Numerical Solution of the Hamilton-Jacobi-Bellman Formulation for Continuous Time Mean Variance Asset Allocation

Peter Forsyth[1] Jian Wang[2]

[1]Cheriton School of Computer Science
University of Waterloo

[2]Aon Benfield, Toronto

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Continuous Time Mean Variance Asset Allocation

Suppose an investor saves for retirement by contributing to a pension account at a rate π per year.

She can divide her wealth W in the pension account into

- A fraction p invested in a risky asset S which follows

$$
\begin{aligned}
dS &= (r + \xi\sigma) S \; dt + \sigma S \; dZ_1 \\
& \quad dZ_1 = \text{ increment of a Wiener process} \\
& \quad \xi = \text{ the market price of risk} \\
& \quad \sigma = \text{ volatility}
\end{aligned}
$$

- A fraction $(1 - p)$ in a riskless asset B which follows

$$
\frac{dB}{dt} = rB
$$

The process followed by $W = B + S$ is

$$
dW = (r + p\xi\sigma) W \; dt + \pi \; dt + p\sigma W \; dZ_1 .
$$

Optimal Strategy

Define

$$
\begin{aligned}
p(W,t) &= \text{dynamic fraction invested in the risky asset} \\
W_T &= \text{terminal wealth}
\end{aligned}
$$

Let

$$
\begin{aligned}
E_{t,w}^{p(\cdot)}[\cdot] &= E[\cdot | W(t) = w] \text{ with } \; p(s, W(s)), s \geq t \\
&\quad \text{being the strategy along path } \; W(s), s \geq t \\
\mathrm{Var}_{t,w}^{p(\cdot)}[\cdot] &= \mathrm{Var}[\cdot | W(t) = w] \text{ Variance under strategy } p(\cdot) \\
&\quad \text{along path } \; W(s), s \geq t
\end{aligned}
$$

So that

$$
\mathrm{Var}_{t,w}^{p(\cdot)}[W_T] = E_{t,w}^{p(\cdot)}[(W_T)^2] - \left(E_{t,w}^{p(\cdot)}[W_T] \right)^2
$$

Minimum Variance: Standard Formulation

The objective is to determine the strategy $p(\cdot)$ such that

$$\begin{aligned} J(w,t) &= \sup_{p(s \geq t, W(s))} \left\{ E_{t,w}^{p}[W_T] - \lambda Var_{t,w}^{p}[W_T] \right\} , \\ & \lambda = \text{ Lagrange multiplier} \end{aligned} \tag{1}$$

Solving (1) for various λ traces out a curve in the expected value, standard deviation plane.

• Let $p_t^*(s,w), s \geq t$ be the optimal policy for (1).

Then $p_{t+\Delta t}^*(s,w), s \geq t + \Delta t$ is the optimal policy for

$$J(W(t+\Delta t), t+\Delta t) = \sup_{p(s \geq t+\Delta t, W(s))} \left\{ E_{t+\Delta t, W(t+\Delta t)}^{p}[W_T] - \lambda Var_{t+\Delta t, W(t+\Delta t)}^{p}[W_T] \right\} .$$

Pre-Commitment Policy

However, in general

$$p_t^*(s, W(s)) \neq p_{t+\Delta t}^*(s, W(s)) \ ; \ s \geq t + \Delta t \ , \qquad (2)$$

$\hookrightarrow$ Optimal policy is not *time-consistent*.
The strategy which solves problem (1) has been called the *pre-commitment* policy (Basak,Chabakauri: 2010; Bjork et al: 2010)

- Much discussion on the economic meaning of such strategies.
- Possible to formulate a time-consistent version of mean-variance.
- Or other strategies: mean quadratic variation
- Different applications may require different strategies.
- We focus on pre-commitment solution today, with a brief discussion of alternative strategies

Pre-Commitment

Problem:

- Since the pre-commitment strategy is not time consistent, there is no natural dynamic programming principle
- We would like to formulate this problem as the solution of an HJB equation.
- How are we going to do this?

