.40), we can write

$$\begin{aligned}&\mathcal{G}_{i,j}^{n+1}\big(h, \phi_{i,j}^{n+1}+\xi, \{\phi_{l,m}^{n+1}+\xi\}_{\substack{l\neq i\\ m\neq j}}, \{\phi_{i,j}^{n}+\xi\}\big) - F_{in}\big(D^2\phi(\mathbf{x}), D\phi(\mathbf{x}), \phi(\mathbf{x}), \mathbf{x}\big)\\ &= O(h) + c(\mathbf{x})\xi, \quad \text{if } 0 < W_i < W_{i_{\max}}, G_r\Delta\tau < A_j \le A_{j_{\max}},\ 0 < \tau^{n+1} \le T,\end{aligned} \tag{6.41}$$

where $c(\mathbf{x})$ is bounded satisfying $0 \le c(\mathbf{x}) \le r$. This proves the first equation in (6.32).

Following similar arguments as above, we can prove the rest of equations in (6.32). We omit the details here.

□

Remark 6.3. *To ease the presentation of the scheme, we impose the grid size condition (6.31) for the purpose of making $V(\max(W_i - \gamma_{i,j}^n, 0), A_j - \gamma_{i,j}^n, \tau^n) = V(W_i - \gamma_{i,j}^n, A_j - \gamma_{i,j}^n, \tau^n)$ for any $\gamma_{i,j}^n \in [0, G_r\Delta\tau]$ and all nodes $W_i > 0$. However, we can avoid this condition by modifying the scheme according to the following ideas: at first we extend the W grid in the $W < 0$ direction, that is, the extended grid includes nodes with negative W values. Then at each timestep $\tau^{n+1} > 0$, we first compute $V_{0,j}^{n+1}$ at $W = 0$ using discrete equation (6.6) (this is possible since we do not require information from other grid nodes in W direction), and then we set $V_{i,j}^{n+1} = V_{0,j}^{n+1}$ for all $W_i < 0$. Finally, we compute $V_{i,j}^{n+1}$ using a modification of equation (6.6):*

$$V_{i,j}^{n+1} - \sup_{\gamma_{i,j}^n \in [0, A_j]} \Big[V_{\tilde{i},\hat{j}}^n + f\big(\gamma_{i,j}^n\big)\Big] - \Delta\tau\big(\mathcal{L}_h V\big)_{i,j}^{n+1} = 0, \tag{6.42}$$

where the term $V_{\tilde{i},\hat{j}}^n$ is the approximation of $V(W_i - \gamma_{i,j}^n, A_j - \gamma_{i,j}^n, \tau^n)$ by linear interpolation. Since $V_{\tilde{i},\hat{j}}^n$ exists in the case when $W_i - \gamma_{i,j}^n < 0$ and is equal to the approximation of $V(0, A_j - \gamma_{i,j}^n, \tau^n)$, the modified scheme is identical to the original one. Therefore, with respect to the modified scheme, (6.33) follows without imposing condition (6.31). Hence condition (6.31) can be eliminated.

Remark 6.4. *It can be verified that the operators $F_{in}(M, p, g, \mathbf{x})$, $F_{W_0}(M, p, g, \mathbf{x})$, $F_{A_0}(M, p, g, \mathbf{x})$, $F_{A'}(M, p, g, \mathbf{x})$, and $F_{W'}(M, p, g, \mathbf{x})$ defined in (6.27) and (6.30) are continuous on $(M, p, g, \mathbf{x})$, given a smooth function $\phi(\mathbf{x})$; meanwhile, operators $F_{W_m}(g, \mathbf{x})$ and $F_{\tau^0}(g, \mathbf{x})$ are continuous on $(g, \mathbf{x})$. In particular, $\phi - \sup_{\gamma \in (0,A]}\big[\phi(\max(W-\gamma, 0), A-\gamma, \tau) + (1-\kappa)\gamma - c\big]$ is continuous on $\mathbf{x}$ based on an argument similar to the proof of Lemma 4.2.*

In order to verify the consistency of scheme (6.25), following [3], we define

Definition 6.5. *If C is a topological space and $u : C \to \mathbb{R}$ is a function, then the upper semi-continuous (usc) envelope $u^* : C \to \mathbb{R}$ and the lower semi-continuous (lsc) envelope $u_* : C \to \mathbb{R}$ of u are defined by*

$$u^* = \limsup_{\substack{y\to x\\ y\in C}} u(y) \qquad \text{and} \qquad u_* = \liminf_{\substack{y\to x\\ y\in C}} u(y), \tag{6.43}$$

respectively.

Lemma 6.6 (Consistency)**.** *Assuming all the conditions in Lemma 6.2 are satisfied, then the scheme (6.25) is consistent to the impulse control problem (3.15-3.19) in $\bar{\Omega}$ according to the definition in [3, 2]. That is, for all $\hat{\mathbf{x}} = (\hat{W}, \hat{A}, \hat{\tau}) \in \bar{\Omega}$ and any function $\phi(W, A, \tau)$ having bounded* *derivatives of all orders in* $(W, A, \tau) \in \bar{\Omega}$ *with* $\phi_{i,j}^{n+1} = \phi(W_i, A_j, \tau^{n+1})$ *and* $\mathbf{x} = (W_i, A_j, \tau^{n+1})$*, we have*

$$\limsup_{\substack{h\to 0\\ \mathbf{x}\to\hat{\mathbf{x}}\\ \xi\to 0}} \mathcal{G}_{i,j}^{n+1}\big(h, \phi_{i,j}^{n+1}+\xi, \big\{\phi_{l,m}^{n+1}+\xi\big\}_{\substack{l\neq i\\ m\neq j}}, \big\{\phi_{i,j}^{n}+\xi\big\}\big) \le F^*\big(D^2\phi(\hat{\mathbf{x}}), D\phi(\hat{\mathbf{x}}), \phi(\hat{\mathbf{x}}), \hat{\mathbf{x}}\big), \tag{6.44}$$

and

$$\liminf_{\substack{h\to 0\\ \mathbf{x}\to\hat{\mathbf{x}}\\ \xi\to 0}} \mathcal{G}_{i,j}^{n+1}\big(h, \phi_{i,j}^{n+1}+\xi, \big\{\phi_{l,m}^{n+1}+\xi\big\}_{\substack{l\neq i\\ m\neq j}}, \big\{\phi_{i,j}^{n}+\xi\big\}\big) \ge F_*\big(D^2\phi(\hat{\mathbf{x}}), D\phi(\hat{\mathbf{x}}), \phi(\hat{\mathbf{x}}), \hat{\mathbf{x}}\big). \tag{6.45}$$

