
Numerical Methods for Nonlinear PDEs in Finance

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1 Introduction

Many problems in finance can be posed in terms of an optimal stochastic control. Some well-known examples include transaction cost/uncertain volatility models [17, 2, 25], passport options [1, 26], unequal borrowing/lending costs in option pricing [9], risk control in reinsurance [23], optimal withdrawals in variable annuities [13], optimal execution of trades [20, 19], and asset allocation [28, 18]. A recent survey on the theoretical aspects of this topic is given in [24].

These optimal stochastic control problems can be formulated as nonlinear Hamilton-Jacobi-Bellman (HJB) partial differential equations (PDEs). In general, especially in realistic situations where the controls are constrained (e.g. in the case of asset allocation, we may require that trading must cease upon insolvency, that short positions are not allowed, or that position limits are imposed), there are no analytical solutions to the HJB PDEs. At first glance, it would appear to be a formidable task to develop a numerical method for solving such complex PDEs. In addition, there may be no smooth classical solutions to the HJB equations. In this case, we must seek the viscosity solution [12] of these equations.

However, using the powerful theory developed in [7, 5, 3] we can devise a general approach for numerically solving these HJB PDEs. This approach ensures convergence to the viscosity solution.

The contributions of this article are as follows:

- We discuss several examples of optimal stochastic control in finance.
- We give an intuitive description of the concept of a viscosity solution.
- We present a general approach for discretizing the HJB PDEs. This technique ensures that the discrete solutions converge to the viscosity solution [7, 5, 3]. The method uses fully implicit time stepping. Consequently, there

are no time step restrictions due to stability considerations, an advantage over the Markov chain approach [16].

- We also discuss some techniques for the solution of the nonlinear discretized algebraic equations and an important property of the discrete solutions (i.e. preservation of arbitrage inequalities).
- Finally, we present a numerical example, illustrating that seemingly reasonable discretization methods, which do not satisfy the conditions in [7] can converge to incorrect (i.e. non-viscosity) solutions, and even solutions which embed arbitrage opportunities.

2 Examples

2.1 Uncertain Volatility

Let $V(S, t)$ be the value of a contingent claim written on an asset which has a price S that evolves according to the stochastic process

$$dS = \mu S dt + \sigma S dZ, \quad (1)$$

where μ is the drift rate, σ is volatility, and dZ is the increment of a Wiener process. There are a number of situations where $V(S, t)$ must be determined by solving an optimal control problem.

Consider for example, the uncertain volatility model developed in [2, 21]. This provides a pricing mechanism for cases where volatility is uncertain, but lies within a band, $\sigma \in [\sigma_{\min}, \sigma_{\max}]$. In this case, the PDE which is used to determine the value of a contingent claim is determined by the two extremal volatilities. Let the expiry time of the claim be T , and let $\tau = T - t$. For a short position the optimal control problem is given by

$$V_\tau = \sup_{Q \in \hat{Q}} \left\{ \frac{Q^2 S^2}{2} V_{SS} + SV_S - rV \right\} = 0 \quad (2)$$

where $\hat{Q} = \{\sigma_{\min}, \sigma_{\max}\}$ and r is the borrowing/lending rate. Replacing the sup by an inf gives the corresponding pricing equation for a long position. It should also be pointed out that a PDE of precisely the same form as (2) arises in the completely different context of option valuation under transaction costs [17].

2.2 Continuous Time Mean-Variance Asset Allocation

We suppose that an investor may divide his wealth W into a fraction p in a risky asset, the price of which follows process (1), and a fraction $(1 - p)$ in a risk-free bond, the value of which follows

$$\frac{dB}{dt} = rB, \quad (3)$$

where r is the risk-free rate. If α is the number of units of S owned, then $W = \alpha S + B$, and the process followed by W is

$$dW = [p\mu + (1 - p)r]W dt + p\sigma W dZ. \quad (4)$$

We suppose that the investor follows an asset allocation strategy $p(t)$ for time $t \in [0, T]$. If W_T is the wealth at the terminal time T , then the optimal strategy may be posed as finding the $p(t)$ that maximizes the expected return less a penalty for risk (as measured by variance), i.e.

$$\sup_{p(t) \in z} \{E^{t=0}[W_T] - \lambda \text{var}^{t=0}[W_T]\}, \quad (5)$$

where

- $E^{t=0}[\cdot]$ is the expectation as seen at $t = 0$
- $\text{var}^{t=0}[\cdot]$ is the variance as seen at $t = 0$
- z is the set of admissible controls, and
- λ is the risk aversion parameter.

Varying λ allows us to generate a set of points $(\sqrt{\text{var}^{t=0}[W_T]}, E^{t=0}[W_T])$ on the mean-variance efficient frontier.

Problem (5) is the *pre-commitment* version of the mean-variance trade-off [8]. There is no direct dynamic programming formulation of problem (5). However, we can solve a different problem which has the same optimal control $p(t)$ and which is easier to solve.

We would like to use dynamic programming to determine the efficient frontier, given by equation (5). However, the presence of the variance term causes some difficulty. This can be avoided with the help of the results in [18, 28]:

Theorem 1 (Equivalent Linear Quadratic (LQ) problem). *If $p^*(t)$ is the optimal control of problem (5), then $p^*(t)$ is also the optimal control of problem*

$$\sup_{p(t) \in z} \{E^{t=0}[\mu W_T - \lambda W_T^2]\}, \quad (6)$$

where

$$\mu = 1 + 2\lambda E_{p^*}^{t=0}[W_T], \quad (7)$$

with p^* being the optimal control of problem (6).

The notation $E_{p^*}^{t=0}[\cdot]$ refers to the expected value given the strategy $p^*(t)$. This result seems at first sight to be not very useful, since the parameter μ is a function of the optimal control p^* , which is not known until the problem is solved. However, we can write equation (6) in the form

$$-\lambda \inf_{p(t) \in z} E^{t=0}[W_T^2 - \gamma W_T] \quad (8)$$

with $\gamma = \mu/\lambda$, since $\lambda > 0$. Consequently, for fixed γ , an optimal control of problem (8) is an optimal control of

$$\inf_{p(t) \in z} \left\{ E^{t=0} \left[(W_T - \frac{\gamma}{2})^2 \right] \right\}. \quad (9)$$

As a result, for fixed γ , we can determine the optimal control $p(t)$ of problem (5) as follows. Let

$$V(W, \tau) = E^{T-\tau} [(W_T - \gamma)^2]. \quad (10)$$

Then, V is given from the solution to

$$V_\tau = \inf_{p \in z} \{ (p\mu + (1-p)r)WV_W + (p\sigma)^2 W^2 V_{WW} \} \quad (11)$$

$$V(W, \tau = 0) = (W - \gamma/2)^2. \quad (12)$$

Having solved equation (12), we then have the optimal control $p^*(W, t)$. This can be used to determine a pair $(\sqrt{\text{var}^{t=0}[W_T]}, E^{t=0}[W_T])$. Varying γ allows us to trace out an efficient frontier.