Solution:

- Go back to first principles

Minimum Variance: Basic Principle

Equivalent formulation: determine the strategy $p(\cdot)$ such that

$$\min Var^{p(\cdot)}_{t=0,w}[W_T] = E^{p(\cdot)}_{t=0,w}[(W_T)^2] - d^2$$
$$\text{subject to } \begin{cases} E^{p(\cdot)}_{t=0,w}[W_T] = d \\ p(\cdot) \in \mathbb{P} \end{cases}$$
$$\mathbb{P} = \text{ set of admissible controls}$$

Given an expected return $d = E^{p(\cdot)}_{t=0,w}[W_t]$, strategy $p(\cdot)$ produces the smallest possible variance.

Varying the parameter d traces out a curve in the expected value - standard deviation plane.

Eliminate Constraint

Original problem is convex optimization, use Lagrange multiplier γ to eliminate constraint.

$$\max_{\gamma} \min_{p(\cdot)\in\mathbb{P}} E^{p(\cdot)}_{t=0,w}\left[(W_T)^2 - d^2 - \gamma(E^{p(\cdot)}_{t=0,w}[W_T] - d)\right] . \qquad (3)$$

Suppose somehow we know the γ which solves (3), for fixed d.

Then the optimal strategy $p^*(\cdot)$ which solves (3) is given by (for fixed γ)

$$\min_{p(\cdot)\in\mathbb{P}} E^{p(\cdot)}_{t=0,w}\left[\left(W_T - \frac{\gamma(t,w,d)}{2}\right)^2\right] . \qquad (4)$$

Note we have effectively replaced parameter d by γ in (4).

Construction of Efficient Frontier

We can alternatively regard γ as a parameter, and determine the optimal strategy $p^*(\cdot)$ which solves

$$\min_{p(\cdot)\in\mathbb{P}} E^{p(\cdot)}_{t=0,w}\left[(W_T - \frac{\gamma}{2})^2\right] . \tag{5}$$

Once $p^*(\cdot)$ is known, we can easily determine $E^{p^*(\cdot)}_{t=0,w}[W_T]$, $E^{p^*(\cdot)}_{t=0,w}[(W_T)^2]$, by solving an additional linear PDE.

For given γ, this gives us $(E^{p^*(\cdot)}_{t=0,w}[W_T], \mathrm{Std}^{p^*(\cdot)}_{t=0,w}[W_T])$, a single point on the efficient frontier.

Repeating the above for different γ generates points on the efficient frontier.
$\hookrightarrow$ Efficient frontier construction reduces to repeated solves of (5).

HJB Equation: Wealth Case

Let $V(w,\tau) = E_{t=T-\tau,w}[(W_T - \gamma/2)^2]$, then standard dynamic programming can be used to determine the HJB equation satisfied by $V(w,\tau)$.

$p^*(\cdot)$ is determined from solution of HJB equation

$$V_\tau = \inf_{p \in \mathbb{P}} \left\{ \mu_w^p V_w + \frac{(\sigma_w^p)^2}{2} w^2 V_{ww} \right\}$$

$$\mu_w^p = \pi + w(r + p\sigma\xi)$$

$$\sigma_w^p = p\sigma w$$

$$V(w, \tau = 0) = (w - \frac{\gamma}{2})^2$$

Wealth to Income Ratio Case

As pointed out in (Cairns et al, 2006) it is perhaps more relevant to consider optimizing the wealth to income ratio at retirement.
Let the risky asset follow (as before)

$$dS = (r + \xi\sigma)S\ dt + \sigma S\ dZ_1$$

The investor's yearly stochastic salary Y follows

$$dY = (r + \mu_Y)Y\ dt + \sigma_{Y_0} Y\ dZ_0 + \sigma_{Y_1} Y\ dZ_1$$
$$dZ_0 \text{ is independent of } dZ_1$$

The investor contributes into the pension plan at a rate of πY.
The investor allocates a fraction p to the risky asset and $(1 - p)$ to the riskless asset.
The total wealth W then follows

$$dW = (r + p\xi\sigma)W\ dt + p\sigma W dZ_1 + \pi Y dt$$

Wealth to Income Ratio: II

Define new state variable $X(t) = W(t)/Y(t)$, then by Ito's Lemma, we obtain

$$\begin{aligned} dX &= [\pi + X(-\mu_Y + p\sigma(\xi - \sigma_{Y_1}) + \sigma_{Y_0}^2 + \sigma_{Y_1}^2)]dt \\ &\quad -\sigma_{Y_0} X dZ_0 + X(p\sigma - \sigma_{Y_1})dZ_1 \end{aligned}$$

