A Numerical Scheme for Guaranteed Minimum Withdrawal Benefit (GMWB) Variable Annuities

Peter Forsyth*, Zhuliang Chen†
*University of Waterloo, †Morgan Stanley

High Performance Workstations: Toronto August 4-6

Washington Post, May, 2004

"They have stumbled onto a killer app for the financial needs of today's boomers. It's called a GMWB. The deal is that for a half-percentage point per year, you can invest with a guarantee that your entire principal will be returned to you, provided you do not withdraw at a rate greater than 7% annually."

The Retirement Risk Zone

Consider an investor with a retirement account, which is invested in the stock market

Over the long run (before retirement), it does not matter if

- the market first drops by 10% per year over several years and then goes up by 20% per year for several years; or
- the market first goes up by 20% per year and then drops by 10% per year

$$(.9)(.9)...(1.2)(1.2)... = (1.2)(1.2)...(.9)(.9)...$$

The Retirement Risk Zone II

This is not the case once the investor retires, and begins to make withdrawals from the retirement account

The outcomes will be very different in the cases:

- in the first few years after retirement, the market has losses, and the account is further depleted by withdrawals, followed by some years of good market returns; compared to
- a few years of good market returns, after retirement (including withdrawals), followed by some years of losses

Losses in the early years of retirement can be devastating in the long run! Early bad returns can cause complete depletion of the account.

A Typical GMWB Example

Investor pays \$100 to an insurance company, which is invested in a risky asset.

Denote amount in risky asset sub-account by $W = 100$.

The investor also has a virtual guarantee account $A = 100$.

Suppose that the contract runs for 10 years, and the guaranteed withdrawal rate is \$10 per year.

A Typical GMWB Example II

At the end of each year, the investor can choose to withdraw up to \$10 from the account. If $\$\gamma \in [0, 10]$ is withdrawn, then

$$
\begin{aligned}
W_{new} &= \max(W_{old} - \gamma, 0) \quad ; \quad Actual\ investment \\
A_{new} &= A_{old} - \gamma \qquad\qquad\quad ; \quad Virtual\ account
\end{aligned}
$$

This continues for 10 years. At the end of 10 years, the investor can withdraw anything left, i.e. $\max(W_{new}, A_{new})$.

Note: the investor can continue to withdraw cash as long as $A > 0$, *even if* $W = 0$ (recall that W is invested in a risky asset).

Example: Order of Random Returns

Good Returns at Start

Time	Return (%)	Balance ($)	Withdrawal ($)
1	41.65	141.65	10
2	31.12	172.62	10
3	20.15	195.39	10
4	-30.25	129.31	10
5	18.05	140.85	10
6	16.82	152.86	10
7	20.12	171.60	10
8	7.44	173.62	10
9	-40.90	96.70	10
10	-7.5	80.20	10
Total Withdrawal Amount ($)			170.20
Ten year balance if no withdrawal ($)			151.37

Same Random Returns: Different Order

No GMWB

Time	Return (%)	Balance ($)	Withdrawal ($)
1	-30.25	69.75	10
2	-40.90	35.31	10
3	16.82	29.57	10
4	7.44	21.03	10
5	41.65	15.62	10
6	20.12	6.75	6.75
7	31.12	0	0
8	18.05	0	0
9	20.15	0	0
10	-7.5	0	0
Total Withdrawal Amount ($)			56.75
Ten year balance if no withdrawal ($)			151.37

Unlucky Order of Returns: With GMWB

GMWB Protection

Time	Return (%)	Balance ($)	Withdrawal ($)
1	-30.25	69.75	10
2	-40.90	35.31	10
3	16.82	29.57	10
4	7.44	21.03	10
5	41.65	15.62	10
6	20.12	6.75	10
7	31.12	0	10
8	18.05	0	10
9	20.15	0	10
10	-7.5	0	10
Total Withdrawal Amount ($)			100
Ten year balance if no withdrawal ($)			151.37

Why is this useful

The investor can participate in market gains, but still has a guaranteed cash flow, in the case of market losses

This insulates pensioners from losses in the early years of retirement.

This protection is paid for by deducting a yearly fee α from the amount in the risky account W each year.

In 2004, 70% of all variable annuities sold in the US included a GMWB option.

Some More Details

The investor can choose to withdraw up to the specified contract *rate* G_r without penalty.