Proof. Let us first prove (6.44). According to the definition of lim sup, there exist sequences $h_k, i_k, j_k, n_k, \xi_k$ such that

$$h_k \to 0, \quad \xi_k \to 0, \quad \mathbf{x}_k \equiv \big(W_{i_k}, A_{j_k}, \tau^{n_k+1}\big) \to (\hat{W}, \hat{A}, \hat{\tau}) \ \text{ as } k\to\infty, \tag{6.46}$$

and

$$\begin{aligned}
&\limsup_{k\to\infty} \mathcal{G}_{i_k,j_k}^{n_k+1}\big(h_k, \phi_{i_k,j_k}^{n_k+1}+\xi_k, \big\{\phi_{l,m}^{n_k+1}+\xi_k\big\}_{\substack{l\neq i_k\\ m\neq j_k}}, \big\{\phi_{i,j}^{n_k}+\xi_k\big\}\big)\\
&= \limsup_{\substack{h\to 0,\xi\to 0,\\ \mathbf{x}\to\hat{\mathbf{x}}}} \mathcal{G}_{i,j}^{n+1}\big(h, \phi_{i,j}^{n+1}+\xi, \big\{\phi_{l,m}^{n+1}+\xi\big\}_{\substack{l\neq i\\ m\neq j}}, \big\{\phi_{i,j}^{n}+\xi\big\}\big).
\end{aligned} \tag{6.47}$$

At first, we consider the case when $\hat{\mathbf{x}} \in \Omega_{in}$. Let $\Delta\tau_k$ denote the timestep corresponding to parameter h_k. Then if h_k is sufficiently small, we have $0 < W_{i_k} < W_{i_{\max}}$, $G_r\Delta\tau_k < A_{j_k} \le A_{j_{\max}}$ and $0 < \tau^{n_k+1} \le T$. According to (6.32), we have

$$\mathcal{G}_{i_k,j_k}^{n_k+1}\big(h_k, \phi_{i_k,j_k}^{n_k+1}+\xi_k, \big\{\phi_{l,m}^{n_k+1}+\xi_k\big\}_{\substack{l\neq i_k\\ m\neq j_k}}, \big\{\phi_{i,j}^{n_k}+\xi_k\big\}\big) = F_{in}\big(D^2\phi(\mathbf{x}_k), D\phi(\mathbf{x}_k), \phi(\mathbf{x}_k), \mathbf{x}_k\big) + O(h_k) + c(\mathbf{x}_k)\xi_k. \tag{6.48}$$

Thus, (6.47-6.48) and continuity of F_{in} (see remark 6.4) lead to

$$\begin{aligned}
&\limsup_{\substack{h\to 0,\xi\to 0,\\ \mathbf{x}\to\hat{\mathbf{x}}}} \mathcal{G}_{i,j}^{n+1}\big(h, \phi_{i,j}^{n+1}+\xi, \big\{\phi_{l,m}^{n+1}+\xi\big\}_{\substack{l\neq i\\ m\neq j}}, \big\{\phi_{i,j}^{n}+\xi\big\}\big)\\
&\le \limsup_{k\to\infty} F_{in}\big(D^2\phi(\mathbf{x}_k), D\phi(\mathbf{x}_k), \phi(\mathbf{x}_k), \mathbf{x}_k\big) + \lim_{k\to\infty}\big[O(h_k) + c(\mathbf{x}_k)\xi_k\big]\\
&= F_{in}\big(D^2\phi(\hat{\mathbf{x}}), D\phi(\hat{\mathbf{x}}), \phi(\hat{\mathbf{x}}), \hat{\mathbf{x}}\big)\\
&= F^*\big(D^2\phi(\hat{\mathbf{x}}), D\phi(\hat{\mathbf{x}}), \phi(\hat{\mathbf{x}}), \hat{\mathbf{x}}\big),
\end{aligned} \tag{6.49}$$

which verifies condition (6.44) for $\hat{\mathbf{x}} \in \Omega_{in}$.

We then consider the case when $\hat{\mathbf{x}} \in \Omega_{A_0}\backslash\{0\}\times\{0\}\times(0,T]$, that is, $\hat{\mathbf{x}}$ resides in region Ω_{A_0} excluding a corner line $\hat{\mathbf{x}} \in \{0\}\times\{0\}\times(0,T]$. When k is sufficiently large so that $\mathbf{x}_k$ is sufficiently close to $\hat{\mathbf{x}}$, each element in the convergent sequence $\mathbf{x}_k = (W_{i_k}, A_{j_k}, \tau^{n_k+1})$ satisfies $0 < \tau^{n_k+1} \le T$, $0 < W_{i_k} < W_{i_{\max}}$, and either $G_r\Delta\tau_k < A_{j_k} \le A_{j_{\max}}$, or $0 < A_{j_k} \le G_r\Delta\tau_k$, or $A_{j_k} = 0$. Thus from (6.32), we have

$$\begin{aligned}
&\mathcal{G}_{i_k,j_k}^{n_k+1}\big(h_k, \phi_{i_k,j_k}^{n_k+1}+\xi_k, \big\{\phi_{l,m}^{n_k+1}+\xi_k\big\}_{\substack{l\neq i_k\\ m\neq j_k}}, \big\{\phi_{i,j}^{n_k}+\xi_k\big\}\big)\\
&= \begin{cases} F_{in}\big(D^2\phi(\mathbf{x}_k), D\phi(\mathbf{x}_k), \phi(\mathbf{x}_k), \mathbf{x}_k\big) + O(h_k) + c(\mathbf{x}_k)\xi_k & \text{if } \ G_r\Delta\tau_k < A_{j_k} \le A_{j_{\max}};\\ F_{A'}\big(D^2\phi(\mathbf{x}_k), D\phi(\mathbf{x}_k), \phi(\mathbf{x}_k), \mathbf{x}_k\big) + O(h_k) + c(\mathbf{x}_k)\xi_k & \text{if } \ 0 < A_{j_k} \le G_r\Delta\tau_k;\\ F_{A_0}\big(D^2\phi(\mathbf{x}_k), D\phi(\mathbf{x}_k), \phi(\mathbf{x}_k), \mathbf{x}_k\big) + O(h_k) + c(\mathbf{x}_k)\xi_k & \text{if } \ A_{j_k} = 0. \end{cases}
\end{aligned} \tag{6.50}$$

From definitions of F_{A_0} and $F_{A'}$ in (6.27) and (6.30), and from $\sup_{\hat{\gamma}\in[0,A_{j_k}/\Delta\tau_k]}\big[\hat{\gamma} - \hat{\gamma}\phi_W(\mathbf{x}_k) - \hat{\gamma}\phi_A(\mathbf{x}_k)\big] \geq 0$, we observe that

$$\begin{aligned} &F_{A'}\big(D^2\phi(\mathbf{x}_k), D\phi(\mathbf{x}_k), \phi(\mathbf{x}_k), \mathbf{x}_k\big) \\ &\leq F_{A_0}\big(D^2\phi(\mathbf{x}_k), D\phi(\mathbf{x}_k), \phi(\mathbf{x}_k), \mathbf{x}_k\big), \quad \text{if } \ 0 < A_{j_k} \leq G_r\Delta\tau_k\ . \end{aligned} \tag{6.51}$$

As a result, (6.28-6.29) and (6.50-6.51) lead to

$$\begin{aligned} &\limsup_{k\to\infty} \mathcal{G}_{i_k,j_k}^{n_k+1}\big(h_k, \phi_{i_k,j_k}^{n_k+1} + \xi_k, \{\phi_{l,m}^{n_k+1} + \xi_k\}_{\substack{l\neq i_k\\ m\neq j_k}}, \{\phi_{i,j}^{n_k} + \xi_k\}\big) \\ &\leq \limsup_{k\to\infty} F\big(D^2\phi(\mathbf{x}_k), D\phi(\mathbf{x}_k), \phi(\mathbf{x}_k), \mathbf{x}_k\big) + \lim_{k\to\infty}\big[O(h_k) + c(\mathbf{x}_k)\xi_k\big] \\ &\leq F^*\big(D^2\phi(\hat{\mathbf{x}}), D\phi(\hat{\mathbf{x}}), \phi(\hat{\mathbf{x}}), \hat{\mathbf{x}}\big). \end{aligned} \tag{6.52}$$

This together with (6.47) verify (6.44) for $\hat{\mathbf{x}} \in \Omega_{A_0}\backslash\{0\}\times\{0\}\times(0,T]$.