2.3 Guaranteed Minimum Withdrawal Benefit Variable Annuity

Guaranteed Minimum Withdrawal Benefit (GMWB) variable annuities are discussed at length in [22, 13, 11]. We briefly review the final equations here. Let $W \equiv W(t)$ be the stochastic process of the personal variable annuity account and $A \equiv A(t)$ be the stochastic process of the account balance of the guarantee. We assume that the reference portfolio $S \equiv S(t)$, which underlies the variable annuity policy before the deduction of any proportional fees, follows a geometric Brownian motion under the risk-neutral measure with a volatility of σ and a risk-free interest rate of r :

$$dS = rS dt + \sigma S dZ. \quad (13)$$

The major feature of the GMWB is the guarantee on the return of the entire premium via withdrawal. The insurance company charges the policy holder a proportional annual insurance fee η for this guarantee. Therefore we have the following stochastic differential equation for W :

$$dW = \begin{cases} (r - \eta)W dt + \sigma W dZ + dA & \text{if } W > 0, \\ 0 & \text{if } W = 0. \end{cases} \quad (14)$$

Let $\gamma \equiv \gamma(t)$ denote the withdrawal rate at time t and assume $0 \leq \gamma \leq \lambda$ (λ is the maximum possible withdrawal rate). The policy guarantees that the accumulated sum of withdrawals throughout the policy's life is equal to the premium paid up front, which is denoted by ω_0 . Consequently, we have $A(0) = \omega_0$, and

$$A(t) = \omega_0 - \int_0^t \gamma(u) du. \quad (15)$$

In addition, almost all policies with GMWB put a cap on the maximum allowed withdrawal rate without penalty. Let G be such a contractual withdrawal rate, and κ be the proportional penalty charge applied on the portion of withdrawal exceeding G . The net withdrawal rate $f(\gamma)$ received by the policy holder is then

$$f(\gamma) = \begin{cases} \gamma & 0 \leq \gamma \leq G, \\ G + (1 - \kappa)(\gamma - G) & G < \gamma \leq \lambda. \end{cases} \quad (16)$$

The no-arbitrage value $V(W, A, t)$ of the variable annuity with GMWB therefore is given by

$$V(W, A, t) = \max_{\gamma \in [0, \lambda]} E_t \left[e^{-r(T-t)} \max(W(T), 0) + \int_t^T e^{-r(u-t)} f(\gamma(u)) du \right] \quad (17)$$

where T is the policy maturity time and the expectation E_t is taken under the risk-neutral measure. The withdrawal rate γ is the control variable chosen to maximize the value of $V(W, A, t)$.

Define

$$\mathcal{L}V = \frac{\sigma^2}{2} W^2 V_{WW} + (r - \eta) W V_W - rV, \quad (18)$$

and

$$\mathcal{F}V = 1 - V_W - V_A. \quad (19)$$

If we let the maximum possible withdrawal rate $\lambda \rightarrow \infty$ (withdrawing instantaneously a finite amount), then we obtain the singular control problem [13]

$$\min [V_\tau - \mathcal{L}V - G \max(\mathcal{F}V, 0), \kappa - \mathcal{F}V] = 0. \quad (20)$$

3 Viscosity Solutions

The highly nonlinear PDEs (2,12,20) do not have smooth (i.e. differentiable) solutions in general. In this case, it is not obvious what we mean by the solution to a differential equation. To clarify, it is useful to give an intuitive description of the concept of a *viscosity solution*. For sake of illustration, consider equation (2).

We can write our PDE as

$$g(V, V_S, V_{SS}, V_\tau) = V_\tau - \sup_{Q \in \hat{Q}} \left\{ \frac{Q^2 S^2}{2} V_{SS} + S V_S - rV \right\} = 0. \quad (21)$$

We assume that $g(x, y, z, w)$ ($x = V, y = V_S, z = V_{SS}, w = V_\tau$) satisfies the ellipticity condition

$$g(x, y, z + \epsilon, w) \leq g(x, y, z, w) \quad \forall \epsilon \geq 0, \quad (22)$$

which in our case usually means that the coefficient of the V_{SS} term in $\mathcal{L}V$ is non-negative. Suppose for the moment that smooth solutions to equation (21) exist, i.e. $V \in C^{2,1}$, where $C^{2,1}$ refers to a continuous function $V = V(S, \tau)$ having continuous first and second derivatives in S , and a continuous first derivative in τ . Let ϕ be a set of $C^{2,1}$ test functions. Suppose $V - \phi \leq 0$, and that $\phi(S_0, \tau_0) = V(S_0, \tau_0)$ at the single point (S_0, τ_0) . Then the single point (S_0, τ_0) is a global maximum of $(V - \phi)$,

$$\begin{aligned} V - \phi &\leq 0, \\ \max(V - \phi) &= V(S_0, \tau_0) - \phi(S_0, \tau_0) = 0. \end{aligned} \quad (23)$$

Consequently, at (S_0, τ_0)

$$\begin{aligned} \phi_\tau &= V_\tau \\ \phi_S &= V_S \\ (V - \phi)_{SS} &\leq 0 \quad \Rightarrow \quad \phi_{SS} \geq V_{SS}. \end{aligned} \quad (24)$$

Hence, from equations (22,24), we have

$$\begin{aligned} &g(V(S_0, \tau_0), \phi_S(S_0, \tau_0), \phi_{SS}(S_0, \tau_0), \phi_\tau(S_0, \tau_0)) \\ &= g(V(S_0, \tau_0), V_S(S_0, \tau_0), \phi_{SS}(S_0, \tau_0), V_\tau(S_0, \tau_0)) \\ &\leq g(V(S_0, \tau_0), V_S(S_0, \tau_0), V_{SS}(S_0, \tau_0), V_\tau(S_0, \tau_0)) = 0, \end{aligned} \quad (25)$$

or, to summarize,

$$\begin{aligned} g(V(S_0, \tau_0), \phi_S(S_0, \tau_0), \phi_{SS}(S_0, \tau_0), \phi_\tau(S_0, \tau_0)) &\leq 0 \\ V - \phi &\leq 0 \\ \max(V - \phi) &= V(S_0, \tau_0) - \phi(S_0, \tau_0) = 0. \end{aligned} \quad (26)$$

If this is true for any test function ϕ , then we say that V is a *viscosity subsolution* of equation (21).