Usually, a penalty $(\kappa > 0)$ is charged for withdrawals above G_r

Let $\hat{\gamma}$ be the rate of withdrawal selected by the holder.

Then, the rate of actual cash received by the holder of the GMWB is

$$\hat{f}(\hat{\gamma}) = \begin{cases} \hat{\gamma} & \text{if } 0 \le \hat{\gamma} \le G_r, \\ \hat{\gamma} - \kappa(\hat{\gamma} - G_r) & \text{if } \hat{\gamma} > G_r. \end{cases}$$

Stochastic Process

Let S denote the value of the risky asset, we assume that the risk neutral process followed by S is

$$
\begin{aligned}
dS &= rSdt + \sigma S dZ \\
&\quad r = \text{ risk free rate}; \quad \sigma = \text{ volatility} \\
&\quad dZ = \phi\sqrt{dt} \;\; ; \;\; \phi \sim \mathcal{N}(0,1)
\end{aligned}
$$

The risk neutral process followed by W is then (including withdrawals dA).

$$
\begin{aligned}
dW &= (r-\alpha)W dt + \sigma W dZ + dA, \quad \text{if } W > 0 \\
dW &= 0, \quad \text{if } W = 0 \\
&\quad \alpha = \text{ fee paid for guarantee} \;\; ; \;\; A = \text{ guarantee account}
\end{aligned}
$$

No-arbitrage Value

Let $V(W, A, \tau)$ ($\tau = T - t$, T is contract expiry) be the no-arbitrage value of the GMWB contract (i.e. the cost of hedging).

At contract expiry ($\tau = 0$) we have (payoff = withdrawal)

$$V(W, A, \tau = 0) \;=\; \max(W, A(1 - \kappa))$$

It turns out that it is optimal to withdraw at a rate $\hat{\gamma}$

- $\hat{\gamma} \in [0, G_r]$, or
- $\hat{\gamma} = \infty$ (instantaneously withdraw a finite amount)

Impulse Control

Let

$$\mathcal{L}V = \frac{1}{2}\sigma^2 W^2 V_{WW} + (r - \alpha) W V_W - rV.$$

Since we have the option of withdrawing at a finite rate at each point in (W, A, τ), Ito's Lemma and no-arbitrage arguments give

$$V_\tau - \mathcal{L}V - \max_{\hat{\gamma} \in [0, G_r]} \left(\hat{\gamma} - \hat{\gamma} V_W - \hat{\gamma} V_A \right) \geq 0$$

Note that $\hat{\gamma}$ is a finite withdrawal *rate*. Withdrawals only allowed if $A > 0$.

Impulse Control II

We also have the option of withdrawing a finite amount instantaneously (withdrawing at an infinite rate) at each point in (W, A, τ)

$$V(W, A, \tau) - \sup_{\gamma \in (0, A]} \left[V(\max(W - \gamma, 0), A - \gamma, \tau) + (1 - \kappa)\gamma - c \right] \geq 0 \ .$$

where γ is a finite withdrawal *amount*. Note that $c > 0$ is a fixed cost (which can be very small).

Note that this equation specifies that any amount in the remaining guarantee account can be withdrawn instantaneously (i.e. $\gamma \in (0, A]$) with a penalty.

HJB Variational Inequality

Since it must be optimal to either withdraw at a finite rate or withdraw a finite amount at each point, then this can all be written compactly as a Hamilton Jacobi Bellman Variational Inequality

$$
\begin{aligned}
&\min\Bigg\{ V_\tau - \mathcal{L}V - \max_{\hat{\gamma}\in[0,G_r]} \big(\hat{\gamma} - \hat{\gamma}V_W - \hat{\gamma}V_A\big), \\
&V - \sup_{\gamma\in(0,A]} \big[V(\max(W-\gamma,0), A-\gamma, \tau) + (1-\kappa)\gamma - c\big] \Bigg\} \\
&= 0
\end{aligned}
$$

Alternative Approach: Discrete Withdrawal Times

Rather than attempt to solve the HJB Impulse Control problem directly, let's replace this problem by a *discrete withdrawal* problem