Now, we seek to find the control $p(\cdot)$ which solves

$$\min Var_{t=0,X(t)=x}^{p(\cdot)}[X_T] = E_{t=0,X(t)=x}^{p(\cdot)}[(X_T)^2] - d^2$$
$$\text{subject to } \begin{cases} E_{t=0,X(t)=x}^{p(\cdot)}[W_T] = d \\ p(\cdot) \in \mathbb{P} \end{cases} \tag{6}$$

Wealth to Income Ratio: III

Following same steps as before, the optimal control which solves the mean variance problem is also the optimal control which solves

$$\min_{p(\cdot)\in\mathbb{P}} E^{p(\cdot)}_{t=0,x}\left[(X_T - \frac{\gamma}{2})^2\right] . \tag{7}$$

Let $V(x,\tau) = E_{\tau=T-t,x}[(X_T - \gamma/2)^2]$, then usual steps determine the HJB equation for $V(x,\tau)$

$$V_\tau = \inf_{p\in\mathbb{P}}\left\{\mu_x^p V_x + \frac{1}{2}(\sigma_x^p)^2 V_{xx}\right\}$$

$$\mu_x^p = \pi + x(-\mu_Y + p\sigma(\xi_1 - \sigma_{Y_1}) + \sigma_{Y_0}^2 + \sigma_{Y_1}^2)$$

$$(\sigma_x^p)^2 = x^2(\sigma_{Y_0}^2 + (p\sigma - \sigma_{Y_1})^2) \tag{8}$$

$$V(x,\tau=0) = (x - \frac{\gamma}{2})^2 \tag{9}$$

General Form

Both these problems have the same mathematical form.

Let

$$z = \begin{cases} w & \text{wealth case} \\ x & \text{wealth to income ratio case} \end{cases} \qquad (10)$$

Then, we can write both problems in the general form

$$\begin{aligned} V_\tau &= \inf_{p \in \mathbb{P}} \{\mathcal{L}^p V\} \\ &\mathcal{L}^p V \equiv a(z,p) V_{zz} + b(z,p) V_z \end{aligned} \qquad (11)$$

Admissible Controls

Recall that $\mathbb{P}$ is the set of admissible controls.
Let $\mathbb{D}$ be the set of possible values of z (wealth or wealth/income)
We consider the following cases:

Case	$\mathbb{D}$	$\mathbb{P}$
Allow Bankruptcy	$[-\infty, +\infty]$	$(-\infty, +\infty)$
No Bankruptcy	$[0, \infty]$	$[0, +\infty)$
Bounded Control	$[0, \infty]$	$[0, p_{\max}]$

• Bankruptcy: trading continues even if insolvent (not realistic, but analytic solutions available in some cases)

• No bankruptcy: also not shorting stock

• Bounded control: realistic case, i.e. $p_{\max} = 1.5$ corresponds to maximum borrowing of 50% of wealth

Boundary Conditions

Localize domain $z \in [-\infty, +\infty]$ to $z \in [z_{\text{min}}, z_{\text{max}}]$.
Assume $V(z, \tau) \simeq A(\tau) z^2$, $z \pm \infty$.
• Substitute asymptotic form into HJB PDE, solve for a Dirichlet condition at $z = z_{\text{min}}, z_{\text{max}}$.

In the No Bankruptcy/Bounded Control case, we need to apply a boundary condition at $z = 0$.
• Assume $pz \to 0$ as $z \to 0$, $\Rightarrow Z(t) \geq 0$
$\hookrightarrow$ Then take limit of PDE as $z \to 0$

$$V_\tau = \pi V_z \quad ; \quad z = 0$$

Numerical scheme will use boundary condition if required, or ignore it if it is not needed (Ekstrom et al, 2009).

Admissible Controls: Allow Bankruptcy

Consider the SDE for the Wealth case:

$$dW = (r + p\xi\sigma)W\ dt + \pi\ dt + p\sigma W\ dZ_1.$$

If p is bounded as $W \to 0$, then SDE becomes

$$dW \simeq \pi\ dt \quad ; \quad W \to 0$$

$\Rightarrow$ W cannot be become negative

But we allow bankruptcy ($W < 0$) in this case

Conclusion: In the Allow Bankruptcy case, p becomes unbounded as $w, x \to 0$.