Now, suppose that χ is a $C^{2,1}$ test function, with $V - \chi \geq 0$, and $V(S_0, \tau_0) = \chi(S_0, \tau_0)$ at the single point (S_0, τ_0) . Then, (S_0, τ_0) is the global minimum of $V - \chi$,

$$\begin{aligned} V - \chi &\geq 0 \\ \min(V - \chi) &= V(S_0, \tau_0) - \chi(S_0, \tau_0) = 0. \end{aligned} \quad (27)$$

Consequently, at (S_0, τ_0)

$$\begin{aligned}
 \chi_\tau &= V_\tau \\
 \chi_S &= V_S \\
 (V - \chi)_{SS} &\geq 0 \quad \Rightarrow \quad \chi_{SS} \leq V_{SS}.
 \end{aligned} \tag{28}$$

Hence, from equations (27,28), we have

$$\begin{aligned}
 &g(V(S_0, \tau_0), \chi_S(S_0, \tau_0), \chi_{SS}(S_0, \tau_0), \chi_\tau(S_0, \tau_0)) \\
 &= g(V(S_0, \tau_0), V_S(S_0, \tau_0), \chi_{SS}(S_0, \tau_0), V_\tau(S_0, \tau_0)) \\
 &\geq g(V(S_0, \tau_0), V_S(S_0, \tau_0), V_{SS}(S_0, \tau_0), V_\tau(S_0, \tau_0)) = 0.
 \end{aligned} \tag{29}$$

Summarizing,

$$\begin{aligned}
 &g(V(S_0, \tau_0), \chi_S(S_0, \tau_0), \chi_{SS}(S_0, \tau_0), \chi_\tau(S_0, \tau_0)) \geq 0 \\
 &V - \chi \geq 0 \\
 &\min(V - \chi) = V(S_0, \tau_0) - \chi(S_0, \tau_0) = 0.
 \end{aligned} \tag{30}$$

If this is true for any test function χ , we say that V is a *viscosity supersolution* of equation (21). A solution which is both a viscosity subsolution and a viscosity supersolution is a viscosity solution.

Now, suppose that V is continuous but not smooth. This means that we cannot define V as the solution to $g(V, V_S, V_{SS}, V_\tau) = 0$. However, we can still use conditions (26) and (30) to define a viscosity solution to equation (21), since all derivatives are applied to smooth test functions. Informally, a viscosity solution V to equation (21) is defined such that

- For any $C^{2,1}$ test function ϕ , such that

$$V - \phi \leq 0; \quad \phi(S_0, \tau_0) = V(S_0, \tau_0), \tag{31}$$

(ϕ touches V at the single point (S_0, τ_0)), then

$$g(V(S_0, \tau_0), \phi_S(S_0, \tau_0), \phi_{SS}(S_0, \tau_0), \phi_\tau(S_0, \tau_0)) \leq 0. \tag{32}$$

- As well, for any $C^{2,1}$ test function χ such that

$$V - \chi \geq 0; \quad V(S_0, \tau_0) = \chi(S_0, \tau_0), \tag{33}$$

(χ touches V at the single point (S_0, τ_0)), then

$$g(V(S_0, \tau_0), \chi_S(S_0, \tau_0), \chi_{SS}(S_0, \tau_0), \chi_\tau(S_0, \tau_0)) \geq 0. \tag{34}$$

An example of a subsolution and a typical test function is shown in Figure 1. Similarly, the supersolution case is shown in Figure 2.

Note that there may be some points where a smooth test function can touch the viscosity solution only from above or below, but not both. The kink

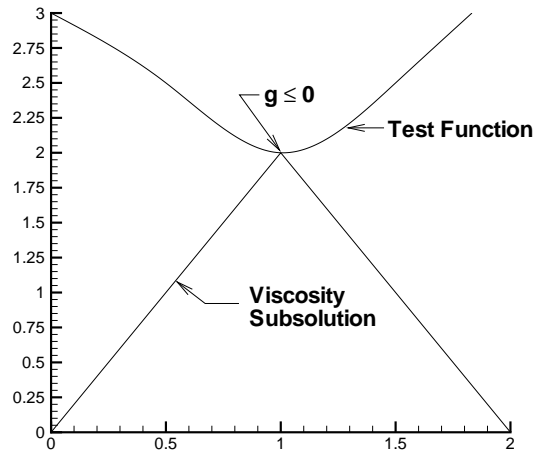


Fig. 1. Illustration of viscosity subsolution definition.

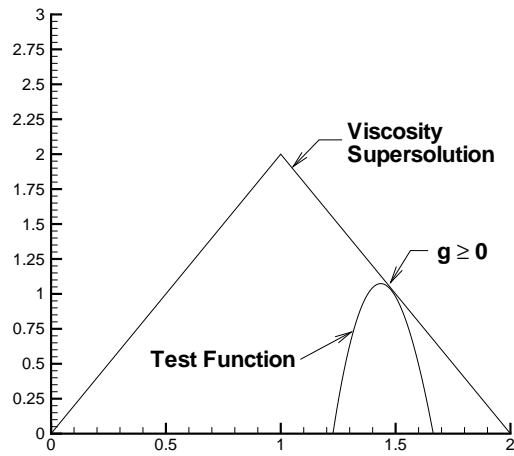


Fig. 2. Illustration of viscosity supersolution definition.

at $S = 1$ in Figure 2 is an example of such a situation. It is not possible for a smooth $C^{2,1}$ test function χ satisfying $V - \chi \geq 0$, $\chi(1, \tau_0) = V(1, \tau_0)$ to exist.

There may also be some points where a smooth $C^{2,1}$ test function cannot touch the solution from either above or below. As a pathological example, consider the function

$$f(x) = \begin{cases} \sqrt{x} & x \geq 0, \\ -\sqrt{-x} & x < 0. \end{cases} \quad (35)$$

This function cannot be touched at the origin from below (or above) by any smooth function with bounded derivatives. Note that the definition of a viscosity solution only specifies what happens when the test function touches the viscosity solution at a single point (from either above or below). The definition is silent about cases where this cannot happen.

4 General Form for the Example Problems

We can treat many control problems in finance using a similar approach. Even singular control problems, as in equation (20), can be solved using the methods described here, if we use the penalty technique described in [13].