- Assume that the holder can only withdraw at discrete *withdrawal* times $\tau_1, ..., \tau_N$, with $\tau_{i+1} - \tau_i = \Delta t_w$
- Use dynamic programming idea, work backwards from $t = T(\tau = 0)$, so that $V(W, A, 0) = \max(W, A(1 - \kappa))$
- During the interval from $\tau = 0$ to $\tau = \tau_1$ (the first withdrawal time going backwards) we solve

$$V_\tau - \mathcal{L}V = 0 \ \ ; \ \ \mathcal{L}V = \frac{1}{2}\sigma^2 W^2 V_{WW} + (r - \alpha)WV_W - rV.$$

Optimum Strategy: Discrete Withdrawals

At τ_1, we assume that the holder withdraws the optimum *amount* γ

$$V(W, A, \tau_1^+) = \max_{\gamma \in [0,A]} \left[V\big(\max(W - \gamma, 0), A - \gamma, \tau_1\big) + f(\gamma) \right],$$

where now the cash flow term is

$$f(\gamma) = \begin{cases} \gamma & \text{if } 0 \leq \gamma \leq G, \\ \gamma - \kappa(\gamma - G) - c & \text{if } \gamma > G. \end{cases}$$

$$G = G_r \Delta t_w$$

Discrete Withdrawals

Then, from τ_1^+ to τ_2, we solve

$$V_\tau - \mathcal{L}V = 0 \;\; ; \quad \text{No A dependence in } \mathcal{L}V$$

Then, we determine the optimum withdrawal at τ_2^+, and so on, back down to $\tau = T(t = 0)$ today.

This would appear to be a reasonable approximation to reality.

In fact, most real contracts allow only discrete withdrawals.

Discrete Withdrawal: A Numerical Scheme

Define nodes in the W direction $[W_0, W_1, \ldots, W_{i_{\max}}]$, and in the A direction $[A_0, A_1, \ldots, A_{j_{\max}}]$.

Let $V(W_i, A_j, \tau^n) = V_{i,j}^n$.

Let $(\mathcal{L}_h V)_{i,j}^n$ be a discrete form of the operator $\mathcal{L}V$.

Away from withdrawal times, we solve

$$\frac{V_{i,j}^{n+1} - V_{i,j}^n}{\Delta\tau} \;=\; (\mathcal{L}_h V)_{i,j}^{n+1}$$

A Numerical Scheme II

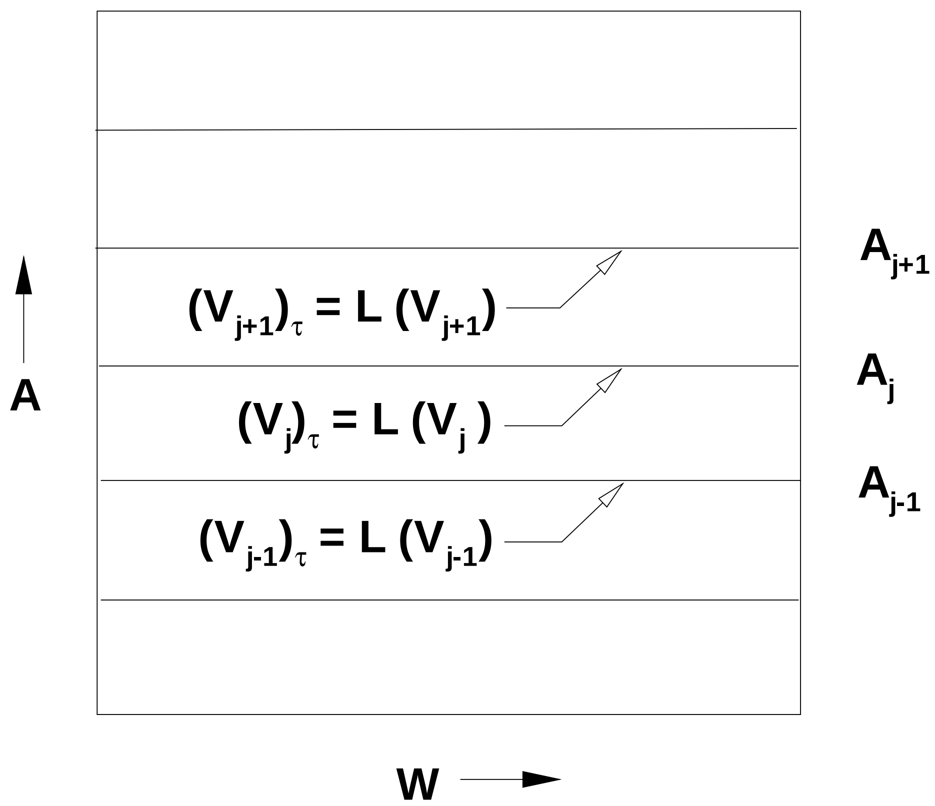
At withdrawal time τ_n, we then solve the optimization problem