Following arguments similar to the above, we can prove (6.44) for the corner line $\hat{\mathbf{x}} \in \{0\} \times \{0\}\times(0,T]$ as well as for the boundary regions Ω_{W_0}, Ω_{W_m} and Ω_{τ^0}. We omit the details here.

Showing condition (6.45) follows in the same manner as above: we verify the condition for different regions defined in (6.26). Here we only show (6.45) for $\hat{\mathbf{x}} \in \Omega_{A_0}\backslash\{0\}\times\{0\}\times(0,T]$. Let $h_k, i_k, j_k, n_k, \xi_k$ be sequences satisfying (6.46) such that

$$\begin{aligned} &\liminf_{k\to\infty} \mathcal{G}_{i_k,j_k}^{n_k+1}\big(h_k, \phi_{i_k,j_k}^{n_k+1} + \xi_k, \{\phi_{l,m}^{n_k+1} + \xi_k\}_{\substack{l\neq i_k\\ m\neq j_k}}, \{\phi_{i,j}^{n_k} + \xi_k\}\big) \\ &= \liminf_{\substack{h\to 0,\xi\to 0,\\ \mathbf{x}\to\hat{\mathbf{x}}}} \mathcal{G}_{i,j}^{n+1}\big(h, \phi_{i,j}^{n+1} + \xi, \{\phi_{l,m}^{n+1} + \xi\}_{\substack{l\neq i\\ m\neq j}}, \{\phi_{i,j}^{n} + \xi\}\big). \end{aligned} \tag{6.53}$$

Then for sufficiently large k, from (6.32), equation (6.50) holds as discussed above. Then from definitions of F_{in} and $F_{A'}$ in (6.27) and (6.30), and from $\sup_{\hat{\gamma}\in[0,A_{j_k}/\Delta\tau_k]}\big[\hat{\gamma} - \hat{\gamma}\phi_W(\mathbf{x}_k) - \hat{\gamma}\phi_A(\mathbf{x}_k)\big] \leq \sup_{\hat{\gamma}\in[0,G_r]}\big[\hat{\gamma} - \hat{\gamma}\phi_W(\mathbf{x}_k) - \hat{\gamma}\phi_A(\mathbf{x}_k)\big]$ if $0 < A_{j_k} \leq G_r\Delta\tau_k$, we obtain

$$\begin{aligned} &F_{A'}\big(D^2\phi(\mathbf{x}_k), D\phi(\mathbf{x}_k), \phi(\mathbf{x}_k), \mathbf{x}_k\big) \\ &\geq F_{in}\big(D^2\phi(\mathbf{x}_k), D\phi(\mathbf{x}_k), \phi(\mathbf{x}_k), \mathbf{x}_k\big), \quad \text{if } \ 0 < A_{j_k} \leq G_r\Delta\tau_k\ . \end{aligned} \tag{6.54}$$

Consequently, (6.28-6.29), (6.50) and (6.54) lead to

$$\begin{aligned} &\liminf_{k\to\infty} \mathcal{G}_{i_k,j_k}^{n_k+1}\big(h_k, \phi_{i_k,j_k}^{n_k+1} + \xi_k, \{\phi_{l,m}^{n_k+1} + \xi_k\}_{\substack{l\neq i_k\\ m\neq j_k}}, \{\phi_{i,j}^{n_k} + \xi_k\}\big) \\ &\geq \liminf_{k\to\infty} F\big(D^2\phi(\mathbf{x}_k), D\phi(\mathbf{x}_k), \phi(\mathbf{x}_k), \mathbf{x}_k\big) + \lim_{k\to\infty}\big[O(h_k) + c(\mathbf{x}_k)\xi_k\big] \\ &\geq F_*\big(D^2\phi(\hat{\mathbf{x}}), D\phi(\hat{\mathbf{x}}), \phi(\hat{\mathbf{x}}), \hat{\mathbf{x}}\big), \end{aligned} \tag{6.55}$$

which together with (6.53) conclude (6.45) for $\hat{\mathbf{x}} \in \Omega_{A_0}\backslash\{0\}\times\{0\}\times(0,T]$.

$\square$

6.1.3 Monotonicity

It is straightforward to verify that scheme (6.25) is monotone. We omit the proof.

Lemma 6.7 (Monotonicity)**.** *If the discretization (5.2) satisfies the positive coefficient condition (5.3) and linear interpolation is used to compute $V^n_{\hat{i},\hat{j}}$, then the discretization (6.25) is monotone according to the definition*

$$\begin{aligned} &\mathcal{G}^{n+1}_{i,j}\big(h, V^{n+1}_{i,j}, \{X^{n+1}_{l,m}\}_{\substack{l\neq i\\ m\neq j}}, \{X^n_{i,j}\}\big) \\ &\leq \mathcal{G}^{n+1}_{i,j}\big(h, V^{n+1}_{i,j}, \{Y^{n+1}_{l,m}\}_{\substack{l\neq i\\ m\neq j}}, \{Y^n_{i,j}\}\big); \quad \text{for all } X^n_{i,j} \geq Y^n_{i,j}, \forall i,j,n. \end{aligned} \tag{6.56}$$

6.1.4 Convergence

In order to prove the convergence of our scheme using the results in [3, 2], we need to assume the following strong comparison result, as defined in [3, 2], for equation (3.15).