For ease of exposition, we will focus on single factor optimal control problems. We give a brief overview of the methods here—see [15] for more details. Let the value function be denoted by $V = V(S, \tau)$, where $\tau = T - t$, with T being the expiry time of the contract or claim being considered. Set

$$\mathcal{L}^Q V \equiv a(S, \tau, Q)V_{SS} + b(S, \tau, Q)V_S - c(S, \tau, Q)V, \quad (36)$$

where Q is a control parameter. We write our problem in the general form

$$V_\tau = \sup_{Q \in \hat{Q}} \left\{ \mathcal{L}^Q V + d(S, \tau, Q) \right\}, \quad (37)$$

\hat{Q} being a compact set of feasible controls. Note that we can replace the sup in equation (37) by an inf and all the methods remain essentially the same.

We will assume in the following that $a(S, \tau, Q) \geq 0$ and $c(S, \tau, Q) \geq 0$. In a financial context this corresponds to non-negative interest rates and volatilities.

4.1 Boundary Conditions

We will assume that the problem is posed on a bounded domain $[S_{\min}, S_{\max}]$. In many cases, the original problem is posed on an unbounded domain. We assume that the problem has been *localized* for computational purposes. We will assume that the boundary conditions at $[S_{\min}, S_{\max}]$ are either the limit of the PDE as $S \rightarrow S_{\min}, S_{\max}$ or some type of given Dirichlet condition.

4.2 Strong Comparison Result

We assume that the HJB PDE (37) along with appropriate boundary conditions satisfies the *strong comparison property* [12], which then implies that there exists a unique, continuous viscosity solution to equation (37).

5 Discretization

Define a grid $\{S_0, S_1, \dots, S_p\}$ with $S_p = S_{\max}$, and let V_i^n be a discrete approximation to $V(S_i, \tau^n)$. Let $V^n = [V_0^n, \dots, V_p^n]'$, and let $(\mathcal{L}_h^Q V^n)_i$ denote the discrete form of the differential operator (36) at node (S_i, τ^n) . The operator (36) can be discretized using forward, backward or central differencing in the S direction to give

$$\begin{aligned} (\mathcal{L}_h^Q V^{n+1})_i &= \alpha_i^{n+1}(Q) V_{i-1}^{n+1} + \beta_i^{n+1}(Q) V_{i+1}^{n+1} \\ &\quad - (\alpha_i^{n+1}(Q) + \beta_i^{n+1}(Q) + c_i^{n+1}(Q)) V_i^{n+1}. \end{aligned} \quad (38)$$

It is important that central, forward or backward discretizations be used to ensure that (40) is a positive coefficient discretization. To be more precise, this condition is

Condition 1 *Positive Coefficient Condition*

$$\alpha_i^{n+1}(Q) \geq 0, \beta_i^{n+1}(Q) \geq 0, c_i^{n+1}(Q) \geq 0, i = 0, \dots, p-1, \forall Q \in \hat{Q}. \quad (39)$$

We will assume that all models have $c_i^{n+1}(Q) \geq 0$. Consequently, we choose central, forward or backward differencing at each node so as to ensure that $\alpha_i^{n+1}(Q), \beta_i^{n+1}(Q) \geq 0$. Appendix A provides details concerning forward, backward and central differencing. Note that different nodes can have different discretization schemes. If we use forward and backward differencing, then equation (57) in Appendix A guarantees a positive coefficient method. However, since this discretization is only first order correct, it is desirable to use central differencing as much as possible (and yet still obtain a positive coefficient method). This issue is discussed in detail in [27].

Equation (37) can now be discretized using fully implicit time stepping together with the discretization (38) to give

$$\frac{V_i^{n+1} - V_i^n}{\Delta\tau} = \sup_{Q^{n+1} \in \hat{Q}} \left\{ (\mathcal{L}_h^{Q^{n+1}} V^{n+1})_i + d_i^{n+1} \right\}. \quad (40)$$

Of course, an explicit method would involve evaluating the terms on the right hand side of equation (40) at the old time level n instead of $n+1$. A Crank-Nicolson scheme would be an equally-weighted average of the fully implicit scheme (40) and an explicit scheme.

5.1 Matrix Form of the Discrete Equations

Set $V^{n+1} = [V_0^{n+1}, V_1^{n+1}, \dots, V_p^{n+1}]'$ and $Q = [Q_0, Q_1, \dots, Q_p]'$. We can write the discrete operator $(\mathcal{L}_h^Q V^n)_i$ as

$$\begin{aligned} (\mathcal{L}_h^Q V^n)_i &= [A(Q)V^n]_i \\ &= [\alpha_i^n(Q)V_{i-1}^n + \beta_i^n(Q)V_{i+1}^n - (\alpha_i^n(Q) + \beta_i^n(Q) + c_i^n(Q))V_i^n], \quad i < p. \end{aligned} \quad (41)$$

The first and last rows of A are modified as needed to handle the boundary conditions. Let F^{n+1} be a vector which encodes boundary conditions (i.e. $F_i^{n+1} = 0$ except possibly at $i = 0, p$).

Let $D^n(Q)$ be the vector with entries

$$[D(Q)]_i^n = \begin{cases} d_i^n(Q) & \text{for } i < p \rightarrow i \text{ is not a Dirichlet node} \\ 0 & \text{for } i = p \rightarrow i \text{ is a Dirichlet node} \end{cases}.$$

Remark 1 (Matrix Supremum Notational Convention). In the following, we will denote

$$\sup_{Q \in \hat{Q}} \left\{ [A^{n+1}(Q)V^{n+1} + D^{n+1}(Q)]_i \right\}$$

by

$$A^{n+1}(Q^{n+1})V^{n+1} + D^{n+1}(Q^{n+1}),$$

where the optimal control at time level $n + 1$ for node i is

$$Q_i^{n+1} \in \arg \sup_{Q \in \hat{Q}} \left\{ [A^{n+1}(Q)V^{n+1} + D^{n+1}(Q)]_i \right\}.$$

If the local objective function is a continuous function of Q , then the supremum is simply the maximum value (since \hat{Q} is compact), and Q^{n+1} is the point where a maximum is reached. Alternatively, if the local objective function is discontinuous, $A^{n+1}(Q^{n+1})$ is interpreted as the appropriate limiting value of $[A^{n+1}(Q)]_i$ which generates the supremum at the limit point Q^{n+1} . An example of an algorithm for computing this limit point is given in [27] for the case of maximizing the usage of central weighting. Note that Q^{n+1} is not necessarily unique.