$$V_{i,j}^{n+} = \max_{\gamma_{i,j}^n \in [0, A_j]} \left[\mathcal{I}_{i,j}(\gamma_{i,j}^n) V^n + f\left(\gamma_{i,j}^n\right) \right],$$

where $\mathcal{I}$ is a linear interpolation operator

- we use a linear interpolant of $V_{i,j}^n$ to determine the optimum withdrawal at each node $\gamma_{i,j}^n$.

Numerical Scheme III



Away from withdrawal times, We solve a decoupled set of 1-d PDEs.

At withdrawal times, we solve a set of decoupled optimization problems.

Computational Bottleneck

Vast majority of CPU time spent solving the local optimization problem at each node:

$$V_{i,j}^{n+} = \max_{\gamma_{i,j}^{n} \in [0, A_j]} \left[\mathcal{I}_{i,j}(\gamma_{i,j}^{n}) V^{n} + f\left(\gamma_{i,j}^{n}\right) \right] ,$$

- This is *embarrassingly parallel*, but requires access to global data

Obvious Question

If we let $\Delta\tau_w \rightarrow 0$, does this discrete withdrawal approximation converge to the solution of the Impulse Control HJB equation?

Solutions of HJB equations not smooth in general.

What does it mean to solve a differential equation where the *solution* is not differentiable?

We need to look for the *viscosity* solution of the Impulse Control HJB equation.

Viscosity Solution

Briefly, a viscosity solution is defined in terms of smooth test functions.

These smooth test functions touch the *viscosity* solution at a single point, and are always *above* or *below* the solution elsewhere.

The viscosity solution is *squeezed* between these nearby test functions.

The viscosity solution does not necessarily satisfy the PDE in any conventional sense.

Basic Result

Theorem 1 (Convergence to the Viscosity Solution (Barles, Souganidis (1993))). *Any numerical scheme which is consistent, l_∞ stable, and monotone, converges to the viscosity solution.*

Consistent Discrete scheme applied to smooth test function satisfies a $\limsup$, $\liminf$ condition, as mesh, timestep $\rightarrow 0$ (smooth test functions squeeze the solution)

Stability (l_∞) Solution bounded in l_∞ as mesh, timestep $\rightarrow 0$.

Monotonicity: What does it mean?

Let $V_{i,j}^n$, $Q_{i,j}^n$ be two discrete solutions to the same HJB equation.

Lemma 1 (Discrete Arbitrage Inequality)**.** *If* $V_{i,j}^n, Q_{i,j}^n$ *are generated using a monotone scheme, and* $\forall i, j$, $Q_{i,j}^0 \geq V_{i,j}^0$, *then*

$$Q_{i,j}^n \geq V_{i,j}^n \quad ; \quad \forall i, j; \forall n$$

In other words, if the payoff $Q(W, A, 0) \geq V(W, A, 0)$, then this inequality must hold at all earlier times, for the discrete solution, regardless of the timestep or meshsize.

Convergence: Basic Idea

If we allow discrete withdrawals every timestep, then our numerical method is

$$V_{i,j}^{n+1} - \max_{\gamma_{i,j}^n \in [0, A_j]} \left[\mathcal{I}_{i,j}(\gamma_{i,j}^n) V^n + f\left(\gamma_{i,j}^n\right) \right] - \Delta\tau \left(\mathcal{L}_h V \right)_{i,j}^{n+1} = 0 \ .$$

where the cash flow term $f\left(\gamma_{i,j}^n\right)$ is

$$\begin{aligned} f(\gamma) \qquad &= \begin{cases} \gamma & \text{if } 0 \leq \gamma \leq G, \\ \gamma - \kappa(\gamma - G) - c & \text{if } \gamma > G. \end{cases} \\ G &= G_r \Delta\tau \end{aligned}$$

and $\mathcal{I}$ is a linear interpolation operator

Does it Converge?