Assumption 6.8. *If u and v are an usc subsolution and a lsc supersolution of the pricing equation (3.15) associated with the boundary conditions (3.16-3.19), respectively, then*

$$u \leq v \quad \text{on } \Omega_{in}. \tag{6.57}$$

The strong comparison result is proved for other similar (but not identical) impulse control problems in [1, 16, 13, 10]. From Lemmas 6.1, 6.7 and 6.6 and Assumption 6.8, using the results in [3, 2], we can obtain the following convergence result:

Theorem 6.9 (Convergence to the viscosity solution)**.** *Assume that discretization (6.2-6.5) (or scheme (6.2), (6.5), (6.6), or scheme (6.25)) satisfies all the conditions required for Lemmas 6.1, 6.7 and 6.6, and that Assumption 6.8 is satisfied, then scheme (6.2-6.5) converges to the unique continuous viscosity solution of the problem (3.15), (3.16-3.19) in Ω_{in}.*

Remark 6.10 (Domain of convergence)**.** *Note that we only consider convergence in Ω_{in}. As discussed in [16], in general, the strong comparison result may only hold in Ω_{in} for impulse control problems.*

7 Solution of the Local Optimization Problems

As indicated in (5.6) and (6.3), the numerical schemes need to solve a discrete local optimization problem

$$\sup_{\gamma^n_{i,j} \in [0, A_j]} \Big[V^n_{\hat{i},\hat{j}} + f\big(\gamma^n_{i,j}\big) \Big] \tag{7.1}$$

at a mesh node (W_i, A_j, τ^n), where $f\big(\gamma^n_{i,j}\big)$ is a piecewise function of $\gamma^n_{i,j}$ given in (4.2) or (6.1) and $V^n_{\hat{i},\hat{j}}$ is a function of $\gamma^n_{i,j}$ (see Remark 5.1).

It is computationally expensive to directly solve problem (7.1) by constructing the curve $V^n_{\hat{i},\hat{j}}$ and then seeking the maximum of the objective function along the curve. In this section, we present the following consistent approximation to problem (7.1). We first select a sequence of control values $\gamma^n_{i,j}$, denoted by $\mathcal{A}_j$, from the interval $[0, A_j]$, where $\mathcal{A}_j$ includes 0, A_j and $G^k = G_r \Delta\tau^{k+1}_O$ (if $G^k < A_j$), and the distance between two consecutive elements in sequence $\mathcal{A}_j$ is bounded by $O(h)$. We then evaluate the function $V^n_{\hat{i},\hat{j}} + f\big(\gamma^n_{i,j}\big)$ using all elements $\gamma^n_{i,j} \in \mathcal{A}_j$, and return as output the maximum among the set of evaluated values. The above procedure indicates that we actually solve an alternative (and simpler) problem

$$\sup_{\gamma_{i,j}^n \in \mathcal{A}_j} \left[V_{\hat{i},\hat{j}}^n + f\big(\gamma_{i,j}^n\big) \right]. \tag{7.2}$$

In terms of a smooth test function, the solutions to problems (7.1-7.2) satisfy the following conditions:

Proposition 7.1. *Let $\phi(W, A, \tau)$ be a smooth function with $\phi_{i,j}^n = \phi(W_i, A_j, \tau^n)$. Then the optimization procedure introduced above results in*

$$\begin{aligned} \sup_{\gamma_{i,j}^n \in \mathcal{A}_j} \left[\phi_{\hat{i},\hat{j}}^n + f\big(\gamma_{i,j}^n\big) \right] &= \sup_{\gamma_{i,j}^n \in [0,A_j]} \left[\phi_{\hat{i},\hat{j}}^n + f\big(\gamma_{i,j}^n\big) \right] + O(h^2) &\quad (7.3) \\ &= \sup_{\gamma_{i,j}^n \in [0,A_j]} \left[\phi\big(\max(W_i - \gamma_{i,j}^n, 0), A_j - \gamma_{i,j}^n, \tau^n\big) + f\big(\gamma_{i,j}^n\big) \right] + O(h^2). &\quad (7.4) \end{aligned}$$

Proof. Let us define $g(\gamma_{i,j}^n)$ as a piecewise linear function of $\gamma_{i,j}^n \in [0, A_j]$ constructed using the discrete values from the set $\left\{ \phi_{\hat{i},\hat{j}}^n \;\middle|\; \gamma_{i,j}^n \in \mathcal{A}_j \right\}$ by linear interpolation. Without loss of generality, we assume $A_j > G^k$. Since $f(\gamma_{i,j}^n)$ is a piecewise linear function on $\gamma \in [0, G^k]$ and $\gamma \in (G^k, A_j]$ satisfying inequality (5.9), and since all three boundary nodes $0, G^k, A_j$ belong to $\mathcal{A}_j$, then the supremum of function $g(\gamma_{i,j}^n) + f(\gamma_{i,j}^n)$ occurs at a node $\gamma_{i,j}^n \in \mathcal{A}_j$. Consequently, we have

$$\sup_{\gamma_{i,j}^n \in [0,A_j]} \left[g(\gamma_{i,j}^n) + f(\gamma_{i,j}^n) \right] = \sup_{\gamma_{i,j}^n \in \mathcal{A}_j} \left[\phi_{\hat{i},\hat{j}}^n + f\big(\gamma_{i,j}^n\big) \right]. \tag{7.5}$$

For any $\gamma_{i,j}^n \in \mathcal{A}_j$, equations (5.5) and (5.1) imply $g(\gamma_{i,j}^n) = \phi\big(\max(W_i - \gamma_{i,j}^n, 0), A_j - \gamma_{i,j}^n, \tau^n\big) + O(h^2)$. Since in addition ϕ is smooth and the distance between any two consecutive elements in $\mathcal{A}_j$ is bounded by $O(h)$, we have

$$g(\gamma_{i,j}^n) = \phi\big(\max(W_i - \gamma_{i,j}^n, 0), A_j - \gamma_{i,j}^n, \tau^n\big) + O(h^2), \quad \forall \gamma_{i,j}^n \in [0, A_j]. \tag{7.6}$$

Equation (7.6) and the result in [5] imply that

$$\sup_{\gamma_{i,j}^n \in [0,A_j]} \left[g(\gamma_{i,j}^n) + f(\gamma_{i,j}^n) \right] = \sup_{\gamma_{i,j}^n \in [0,A_j]} \left[\phi\big(\max(W_i - \gamma_{i,j}^n, 0), A_j - \gamma_{i,j}^n, \tau^n\big) + f\big(\gamma_{i,j}^n\big) \right] + O(h^2). \tag{7.7}$$

Therefore, (7.4) follows from (7.5) and (7.7).

Finally, equation (7.3) holds according to (7.4) and the following equation implied from (5.5) and (5.1):

$$\sup_{\gamma_{i,j}^n \in [0,A_j]} \left[\phi_{\hat{i},\hat{j}}^n + f\big(\gamma_{i,j}^n\big) \right] = \sup_{\gamma_{i,j}^n \in [0,A_j]} \left[\phi\big(\max(W_i - \gamma_{i,j}^n, 0), A_j - \gamma_{i,j}^n, \tau^n\big) + f\big(\gamma_{i,j}^n\big) \right] + O(h^2). \tag{7.8}$$

□

Remark 7.2. *According to Proposition 7.1, we can easily verify that the proof of the convergence Theorem 5.7 under the discrete withdrawal scenario still holds in the case when the discrete equation (5.6) solves the alternative optimization problem (7.2).*

Similarly, Proposition 7.1 implies that the convergence Theorem 6.9 under the continuous withdrawal scenario also holds when the discrete equation (6.3) or (6.6) solves the optimization problem (7.2). In fact, equations (7.3-7.4) show that the discrete local optimization problem (7.2) results in the same order of discretization error as problem (7.1). As a result, the consistency proof in Section 6.1 follows while solving problem (7.2). Meanwhile, the stability and monotonicity properties described in Section 6.1 are straightforward to verify in this case.