The discrete equations (40) can be written as

$$[I - \Delta\tau A^{n+1}(Q^{n+1})] V^{n+1} = V^n + \Delta\tau D^{n+1}(Q^{n+1}) + (F^{n+1} - F^n), \quad (42)$$

where

$$Q_i^{n+1} \in \arg \sup_{Q \in \hat{Q}} \left\{ [A^{n+1}(Q)V^{n+1} + D^{n+1}(Q)]_i \right\}.$$

For convenience, define

$$(\Delta\tau)_{\max} = \max_n(\tau^{n+1} - \tau^n) \text{ and } (\Delta\tau)_{\min} = \min_n(\tau^{n+1} - \tau^n),$$

where we assume that there are mesh size/time step parameters h_{\min}, h_{\max} such that

$$\begin{aligned} (\Delta S)_{\max} &= C_1 h_{\max}, & (\Delta\tau)_{\max} &= C_2 h_{\max}, \\ (\Delta S)_{\min} &= C_3 h_{\min}, & (\Delta\tau)_{\min} &= C_4 h_{\min}, \end{aligned}$$

with C_1, C_2, C_3, C_4 being positive constants independent of h .

We can then write the discrete equations (40) or (42) at each node in the form

$$G_i^{n+1}(h_{\max}, V_i^{n+1}, V_{i+1}^{n+1}, V_{i-1}^{n+1}, V_i^n, V_{i+1}^n, V_{i-1}^n) = 0,$$

where

$$\begin{aligned} G_i^{n+1} &\equiv \frac{V_i^{n+1} - V_i^n}{\Delta\tau} \\ &- \sup_{Q^{n+1} \in \hat{Q}} \left\{ \left(A^{n+1}(Q^{n+1})V_i^{n+1} + D^{n+1}(Q^{n+1}) \right)_i \right\} - \frac{F_i^{n+1} - F_i^n}{\Delta\tau}. \end{aligned} \quad (43)$$

For notational brevity, we shall occasionally write

$$G_i^{n+1}(h_{\max}, V_i^{n+1}, \{V_j^{n+1}\}_{j \neq i}, V_i^n) \equiv G_i^{n+1}(h_{\max}, V_i^{n+1}, V_{i+1}^{n+1}, V_{i-1}^{n+1}, V_i^n), \quad (44)$$

where $\{V_j^{n+1}\}_{j \neq i}$ is the set of values V_j^{n+1} , for $j = 1, \dots, p$, with $j \neq i$.

6 Convergence to the Viscosity Solution

In [25], examples were given in which seemingly reasonable discretizations of nonlinear option pricing PDEs were either unstable or converged to the incorrect solution. It is important to ensure that we can generate discretizations which are guaranteed to converge to the viscosity solution [3, 12]. Assuming that equation (37) satisfies the strong comparison property [4, 6, 10], then, from [7, 3], a numerical scheme converges to the viscosity solution if the method is (i) consistent, (ii) stable (in the l_∞ norm), and (iii) monotone. To be precise, we define these terms.

Definition 1 (Stability). *Discretization (43) is stable if*

$$\|V^{n+1}\|_\infty \leq C_5,$$

for $0 \leq n \leq N$, $T = N\Delta\tau$, for $(\Delta\tau)_{\min} \rightarrow 0$, $(\Delta S)_{\min} \rightarrow 0$, where C_5 is independent of $(\Delta\tau)_{\min}, (\Delta S)_{\min}$.

Lemma 1 (Stability). *If the discretization (43) satisfies the positive coefficient condition (39), then the scheme is l_∞ stable.*

Proof. This is easily shown using a maximum analysis as in [15].

For ease of exposition, we consider the simple case where we restrict attention to interior nodes. This allows us to use the following definition of consistency.

Definition 2 (Consistency). Let ϕ denote any smooth function with $\phi_i^n = \phi(S_i, \tau^n)$, and let

$$\Phi = \left(\phi_\tau - \sup_{Q \in \hat{Q}} \{ \mathcal{L}^Q \phi + d \} \right)_i^{n+1} - G_i^{n+1} (h_{\max}, \phi_i^{n+1}, \phi_{i+1}^{n+1}, \phi_{i-1}^{n+1}, \phi_i^n, \phi_{i+1}^n, \phi_{i-1}^n).$$

Scheme (43) is consistent if

$$\lim_{h_{\max} \rightarrow 0} |\Phi| = 0. \quad (45)$$

Remark 2. For the general case where the HJB PDE degenerates at the boundary, a more complicated definition of consistency is required in order to handle boundary data [3]. We refer the reader to [3] for this definition, and to [11] for a specific application of this more complex definition.

Remark 3. Note that Definition 2 is given in terms of smooth test functions ϕ , and does not require differentiability of the actual solution.

Lemma 2 (Consistency). If the discrete equation coefficients are as given in Appendix A, then the discrete scheme (43) is consistent as defined in Definition 2.

Proof. This follows from a Taylor series argument.

Definition 3 (Monotonicity). The discrete scheme (43) is monotone if for all $\epsilon_j^l \geq 0$ and i

$$G_i^{n+1} (h_{\max}, V_i^{n+1}, \{V_j^{n+1} + \epsilon_j^{n+1}\}_{j \neq i}, \{V_j^n + \epsilon_j^n\}) \leq G_i^{n+1} (h_{\max}, V_i^{n+1}, \{V_j^{n+1}\}_{j \neq i}, \{V_j^n\}). \quad (46)$$

Lemma 3 (Monotonicity). If the discretization (43) satisfies the positive coefficient condition (39), then it is monotone as defined in Definition 3.

Proof. We write equation (43) out in component form (at the interior nodes so that $F_i = 0$)

$$G_i^{n+1} (h, V_i^{n+1}, V_{i+1}^{n+1}, V_{i-1}^{n+1}, V_i^n) = \frac{V_i^{n+1} - V_i^n}{\Delta\tau} + \inf_{Q^{n+1} \in \hat{Q}} \left\{ (\alpha_i^{n+1}(Q) + \beta_i^{n+1}(Q) + c_i^{n+1}(Q)) V_i^{n+1} - \alpha_i^{n+1}(Q) V_{i-1}^{n+1} - \beta_i^{n+1}(Q) V_{i+1}^{n+1} - d_i^{n+1}(Q) \right\}. \quad (47)$$