We want to show that this scheme converges as $\Delta\tau$, ΔA, ΔW $\longrightarrow 0$ to the viscosity solution of

$$\min\Big\{ V_\tau - \mathcal{L}V - \max_{\hat\gamma \in [0, G_r]} \big(\hat\gamma - \hat\gamma V_W - \hat\gamma V_A\big),$$

$$V - \sup_{\gamma \in (0, A]} \Big[V(\max(W - \gamma, 0), A - \gamma, \tau) + (1 - \kappa)\gamma - c \Big] \Big\}$$

$$= 0$$

This seems intuitively obvious, but it is not so easy!

Convergence Proof

Let $(\mathcal{L}_h V^n)_{i,j}$ denote the discrete form of the differential operator $\mathcal{L}V$ at node (W_i, A_j, τ^n). Use central, forward, backward differencing to obtain

$$
\begin{aligned}
(\mathcal{L}_h V^{n+1})_{i,j} &= \alpha_i^{n+1} V_{i-1,j}^{n+1} + \beta_i^{n+1} V_{i+1,j}^{n+1} \\
&\quad -(\alpha_i^{n+1} + \beta_i^{n+1} + d_i^{n+1}) V_{i,j}^{n+1} \qquad (1)
\end{aligned}
$$

Definition 1 (Positive Coefficient Scheme). *Scheme (1) is a positive coefficient method if*

$$\alpha_i^{n+1} \geq 0 \ ; \ \beta_i^{n+1} \geq 0 \ ; \ d_i^{n+1} \geq 0$$

Monotonicity, Stability and Consistency

Lemma 2 (Monotonicity and Stability). *Provided* $(\mathcal{L}_h V^{n+1})$ *is discretized using a positive coefficient method and linear interpolation is used when solving the local optimization problem at each node, then the scheme is unconditionally* l_∞ *stable and monotone.*

Proof. Straightforward $\square$

Lemma 3 (Consistency). *Provided the discrete operator* $(\mathcal{L}_h V^{n+1})$ *is consistent in the classical sense, and linear interpolation is used to solve the local optimization problem at each node, then the numerical scheme is consistent as defined in (Barles, Souganidis (1993)).*

Proof. Not so straightforward ($\liminf, \limsup$ for boundary conditions) $\square$

Convergence

Theorem 2 (Strong Comparison Result)**.** *The GMWB Impulse Control problem satisfies the Strong Comparison Result, i.e. there is a unique, continuous viscosity solution to the Impulse Control Problem. (Seydel, 2008)*

Theorem 3 (Convergence to the Viscosity Solution)**.** *The discrete withdrawal numerical method, with withdrawal interval* $\Delta t_w \rightarrow 0$ *converges to the unique viscosity solution of the Impulse Control problem.*

Proof. This scheme is consistent, stable, and monotone, hence converges to the viscosity solution (Barles, Souganidis (1993)). □

Convergence

So, we now have a single scheme which

- Can be used to price GMWB contracts with finite withdrawal intervals (the usual case in real contracts, i.e. withdrawals only allowed once or twice a year)
- We can also price GMWB contracts in the limit as the withdrawal interval $\rightarrow 0$

  $\rightarrow$ Convergence to the Impulse Control problem guaranteed
- No need for different method for these two cases!

Examples

Recall that the investor pays no extra up-front fee for the guarantee (only the initial premium w_0).

The insurance company deducts an annual fee α from the balance in the sub-account W.

Problem: let $V(\alpha, W, A, \tau)$ be the value of the GMWB contract, for given yearly guarantee fee α.