Positive Coefficient Discretization

- Define a set of nodes (z_i, τ^n), discrete solution $V^n = [V_1^n, V_2^n, \ldots]'$, controls $P^n = [p_1^n, p_2^n, \ldots]'$
- Discretize the PDE, fully implicit timestepping, central, forward and backward differencing in z direction
  - Central differencing as much as possible (Wang, Forsyth: 2008)
- Differencing may depend on the control p_i^n at node (z_i, τ^n)

$$
\begin{aligned}
\frac{V_i^{n+1} - V_i^n}{\Delta \tau} &= \inf_{p_i^{n+1} \in \mathbb{P}} \left\{ [A^{n+1}(P^{n+1}) V^{n+1}]_i \right\} \\
\left[A^{n+1}(P^{n+1}) V^{n+1}\right]_i &= \alpha_i^{n+1} V_{i-1}^{n+1} + \beta_i^{n+1} V_{i+1}^{n+1} - (\alpha_i^{n+1} + \beta_i^{n+1}) V_i^{n+1}
\end{aligned}
$$

Force positive coefficient condition

$$
\alpha_i^{n+1} \geq 0 \; ; \; \beta_i^{n+1} \geq 0
$$

Convergence to the Viscosity Solution

Theorem (Strong Comparison)

If the control is bounded, then the HJB PDE satisfies the strong comparison property (i.e. a unique, continuous viscosity solution exists) (see Chaumont (2004)).

But what about the No Bankruptcy case? The control is unbounded as $z \to 0$.

- We can formally get around this problem by rewriting the PDE in terms of a control $q = pz$, which always remains bounded (on a finite grid).
- Same idea used in (Bielecki et al, 2005).
- We are primarily interested in developing numerical techniques for bounded control case (no analytic solutions, practical case)

$\rightarrow$ We will use p as the control, more natural for this case.

Convergence to the Viscosity Solution: Bounded Control

Theorem

Provided that (a) the HJB PDE satisfies a strong comparison result (b) fully implicit timestepping is used with a positive coefficient discretization, then the discrete scheme converges to the unique, continuous viscosity solution of the control problem.

Proof.

The scheme is

- Unconditionally monotone
- Unconditionally stable
- Consistent

Hence convergence follows from the results in (Barles, Souganidis (1991)). □

Solution of Discretized Equations at Each Timestep: Policy Iteration

Each timestep requires solution of the nonlinear set of algebraic equations

$$\max_{P \in \mathbb{P}} \left\{ \left[I - \Delta\tau A^{n+1}(P^{n+1}) \right] V^{n+1} - V^n \right\} = 0 \qquad (12)$$

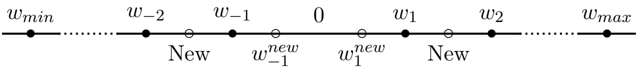
We solve this set of equations using Policy iteration

- If a positive coefficient discretization is used, Policy iteration always converges (Forsyth and Labahn (2008)).

Bankruptcy Allowed: Wealth Case, $w = 0$ problem

$w = 0$ in PDE $\rightarrow V_\tau = \pi V_w \rightarrow$ No Bankruptcy boundary condition.
$\hookrightarrow$ Not what we want.

Construct initial grid, no node at $w = 0$.



As we refine the grid

- Insert new node between each pair of coarse grid nodes, except at $w = 0$
- Add new nodes between $[w_{-1}, 0]$ and $[0, w_1]$
- $w \rightarrow 0$, $p \rightarrow \infty$, pw finite, $w \neq 0$

Numerical Example: Wealth Case (Unbounded Control)