Remark 7.3. *Our implementation uses an unequally spaced (W, A) mesh. As a result, a binary search is required to find the interpolants $V^n_{\hat{i},\hat{j}}$. Let us consider the scheme (6.2-6.5) for the continuous withdrawal case. Since there are $O(1/h^3)$ optimizations performed in total (recall that we need to solve a discrete optimization problem (7.2) at each mesh node (W_i, A_j, τ^n) in this case) and each optimization performs $O(1/h)$ linear interpolations (i.e., there are $O(1/h)$ elements in sequence $\mathcal{A}_j$), resulting in $O(1/h^4)$ binary searches (each costing $O(log(1/h))$).*

We can reduce the number of binary searches as follows. At each timestep, we transform all the discrete values $V^n_{i,j}$ in the original unequally spaced (W, A) mesh to another equally spaced (W, A) mesh by linear interpolation. Then we can solve optimization problems (7.2) for all nodes in the equally spaced mesh without using binary search. The above procedure requires only $O(1/h^3)$ binary searches in total and results in $O(h^2)$ discretization errors for a smooth test function, which hence does not affect the convergence of the numerical scheme to the viscosity solutions. Note that we still require $O(1/h^4)$ interpolation operations. An obvious alternative is to use a one dimensional optimization method which would normally not require $O(1/h)$ function evaluations at each optimization. However, this is not guaranteed to obtain the global maximum along the curve.

8 Numerical Experiments

Having presented a numerical scheme for pricing the GMWB variable annuities in the previous sections, in this section, we conduct numerical experiments based on the scheme.

Under the continuous withdrawal scenario, we observe that the numerical solutions obtained by choosing a sufficiently small fixed cost (e.g., $c = 10^{-8}$) are identical to those obtained by choosing $c = 0$ up to at least seven digits. Since the solutions are also close to that given in [6] (see, e.g., Table 8.4), this suggests that our impulse control formulation (3.15) will converge to the singular control formulation (3.10) as $c \to 0$. It also shows that our scheme can solve both the singular control problem (3.10) with $c = 0$ and the impulse control problem (3.15) with $c > 0$. We will use $c = 10^{-8}$ in the numerical experiments below.

Recall that the computational domain has been localized in the W direction to $[0, W_{\max}]$. Initially, we set $W_{\max} = 1000$. We repeated the computations with $W_{\max} = 5000$. All the numerical results at $t = 0, A = W = w_0$ were the same to seven digits. In the following, all the results are reported with $W_{\max} = 1000$.

Table 8.1 gives the common input parameters for the numerical tests in this section. We first carry out a convergence analysis for the GMWB guarantees with the mesh size/timestep parameters chosen in Table 8.2. Table 8.3 presents the convergence results for the value of the GMWB guarantee with respect to two volatility values, assuming a zero insurance fee and continuous withdrawal. The convergence ratio in the table is defined as the ratio of successive changes in the solution, as the timestep and mesh size are reduced by a factor of two. A ratio of two indicates first order convergence. As shown in Table 8.3, our scheme achieves a first-order convergence as the

Parameter	Value
Expiry time T	10.0 years
Interest rate r	.05
Maximum withdrawal rate G_r	10/year
Withdrawal penalty κ	.10
Initial Lump-sum premium w_0	100
Initial guarantee account balance	100
Initial sub-account value	100

TABLE 8.1: *Common data used in the numerical tests.*

Level	W Nodes	A Nodes	Timesteps
0	65	51	60
1	129	101	120
2	257	201	240
3	513	401	480
4	1025	801	960
5	2049	1601	1920

TABLE 8.2: *Grid and timestep data for convergence tests.*

convergence ratios are approximately two. The table also reveals that a greater volatility produces a higher contract value.

Since no fee is paid at the inception of a GMWB contract, the insurance company needs to charge a proportional insurance fee α so that the contract value V is equal to the initial premium w_0 paid by the investor. This is the no-arbitrage or *fair* fee. That is, let $V(\alpha; W = w_0, A = w_0, t = 0)$ be the value of a GMWB contract at the contract inception as a function of α. Then the fair insurance fee is a solution to the algebraic equation $V(\alpha; w_0, w_0, 0) = w_0$. In this paper, we solve the equation numerically using Newton iteration. Table 8.4 shows the convergence of the fair insurance fees assuming continuous withdrawal for two volatility values $\sigma = .2$ and $\sigma = .3$. Table 8.4 also lists the

Refinement level	$\sigma = .20$		$\sigma = .30$	
	Value	Ratio	Value	Ratio
1	107.6950	n.a.	115.8032	n.a.
2	107.7132	n.a.	115.8457	n.a.
3	107.7232	1.82	115.8678	1.92
4	107.7284	1.92	115.8787	2.03
5	107.7313	1.79	115.8842	1.98

TABLE 8.3: *Convergence study for the value of the GMWB guarantee at* $t = 0$, $W = A = w_0 = 100$. *No insurance fee* ($\alpha = 0$) *is imposed. Data are given in Table 8.1. Continuous withdrawal is permitted.*

Refinement level	$\sigma = .20$	$\sigma = .30$
0	.0152023	.0317364
1	.0145009	.0313861
2	.0141471	.0312579
3	.0139699	.0312536
4	.0138905	.0312584
Value from [6]	.0137	.0304

TABLE 8.4: *Convergence study for the value of the fair insurance fee α, with respect to different values of σ. Data are given in Table 8.1. The value of α is computed so that the option value V satisfies $V = w_0 = 100$ at $t = 0$. Continuous withdrawal is permitted.*

Refinement level	$\sigma = .20$		$\sigma = .30$	
	$\Delta t_O = 1.0$	$\Delta t_O = .50$	$\Delta t_O = 1.0$	$\Delta t_O = .50$
0	.0128893	.0135554	.0291106	.0301345
1	.0128631	.0133379	.0292137	.0301367
2	.0128881	.0133312	.0292781	.0301912
3	.0129025	.0133441	.0293104	.0302238
4	.0129102	.0133516	.0293270	.0302407

TABLE 8.5: *Convergence study for the value of the fair insurance fee α in the discrete withdrawal case. Different withdrawal intervals Δt_O and different values of σ are considered. Data are given in Table 8.1. The value of α is computed so that the option value V satisfies $V = w_0 = 100$ at $t = 0$.*

corresponding fees computed in [6]. These results are close to those reported in [6].

Table 8.5 computes the fair insurance fees under the discrete withdrawal scenario with withdrawal interval being half a year and one year, respectively. Comparing Tables 8.4 and 8.5, we find that the insurance fees increases as the specified withdrawal frequency increases (from once every half a year to an infinite number of times). Furthermore, the insurance fees corresponding to the continuous withdrawal case are very close to those corresponding to the half a year withdrawal case (the difference is less than 6 basis points for $\sigma = .2$ for the fourth refinement level).