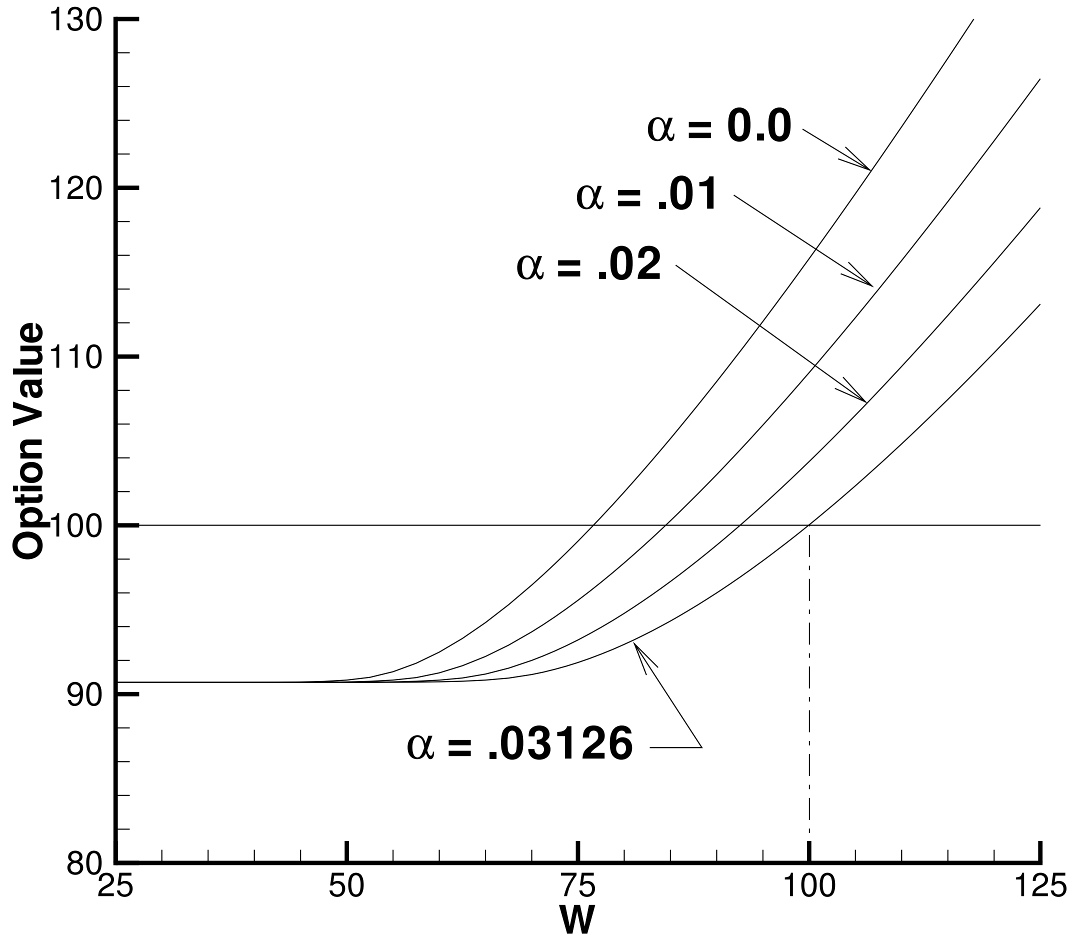
Assume that the investor pays an initial premium w_0 at $t = 0$ $(\tau = T)$.

Find the no-arbitrage fee α such that $V(\alpha, w_0, w_0, T) = w_0$ (we do this by a Newton iteration).