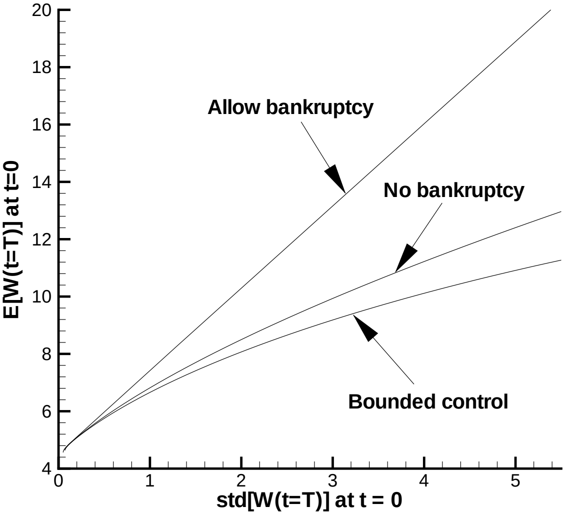
r	0.03	ξ	0.33
σ	0.15	π	0.1
T	20 years	$W(t=0)$	1

Nodes (W)	Time Steps	$Std_{p^*}^{t=0}[W_T]$	$E_{p^*}^{t=0}[W_T]$	Ratio $Std_{p^*}^{t=0}[\cdot]$	Ratio $E[\cdot]$
728	160	0.915441	6.92426		
1456	320	0.872917	6.93442		
2912	640	0.851483	6.93992	1.975	1.847
5824	1280	0.840821	6.94251	2.007	2.124
11648	2560	0.835612	6.94383	2.045	1.962

- Analytic solution: $(Std_{p^*}^{t=0}[W_T], E_{p*}^{t=0}[W_T]) = (0.8307, 6.9454)$
- Wealth can be negative, $p^* \to \infty$, $w \to 0$

Efficient Frontiers: Wealth Case

- Allowing bankruptcy
  $\mathbb{P} = (-\infty, +\infty)$
- No bankruptcy
  $\mathbb{P} = [0, +\infty)$
- bounded control
  $\mathbb{P} = [0., 1.5]$
- $(W = 1, t = 0)$



Wealth to Income Ratio Case

μ_y	0.	ξ_1	0.2
σ	0.2	σ_{Y1}	0.05
σ_{Y0}	0.05	π	0.1
T	20 years	γ	15
$\mathbb{P}$	$[0, 1.5]$	$\mathbb{D}$	$[0, +\infty)$

The total wealth W follows

$$dW = (r + p\xi\sigma)W \; dt + p\sigma W dZ_1 + \pi Y dt$$

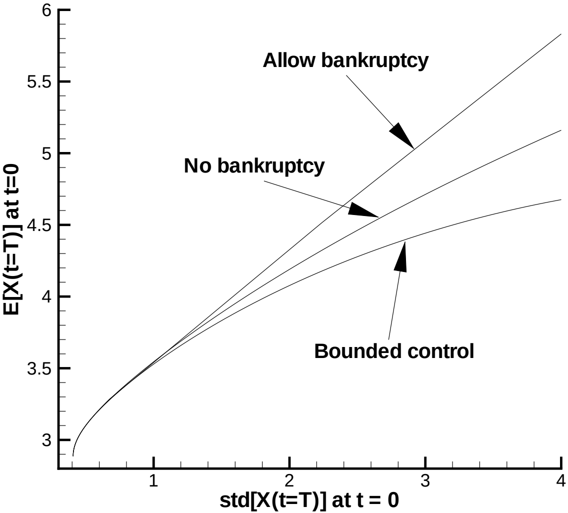
The investor's yearly salary Y follows

$$dY = (r + \mu_Y)Y \; dt + \sigma_{Y_0} Y \; dZ_0 + \sigma_{Y_1} Y \; dZ_1$$
$$dZ_0 \text{ is independent of } dZ_1$$

We seek to maximize $X(T) = W(T)/Y(T)$.
Risky asset proportion $p \in \mathbb{P}$, $x = w/y \in \mathbb{D}$.