In Figure 8.1, we show the value of the GMWB guarantee as a function of W at $t = 0$, $A = 100$, with respect to various values of the insurance fee α including the fair value $\alpha = .03126$. The figure indicates that when W is relatively small, α has no effect on the contract value since in this case, the guarantee component of the contract dominates the equity component (i.e., $A \gg W$). Hence the contract value is determined only by the guarantee account value and is independent of the insurance fee which is imposed on the equity component. As the fee increases, the no-arbitrage value of the contract decreases near $W = 100$. Eventually, the value of the contract is precisely $V = 100$ at $W = 100$ when the fair fee is charged.

Figure 8.2 plots the value surface of the GMWB guarantee at $t = 0$ as a function of W and A assuming a fair insurance fee is imposed. The figure shows that the contract value increases as W and A increase. The value curve along the W direction transforms from a parabolic shape to a straight line as A changes from $A = 100$ to $A = 0$. Note that the surface forms a cusp along the

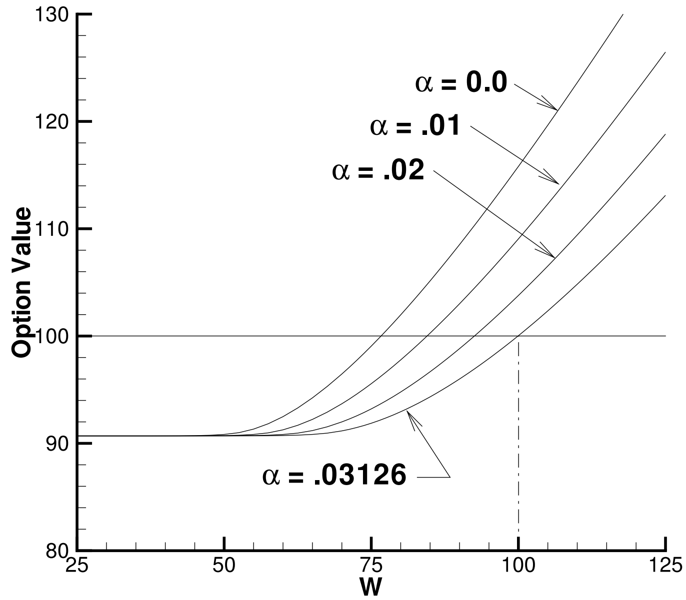

FIGURE 8.1: *The value of the GMWB guarantee as a function of W at $t = 0$, $A = 100$, with respect to various values of the insurance fee α including the fair value $\alpha = .03126$. The fair value of the fee occurs when the value of the guarantee V satisfies $V = w_0 = 100$. Data for this example are given in Table 8.1 with $\sigma = .30$. Continuous withdrawal is allowed.*

line $A = W$ near $A = W = 0$.

We next study the optimal withdrawal strategy for an investor who maximizes the no-arbitrage value of the GMWB guarantee. More precisely, this is the worst case for the provider of the guarantee. According to [6], the optimal strategy is either not to withdraw, or withdraw at the maximum rate G_r, or withdraw a finite amount instantaneously. Figure 8.3 shows a contour plot of the optimal withdrawal strategy at $t = \Delta\tau$ for different values of W and A computed numerically using the data from Table 8.1 with $\sigma = .3$ and using the fair insurance fee. From the figure, the (W, A)-plane is divided into a blank region and three shaded regions. The blank region corresponds to withdrawing continuously at the rate G_r. The upper left and upper right shaded areas correspond to withdrawing a finite amount instantaneously.

Within the elliptical shaded area in the lower left corner, our numerical results suggest zero withdrawals as the optimal strategy. This is unexpected. As is conjectured in [6], based on financial reasoning and numerical tests, it is never optimal not to withdraw since the investor will lose the proportional insurance fee α. To study the control behavior within this region more carefully, we compute the ratio

$$R_{i,j} = \frac{V^h(W_i, A_j, \Delta\tau) - \left[V^h(W_i - G_r\Delta\tau, A_j - G_r\Delta\tau, \Delta\tau) + G_r\Delta\tau\right]}{G_r\Delta\tau}, \tag{8.1}$$

where $V^h(W_i, A_j, \Delta\tau)$ represents the approximate solution at the mesh node $(W, A, t) = (W_i, A_j, \Delta\tau)$ and $V^h(W_i - G_r\Delta\tau, A_j - G_r\Delta\tau, \Delta\tau)$ is the corresponding approximate contract value after a withdrawal of $G_r\Delta\tau$. According to the optimization problem (7.2), if $R_{i,j} > 0$, our numerical scheme chooses a zero control at (W_i, A_j). If $R_{i,j} < 0$, the scheme suggests that it is optimal to withdraw at the rate G_r. We observe that for nodes residing within the shaded elliptical region, the ratios

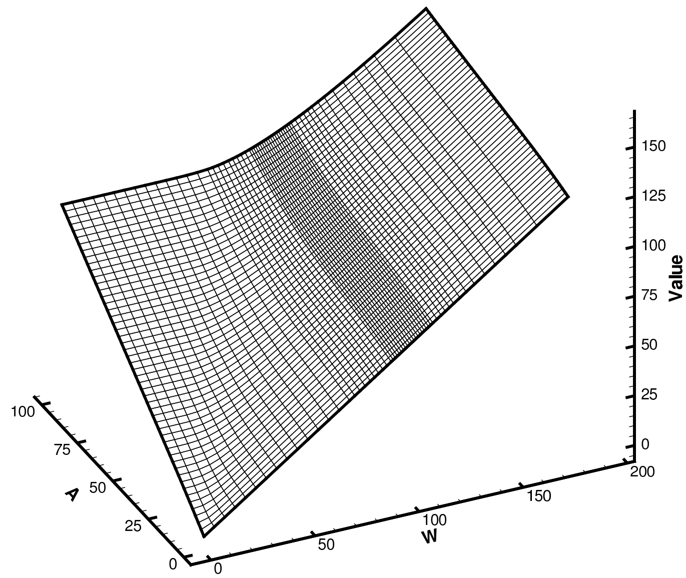

FIGURE 8.2: *The value of the GMWB guarantee at $t = 0$ as a function of sub-account balance W and guarantee account balance A. Data for this example are given in Table 8.1 with $\sigma = .30$ and the fair insurance fee $\alpha = .03126$. Continuous withdrawal is allowed.*

$R_{i,j}$ are positive but decrease towards zero quickly as we refine the mesh size (for example, the ratios are approximately 10^{-3} for the third refinement level). On the one hand, since the value of $|R_{i,j}|$ is insignificant, it is difficult for a numerical scheme to compute the sign of its exact value as $\Delta\tau \to 0$ due to numerical errors. As a result, $R_{i,j}$ may not have the same sign as its exact value and hence the zero withdrawal strategy returned by our scheme may not be correct. On the other hand, since the value of $|R_{i,j}|$ is very small, choosing $\hat{\gamma} = 0$ or $\hat{\gamma} = G_r$ will not affect the value of the guarantee. To verify this, we repeated the computation, but this time, we constrained the mesh nodes within the continuous withdrawal region to use the control value G_r, and disallowed zero as a possible control. The solution at $(W, A, t) = (100, 100, 0)$ resulting from this constraint is identical to the solution without imposing this constraint up to four digits.