Note that, given two functions $X(x), Y(x)$,

$$\inf_x X(x) - \inf_y Y(y) \leq \sup_x (X(x) - Y(x)).$$

Then, for $\epsilon \geq 0$, we have

$$\begin{aligned} & G_i^{n+1}(h, V_i^{n+1}, V_{i+1}^{n+1} + \epsilon, V_{i-1}^{n+1}, V_i^n) - G_i^{n+1}(h, V_i^{n+1}, V_{i+1}^{n+1}, V_{i-1}^{n+1}, V_i^n) \\ &= \inf_{Q \in \hat{Q}} \left\{ (\alpha_i^{n+1}(Q) + \beta_i^{n+1}(Q) + c_i^{n+1}(Q)) V_i^{n+1} \right. \\ &\quad \left. - \alpha_i^{n+1}(Q) V_{i-1}^{n+1} - \beta_i^{n+1}(Q) V_{i+1}^{n+1} - \beta_i^{n+1}(Q) \epsilon - d_i^{n+1}(Q) \right\} \\ &- \inf_{Q^* \in \hat{Q}} \left\{ (\alpha_i^{n+1}(Q^*) + \beta_i^{n+1}(Q^*) + c_i^{n+1}(Q^*)) V_i^{n+1} \right. \\ &\quad \left. - \alpha_i^{n+1}(Q^*) V_{i-1}^{n+1} - \beta_i^{n+1}(Q^*) V_{i+1}^{n+1} - d_i^{n+1}(Q^*) \right\} \\ &\leq \sup_{Q \in \hat{Q}} \left\{ -\beta_i^{n+1}(Q) \epsilon \right\} = -\epsilon \inf_{Q \in \hat{Q}} \left\{ \beta_i^{n+1}(Q) \right\} \leq 0. \end{aligned} \tag{48}$$

This follows from the fact that $\beta_i^{n+1}(Q) \geq 0$. Similarly,

$$G_i^{n+1}(h, V_i^{n+1}, V_{i+1}^{n+1}, V_{i-1}^{n+1} + \epsilon, V_i^n) - G_i^{n+1}(h, V_i^{n+1}, V_{i+1}^{n+1}, V_{i-1}^{n+1}, V_i^n) \leq 0. \tag{49}$$

Finally, it is obvious from equation (47) that

$$G_i^{n+1}(h, V_i^{n+1}, V_{i+1}^{n+1}, V_{i-1}^{n+1}, V_i^n + \epsilon) - G_i^{n+1}(h, V_i^{n+1}, V_{i+1}^{n+1}, V_{i-1}^{n+1}, V_i^n) \leq 0, \tag{50}$$

concluding the proof.

Theorem 2 (Convergence to the Viscosity Solution). *Provided that the original HJB PDE satisfies the strong comparison property, and discretization (42) satisfies all the conditions required for Lemmas 1, 2, and 3, then scheme (42) converges to the viscosity solution of equation (37).*

Proof. This follows directly from the results in [7, 3].

7 Solution of the Nonlinear Discretized Equations

Note that an implicit time stepping method requires the solution of highly nonlinear algebraic equations at each time step. We use a Newton-like form of policy iteration to solve the discrete equations:

Policy Iteration

Let $(V^{n+1})^0 = V^n$
 Let $\hat{V}^k = (V^{n+1})^k$
 For $k = 0, 1, 2, \dots$ until convergence
 Solve $[I - (1 - \theta)\Delta\tau A^{n+1}(Q^k)] \hat{V}^{k+1} =$
 $[I + \theta\Delta\tau A^n(Q^n)] V^n + (F^{n+1} - F^n) +$
 $(1 - \theta)\Delta\tau D^{n+1}(Q^k) + \theta\Delta\tau D^n$
 $Q_i^k \in \arg \sup_{Q \in \hat{Q}} \left\{ [A^{n+1}(Q)\hat{V}^k + D^{n+1}(Q)]_i \right\}$ (51)
 If $k > 0$ and

$$\left(\max_i \frac{|\hat{V}_i^{k+1} - \hat{V}_i^k|}{\max(\text{scale}, |\hat{V}_i^{k+1}|)} < \text{tolerance} \right)$$

 then quit
 EndFor

The term *scale* in scheme (51) is used to preclude unrealistic levels of accuracy when the value is very small. Typically, $scale = 1$ for values expressed in dollars.

Theorem 3 (Convergence of the Policy Iteration). *Provided that the discretization (43) satisfies the positive coefficient condition (39), then the policy iteration (51) converges to the unique solution of equation (42) for any initial iterate \hat{V}^0 . Moreover, the iterates converge monotonically.*

Proof. See [15].

The most fundamental principle of valuation in finance is the absence of arbitrage (i.e. there are no free lunches). One way of stating this is as follows. Imagine that we have two contingent claims with the same expiry time that are written on the same underlying asset, which has a price of S . Denote these two claims by $V(S, \tau)$ and $W(S, \tau)$. No-arbitrage implies that if the terminal payoff for V is always at least as high as that for W , then V must be worth at least as much as W at any time prior to expiry. More succinctly,

$$V(S, 0) \geq W(S, 0) \Rightarrow V(S, \tau) \geq W(S, \tau). \tag{52}$$

Let V^n and W^n denote discrete solutions to equation (42). We would like to ensure that these solutions are arbitrage-free, i.e.

$$V^n \geq W^n \Rightarrow V^{n+1} \geq W^{n+1}. \tag{53}$$

It can be shown that this property is satisfied under certain conditions, which we state in the following theorem:

Theorem 4 (Discrete no-arbitrage principle). *Assume that:*

- (i) *Discretization (43) satisfies the positive coefficient condition (39);*
- (ii) *Fully implicit time stepping is used; and*
- (iii) *Appropriate boundary conditions are imposed at the end-points of the discrete grid (see [15] for details).*

Then the discrete no-arbitrage condition (53) holds.

Proof. See [15].

8 Numerical Example: Uncertain Volatility

As a simple illustration of the methods outlined above, we will consider the case of pricing an option contract in an uncertain volatility model, as described in [2, 21] and outlined above in Section 2.1. Recall that we are interested in valuing an option under the assumption that the volatility σ lies between two bounds, σ_{\min} and σ_{\max} , but is otherwise unknown. From the standpoint of the option writer, the best case is found by solving equation (2), reproduced here for convenience:

$$V_\tau = \sup_{Q \in \hat{Q}} \left\{ \frac{Q^2 S^2}{2} V_{SS} + S V_S - rV \right\} = 0, \quad (54)$$

with $\hat{Q} = \{\sigma_{\min}, \sigma_{\max}\}$. Of course, from the perspective of the purchaser of the option, this would represent the worst possible case. Conversely, the worst case for the writer (found by replacing the sup by an inf in the equation above) corresponds to the best situation for the purchaser. At first glance this problem might appear to be trivial, since option values are increasing in volatility. However, while this is the case for a plain vanilla European option, it is not true in general provided that the option “gamma” V_{SS} can change sign. This can happen, for example, in the case of barrier options. Consider the case of an up-and-out call option, which is just like a regular call option unless the underlying asset price S moves above some barrier H during the contract’s life, in which case the payoff becomes zero. The gamma of this contract can be positive for some values of S and negative for others, as noted, for example, in [14].