Efficient Frontiers: Wealth to Income Case

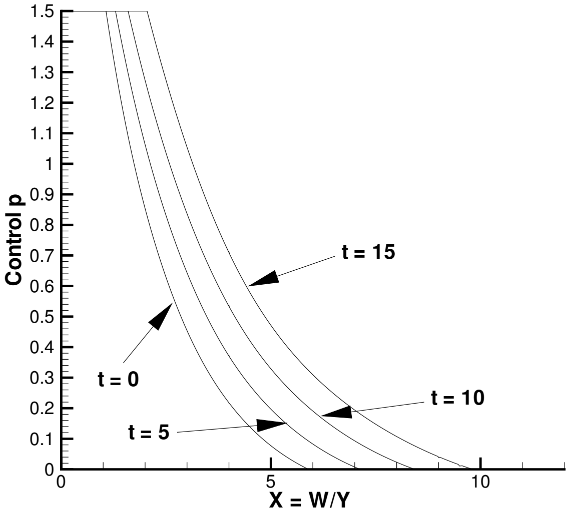
- No analytic solutions available
- No risk free portfolio (due to salary risk)



Optimal Strategy: Wealth to Income Ratio Case

- $X(t=0) = 0.5$, $(\text{Std}_{p^*}^{t=0}[X_T], E_{p^*}^{t=0}[X_T]) = (1.7407, 3.9551)$.

- As time goes on, and if x has not increased, the investor takes on more risk (agrees with analytic solution, wealth case)
- At any time, if (W/Y) is large enough, all wealth is switched to the risk free asset



Alternative Strategies

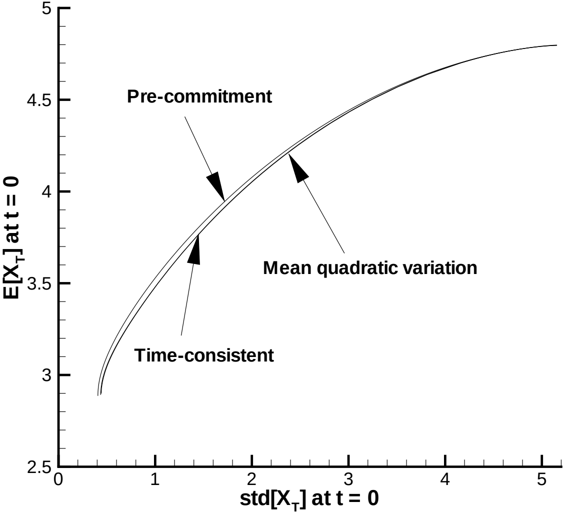
Using similar methods, we can devise a numerical scheme to determine a time consistent mean variance policy (See Wong, Forsyth EJOR (2011)).
In addition, we can also use a mean quadratic variation policy (Brugiere, 1996).

- In the unconstrained control case, the optimal policy for time consistent mean variance is identical to the optimal policy for mean quadratic variation (Bjork, 2010).

In the practical case of bounded controls

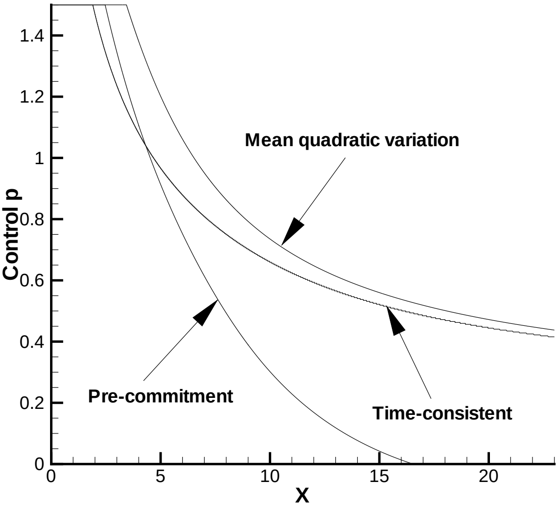
- All efficient frontiers are very similar
- But the optimal controls are noticeably different

Wealth to Income Ratio: Bounded Control



Comparison of Controls

- Control at $t = 0+$
- $\text{Std}^{p^*}_{t=0,x}[X(T)] \simeq 3.24$
- Efficient Frontiers similar
- Controls quite different



Conclusions

- There exists an HJB equation which has the same optimal control as the optimal control for continuous time mean-variance portfolio optimization (*pre-commitment policy*)
- This HJB equation can be reliably solved, and used to generate points on the efficient frontier
    - Any type of constraint can be applied to the control
    - Assuming a strong comparison property holds, then the scheme will converge to the viscosity solution of the HJB PDE
- Similar techniques can be used generate controls for
    - Time consistent mean variance
    - Mean quadratic variation
- For bounded control case, all these strategies give similar efficient frontiers, but controls are different
- Which strategy should we choose?