To see this more clearly, assuming V^h is smooth and $\Delta\tau$ is sufficiently small, using Taylor series expansion leads to the approximation

$$R_{i,j} \approx V_W(W_i, A_j, \Delta\tau) + V_A(W_i, A_j, \Delta\tau) - 1. \tag{8.2}$$

Since we observe that $R_{i,j}$ converges to zero within this region, $V_W + V_A - 1$ also converges to zero in this region. According to the pricing equation (3.15), in this region, the equality

$$V_\tau - \mathcal{L}V - \sup_{\hat{\gamma} \in [0, G_r]} \left[\hat{\gamma}(1 - V_W - V_A)\right] = 0 \tag{8.3}$$

holds since the continuous withdrawal strategy is used. This implies that when $V_W + V_A - 1 \sim 0$, the optimal control $\hat{\gamma}$ can take any value between 0 and G_r, and hence is not unique. From the discussions above, our numerical results seem to suggest that $V_W + V_A - 1 = 0$ for mesh nodes within this region and thus the corresponding optimal control is indeterminate, that is, any $\hat{\gamma} \in [0, G_r]$ is optimal.

The region of withdrawing a finite amount in the upper left of Figure 8.3 is also observed in [6]. In this region, W is less than A before the withdrawal; after the withdrawal, W decreases to zero and the investor carries on withdrawing the remaining balance from the guarantee account at the rate G_r. The strategy can be explained as follows. In this region, the guarantee account balance of the contract dominates the sub-account balance. Hence it is highly probable that the guarantee account value still dominates the sub-account balance, i.e., $A \gg W$, at maturity, and in this case the investor receives $(1-\kappa)A-c$ as the final payoff. In other words, the equity component has a small chance of contributing to the final payoff, but instead requires insurance fee payments. Consequently, it is optimal for the investor to withdraw all the funds from the sub-account (even subject to a penalty).

In Figure 8.3, the upper right region represents withdrawing a finite amount when the sub-account value W dominates the guarantee account value A. In this case, a finite withdrawal is optimal in order to reduce the insurance fee payment, since the guarantee has little value. Note that after the withdrawal, the sub-account balance still dominates the guarantee account value and can contribute to the contract payoff.

In the blank region of Figure 8.3, it is optimal to withdrawal at the rate G_r because this avoids the excessive withdrawal penalty due to withdrawing a finite amount and also avoids the additional insurance fee payment due to zero withdrawal.

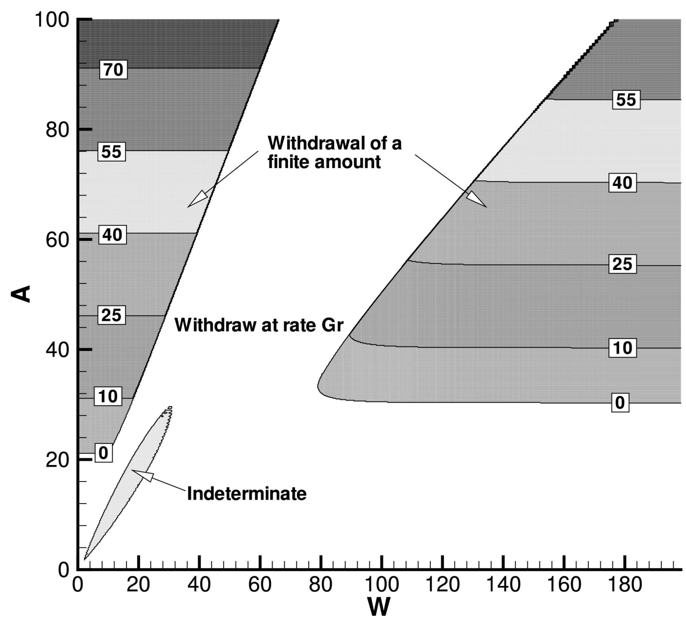

FIGURE 8.3: *The contour plot for the optimal withdrawal strategy of the GMWB guarantee at $t = \Delta\tau$ in the (W, A)-plane. In the regions of withdrawing finite amounts, contour lines representing the same withdrawal levels are shown, where the withdrawal amounts are posted on those contour lines. Data for this example are given in Table 8.1 with $\sigma = .30$ and the fair insurance fee $\alpha = .03216$. Continuous withdrawal is allowed. In the region labeled* indeterminate*, the numerical results indicate that the same value is obtained for any control rate in $[0, G_r]$.*

9 Conclusion

In this paper, we price a typical GMWB variable annuity contract assuming both a continuous withdrawal scenario and a discrete withdrawal scenario. We formulate the continuous withdrawal problem as an impulse stochastic control problem resulting in an HJB variational inequality by introducing a strictly positive fixed cost.

We develop a single numerical scheme for solving both the continuous and discrete withdrawal problems. For the discrete problem, we prove that the scheme converges to the unique viscosity solution of the problem. For the continuous case, provided a strong comparison result holds, we prove the scheme converges to the unique viscosity solution of the HJB variational inequality corresponding to the impulse control problem. The advantage of this approach is that we have a single scheme which can price realistic contracts, as well as the limit of allowing continuous withdrawals.

For the continuous withdrawal case, the numerical results suggest that our impulse control formulation converges to the singular control formulation in [6] as the fixed cost vanishes. The numerical results also demonstrate that our scheme can solve the impulse control problem with a nonzero fixed cost as well as the singular control problem by setting the fixed cost to be zero, although the convergence is proved only for the former case.

Our numerical results appear to show a region where the optimal control is indeterminate and not unique, that is, within the region, all continuous withdrawal strategies can produce the identical solution. The numerical experiments also reveal an optimal strategy of withdrawing a finite amount instantaneously when the equity component of the contract dominates the guarantee component or vice versa. Otherwise, it is optimal to withdraw at the maximum rate G_r.

In the future, we plan to price GMWB contracts with more complex features, such as incorporating various reset provisions. Since these contracts are long term, it is important to consider more realistic stochastic processes for the risky asset. In particular, these types of guarantees will be particularly valuable if the underlying follows a jump diffusion process.

A Proof for Lemma 4.2

The proof of Lemma 4.2 relies on the following Lemma.

Lemma A.1. *This Lemma provides some useful results. Given any $a, b \in \mathbb{R}$, it is straightforward to verify that*

$$|\max(a,0) - \max(b,0)| \leq |a-b|. \tag{A.1}$$

Suppose $X(x), Y(x)$ are functions defined for some bounded compact domain $x \in D$, then according to [9], we have

$$\left|\sup_{x\in D} X(x) - \sup_{y \in D} Y(y)\right| \leq \sup_{x \in D} |X(x) - Y(x)|. \tag{A.2}$$

After presenting Lemma A.1, in the following we prove Lemma 4.2. At first, from equation (4.3), we know $V(W, A, \tau_O^{k+})$ exists for all $(W, A) \in [0, W_{\max}] \times [0, w_0]$. To prove the uniform continuity of $V(W, A, \tau_O^{k+})$ on (W, A), by definition, we need to show that $\forall \epsilon > 0$, $\exists \sigma > 0$, such that $\forall (W', A'), (W'', A'') \in [0, W_{\max}] \times [0, w_0]$ satisfying $\sqrt{(W' - W'')^2 + (A' - A'')^2} < \sigma$, we have $|V(W', A', \tau_O^{k+}) - V(W'', A'', \tau_O^{k+})| < \epsilon$.