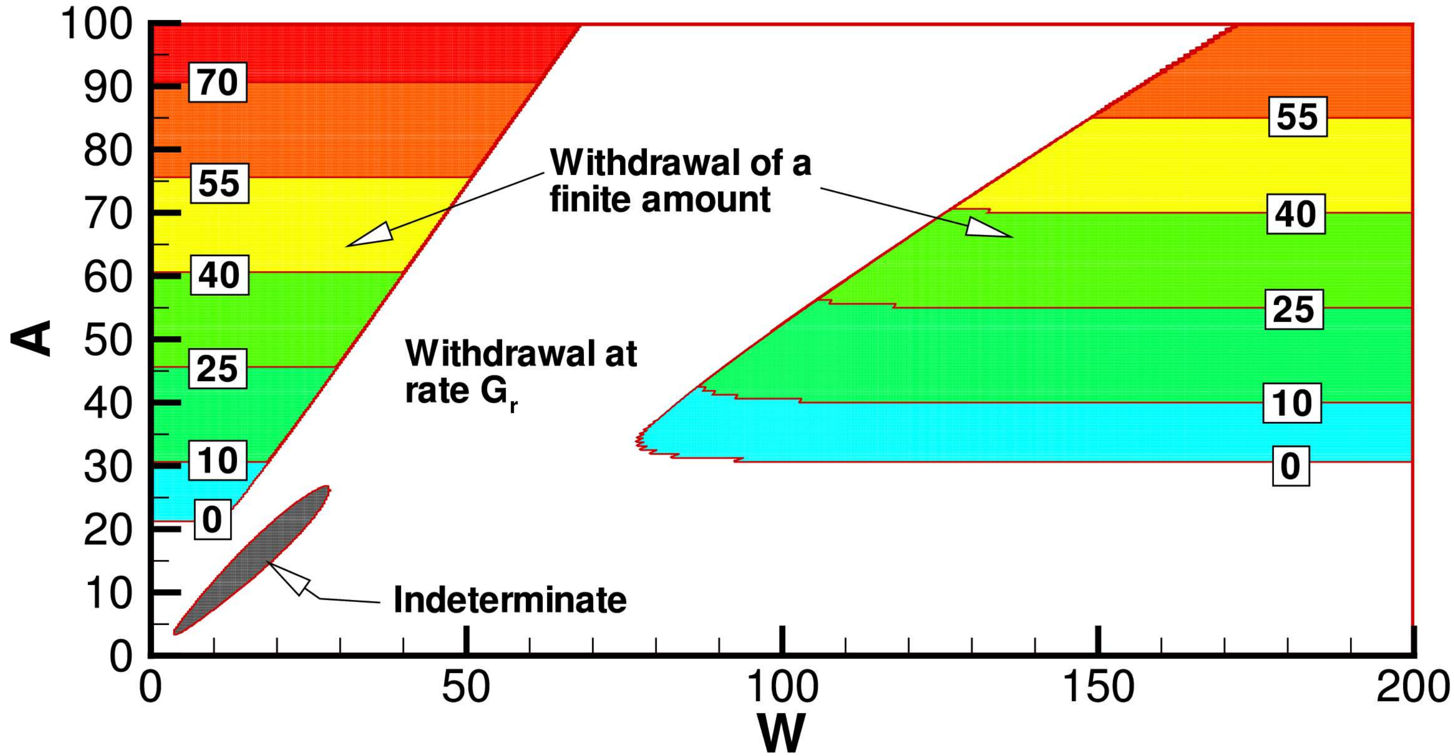
Examples

Parameter	Value
Expiry time T	10.0 years
Interest rate r	.05
Maximum withdrawal rate G_r	10/year
Withdrawal penalty κ	.10
Volatility σ	.30
Initial Lump-sum premium w_0	100
Initial guarantee account balance	100
Initial sub-account value	100
Continuous Withdrawal	Yes

The No-arbitrage Fee ($t = 0$, $A = 100$)



Optimal Withdrawal Strategy



$t = 0$, fair fee charged for $w_0 = 100$. Indeterminate region: appears to converge to optimal withdrawal rate $\hat{\gamma} = 0$?

Indeterminate Region?

In in the continuous withdrawal region, we have

$$V_\tau = \mathcal{L}V + \max_{\hat{\gamma} \in [0, G_r]} \Big[\hat{\gamma}(1 - V_W - V_A) \Big] \tag{2}$$

In the *indeterminate* region, we observe that (mesh, timestep $\rightarrow 0$)

$$\Bigg[1 - V_W(W_i, A_j, \Delta\tau) - V_A(W_i, A_j, \Delta\tau) \Bigg]_{i,j} \rightarrow 0^-$$

If $(1 - V_W - V_A) \rightarrow 0$, then any withdrawal rate in $[0, G_r]$ is optimal.

Control may not be unique (but value is unique).

No-arbitrage Fee

- $\sigma = .15 \rightarrow \alpha = .007$ (70 bps)
- $\sigma = .20 \rightarrow \alpha = .014$ (140 bps)
- $\sigma = .30 \rightarrow \alpha = .031$ (310 bps)
- Current volatility of $S\&P \simeq .25$
- Typical fees charged: $\alpha = .005$ (50 bps) too low for current market conditions.
- Insurance companies seem to be charging fees based on marketing considerations, not hedging costs.
- Fee should be even higher if other (typical) contract options considered

Conclusions

- We have developed a single scheme which can be used to price GMWB contracts with finite withdrawal intervals, and the limiting case of infinitesimal withdrawal intervals
- In case of infinitesimal withdrawal intervals, we have proven convergence to the viscosity solution of the Impulse Control problem
- Insurance companies seem to be charging fees which are too low to cover hedging costs. Another subprime problem?
- Costly part of algorithm is *embarrassingly parallel*, but requires access to global data.

  → Typical of optimal stochastic control problems

  → Can this be implemented efficiently on multi-core machines?