Another example arises in the context of portfolio of plain vanilla European options, and it is this case that we will consider here. Note that this highlights the nonlinear nature of the problem, in that the problem is trivial for each of the options in the portfolio, but not for the linear combination that forms the portfolio. Suppose that an investor purchases a butterfly spread from a financial institution. This involves taking a long position in a low strike (K_1)

Parameter	Value
r	.04
T	0.5
K_1	95
K_2	100
K_3	105
σ_{\min}	0.30
σ_{\max}	0.45

Table 1. Input parameters for test case.

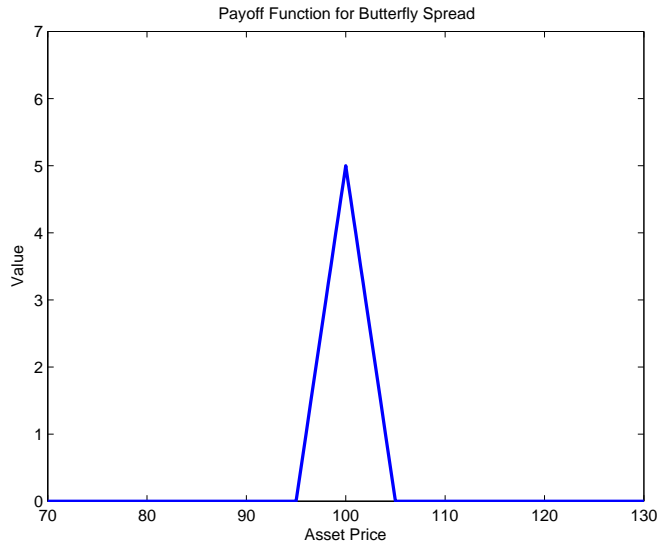


Fig. 3. Payoff function for butterfly spread.

option, a short position in two middle strike (K_2) options, and a long position in a high strike (K_3) option, all with identical maturities. Assume that the strikes are evenly spaced, and that all options are calls.³ Our test case uses the input parameters provided in Table 1.

The payoff function at maturity is plotted in Figure 3. The sharp peak around the middle strike $K_2 = 100$ will generate rapid changes with S in the solution value as we solve over time. This can be expected to cause problems with numerical methods unless we are careful.

Our numerical experiment uses a discrete grid ranging from $S_{\min} = 0$ to $S_{\max} = 500$. The coarsest grid has 94 unevenly spaced nodes (a finer spacing is

³ Actually, it doesn't matter if we form the combined position using three call options or three put options, as the overall payoff is the same either way.

Refinement level	Grid nodes	Time steps	Value at $S = 100$	Change	Ratio	Total iterations	Iterations per step
0	94	100	0.792639			227	2.27
1	187	200	0.796737	0.004098		450	2.25
2	373	400	0.798984	0.002247	1.82	871	2.18
3	745	800	0.800263	0.001279	1.76	1689	2.11
4	1489	1600	0.800957	0.000694	1.84	3260	2.04
5	2977	3200	0.801322	0.000365	1.90	6445	2.01
6	5953	6400	0.801511	0.000189	1.93	12802	2.00

Table 2. Best case for long position, fully implicit time stepping.

Refinement level	Grid nodes	Time steps	Value at $S = 100$	Change	Ratio	Total iterations	Iterations per step
0	94	100	0.130726			227	2.27
1	187	200	0.128638	-0.002088		443	2.22
2	373	400	0.127363	-0.001275	1.64	870	2.18
3	745	800	0.126643	-0.000720	1.77	1685	2.11
4	1489	1600	0.126257	-0.000386	1.87	3297	2.06
5	2977	3200	0.126056	-0.000201	1.92	6488	2.03
6	5953	6400	0.125954	-0.000102	1.97	12844	2.01

Table 3. Worst case for long position, fully implicit time stepping.

placed near the strikes), and uses 100 (constant-sized) time steps. Successive grid refinements involve doubling the number of time steps and inserting new grid points midway between previously existing nodes.

We begin by considering the results for the best case for a long position with fully implicit time stepping. Results are provided in Table 2. In this table, the column labelled “Change” is the difference in the computed solution from the previous grid refinement level, and the column labelled “Ratio” is the change for the current refinement level divided by that for the previous level. Values of “Ratio” around 2 indicate approximate first order convergence. Approximate second order convergence would be shown by values of “Ratio” of about 4. As can be seen from the table, fully implicit time stepping leads asymptotically to approximate first order convergence. The last two columns of the table show the total number of nonlinear iterations taken during the solution, and the average number of nonlinear iterations per time step. For this particular case, about two iterations are required for each time step.

Table 3 repeats the analysis, but for the worst case for a long position. Clearly, the value at $S = 100$ is much lower, but we again see that the algorithm exhibits approximate linear convergence and that around two iterations are needed per time step. Figure 4 plots the solution profile obtained for the best and worst cases for a long position using fully implicit time steps.

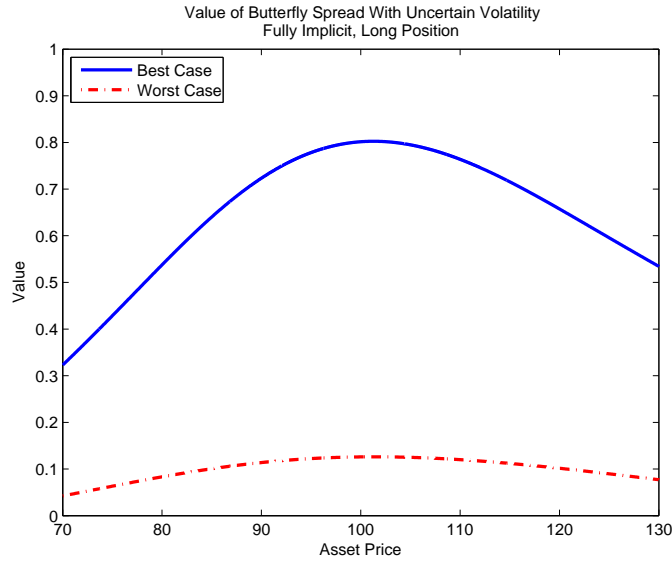


Fig. 4. Value of butterfly spread with uncertain volatility. Fully implicit time stepping, long position.