Let $Y(\gamma; W, A)$ be a function of $\gamma \in [0, A]$ defined as

$$Y(\gamma; W, A) = V(\max(W-\gamma, 0), A-\gamma, \tau_O^k) + f(\gamma), \tag{A.3}$$

where $f(\gamma)$ is given in (4.2). Without loss of generality, we assume $A' \geq A''$. We can write

$$\begin{aligned}
&|V(W', A', \tau_O^{k+}) - V(W'', A'', \tau_O^{k+})| \\
&\leq \left| V(W', A', \tau_O^{k+}) - \sup_{\gamma^k \in [0, A'']} \Big[V\big(\max(W'-\gamma^k, 0), A'-\gamma^k, \tau_O^k\big) + f(\gamma^k) \Big] \right| \\
&\quad + \left| \sup_{\gamma^k \in [0, A'']} \Big[V\big(\max(W'-\gamma^k, 0), A'-\gamma^k, \tau_O^k\big) + f(\gamma^k) \Big] - V(W'', A'', \tau_O^{k+}) \right| \\
&\leq \left[\sup_{\gamma^k \in [0, A']} \Big\{ Y(\gamma^k; W', A') \Big\} - \sup_{\gamma^k \in [0, A'']} \Big\{ Y(\gamma^k; W', A') \Big\} \right] \\
&\quad + \sup_{\gamma^k \in [0, A'']} \Big| V\big(\max(W'-\gamma^k, 0), A'-\gamma^k, \tau_O^k\big) - V\big(\max(W''-\gamma^k, 0), A''-\gamma^k, \tau_O^k\big) \Big|,
\end{aligned} \tag{A.4}$$

where the term inside the bracket of the last inequality above is due to the definition in (A.3)[2], and the last term in the last inequality above is due to (A.2). Next we will consider these two expressions individually.

Let us first consider the expression $\sup_{\gamma^k \in [0, A']} Y(\gamma^k; W', A') - \sup_{\gamma^k \in [0, A'']} Y(\gamma^k; W', A')$. We can write

$$\begin{aligned}
&\sup_{\gamma^k \in [0, A']} Y(\gamma^k; W', A') - \sup_{\gamma^k \in [0, A'']} Y(\gamma^k; W', A') \\
&= \max\left[\sup_{\gamma^k \in [0, A'']} Y(\gamma^k; W', A'), \sup_{\gamma^k \in (A'', A']} Y(\gamma^k; W', A') \right] - \sup_{\gamma^k \in [0, A'']} Y(\gamma^k; W', A') \\
&= \max\left[0, \sup_{\gamma^k \in (A'', A']} Y(\gamma^k; W', A') - \sup_{\gamma^k \in [0, A'']} Y(\gamma^k; W', A') \right]
\end{aligned} \tag{A.5}$$

Since $V(W, A, \tau_O^k)$ is uniformly continuous on (W, A) and since $f(\gamma^k)$ is uniformly continuous at $\gamma^k \in [0, G^k] \cup (G^k, A]$ and satisfies (5.9), we obtain

$$\lim_{\gamma^k \to [\gamma]^-} Y(\gamma^k; W', A') \geq \lim_{\gamma^k \to [\gamma]^+} Y(\gamma^k; W', A'), \quad \forall \gamma \in [0, A']. \tag{A.6}$$

According to (A.6), we have

$$\begin{aligned}
\lim_{\gamma^k \to [A'']^+} Y(\gamma^k; W', A') &\leq \lim_{\gamma^k \to [A'']^-} Y(\gamma^k; W', A') \\
&\leq \sup_{\gamma^k \in [0, A'']} Y(\gamma^k; W', A').
\end{aligned} \tag{A.7}$$

Since we have

$$\lim_{A' \to A''} \sup_{\gamma^k \in (A'', A']} Y(\gamma^k; W', A') = \lim_{\gamma^k \to [A'']^+} Y(\gamma^k; W', A'), \tag{A.8}$$

[2] This term is always positive since $[0, A''] \subseteq [0, A']$ and the functions in the two sup expressions are identical. Thus there is no need to take the absolute value for this term.

inequalities (A.7-A.8) imply that

$$\lim_{A'\to A''}\sup_{\gamma^k\in(A'',A']} Y(\gamma^k;W',A') \le \sup_{\gamma^k\in[0,A'']} Y(\gamma^k;W',A'). \tag{A.9}$$

This together with (A.5) shows that

$$\exists\sigma_0>0,\ \forall|A'-A''|<\sigma_0,\quad \sup_{\gamma^k\in[0,A']} Y(\gamma^k;W',A') - \sup_{\gamma^k\in[0,A'']} Y(\gamma^k;W',A') < \epsilon/2. \tag{A.10}$$

Let us now consider the last term in inequality (A.4). Since $V(W,A,\tau_O^k)$ is uniformly continuous on (W,A), then $\exists\sigma_1>0, \forall(W_1,A_1),(W_2,A_2)\in[0,W_{\max}]\times[0,w_0]$ satisfying $\sqrt{(W_1-W_2)^2+(A_1-A_2)^2}<\sigma_1$, we have

$$\left|V\big(W_1,A_1,\tau_O^k\big)-V\big(W_2,A_2,\tau_O^k\big)\right|<\epsilon/2. \tag{A.11}$$

Let

$$W_1=\max(W'-\gamma^k,0),\ W_2=\max(W''-\gamma^k,0),\ A_1=A'-\gamma^k,\ A_2=A''-\gamma^k. \tag{A.12}$$

Inequality (A.1) implies that $\sqrt{(W_1-W_2)^2+(A_1-A_2)^2}\le\sqrt{(W'-W'')^2+(A'-A'')^2}$. Consequently, if $(W',A'),(W'',A'')$ satisfy

$$\sqrt{(W'-W'')^2+(A'-A'')^2}<\sigma_1, \tag{A.13}$$

then (A.11-A.12) lead to

$$\left|V\big(\max(W'-\gamma^k,0),A'-\gamma^k,\tau_O^k\big)-V\big(\max(W''-\gamma^k,0),A''-\gamma^k,\tau_O^k\big)\right|<\epsilon/2. \tag{A.14}$$

As a result, $\forall(W',A'),(W'',A'')\in[0,W_{\max}]\times[0,w_0]$ satisfying $\sqrt{(W'-W'')^2+(A'-A'')^2}<\min(\sigma_0,\sigma_1)$, then according to (A.4), (A.10) and (A.13-A.14), we obtain

$$|V(W',A',\tau_O^{k+})-V(W'',A'',\tau_O^{k+})|<\epsilon. \tag{A.15}$$

This verifies the uniform continuity of function $V(W,A,\tau_O^{k+})$ on $(W,A)\in[0,W_{\max}]\times[0,w_0]$ by definition.

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