Tables 4 and 5 document the serious problems which can occur when we use numerical methods which are not guaranteed to converge to the viscosity solution and are not necessarily arbitrage-free. The only difference here compared to Tables 2 and 3 is the switch from fully implicit time stepping to Crank-Nicolson. The key results from Table 4 are as follows. Although Crank-Nicolson is in theory second order accurate in time, the convergence rate here is actually less than first order. More importantly, the scheme is converging to a different answer than that obtained in Table 2. Since the fully implicit scheme used in Table 2 is guaranteed to converge to the viscosity solution, the implication here is that the Crank-Nicolson approach is converging to some other (i.e. non-viscosity) solution. Comparing Tables 2 and 3, we can also see that the Crank-Nicolson approach requires more than twice as many nonlinear iterations.

The same general conclusions apply to Table 5: the Crank-Nicolson scheme converges at a rate which is slower than first order, it requires more than twice as many iterations than does the fully implicit approach, and it is converging to an answer which is not the viscosity solution. In fact, the Crank-Nicolson method converges here to a negative value. This represents an obvious arbitrage opportunity and is clearly an absurd result. Cases like this are in a sense reassuring, since it is obvious that the answer makes no sense. From this perspective, the Crank-Nicolson results for the best case long position are possibly of greater concern. Without calculating the correct answer via the fully implicit approach, it is not immediately clear that the Crank-Nicolson answer

Refinement level	Grid nodes	Time steps	Value at $S = 100$	Change	Ratio	Total iterations	Iterations per step
0	94	100	4.410778			428	4.28
1	187	200	4.571876	0.161098		897	4.49
2	373	400	4.687534	0.115658	1.39	1780	4.45
3	745	800	4.765390	0.077856	1.49	3539	4.42
4	1489	1600	4.816438	0.051048	1.53	7161	4.48
5	2977	3200	4.849302	0.032864	1.55	13995	4.37
6	5953	6400	4.870269	0.020967	1.57	27529	4.30

Table 4. Best case for long position, Crank-Nicolson time stepping.

Refinement level	Grid nodes	Time steps	Value at $S = 100$	Change	Ratio	Total iterations	Iterations per step
0	94	100	-6.178730			457	4.57
1	187	200	-6.399983	-0.221253		926	4.63
2	373	400	-6.545795	-0.145812	1.52	1901	4.75
3	745	800	-6.643648	-0.097853	1.49	3815	4.77
4	1489	1600	-6.709119	-0.065471	1.49	7341	4.59
5	2977	3200	-6.751707	-0.042588	1.54	14379	4.49
6	5953	6400	-6.778385	-0.026678	1.60	28317	4.42

Table 5. Worst case for long position, Crank-Nicolson time stepping.

is incorrect. Figure 5 plots the solution profile obtained for the best and worst cases for a long position using the Crank-Nicolson scheme.

In addition to calculating the value of the position, we are often interested in hedging parameters such as delta and gamma. Figures 6 and 7 plot the delta and gamma respectively for the best case for a long position with fully implicit time steps. The corresponding plots for the Crank-Nicolson case for delta and gamma are given in Figures 8 and 9 respectively. Comparing Figure 6 and 8, we see that the plot for delta is much smoother for the fully implicit case (in addition to being far smaller in magnitude). In fact, there appears to be a discontinuity in the delta at $S = 100$ for the Crank-Nicolson case. Figure 7 shows a smooth profile for the option gamma using fully implicit time steps. On the other hand, Figure 9 shows severe oscillations around values of $S = 100$.⁴ Taken collectively, these plots again provide a strong warning against the naïve use of Crank-Nicolson methods in that the calculation of important hedging parameters is prone to serious errors. This is not surprising—if the solution itself is not accurate, we should expect the estimates of its derivatives to be even worse.

⁴ Note that Figure 9 uses a different scale on the x -axis to highlight the oscillations. The rapid changes associated with the oscillations in gamma are one reason why the Crank-Nicolson scheme requires more iterations to solve the nonlinear discrete equations.

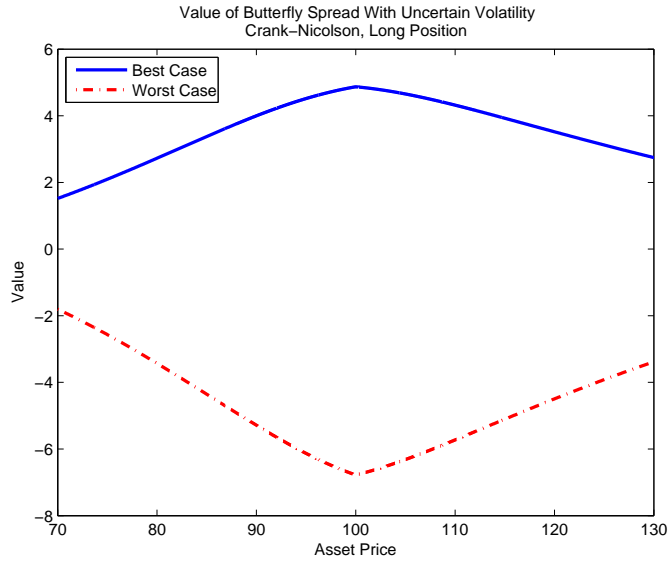


Fig. 5. Value of butterfly spread with uncertain volatility. Crank-Nicolson time stepping, long position.

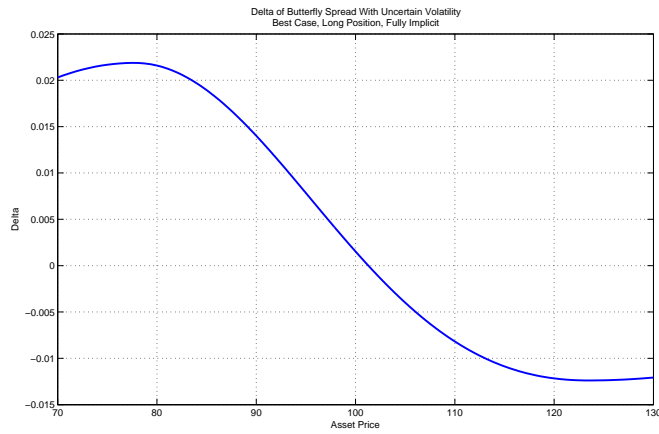


Fig. 6. Delta of butterfly spread with uncertain volatility. Fully implicit time stepping, long position, best case.

9 Conclusions

Many problems of practical interest in finance can be cast as stochastic optimal control problems. These problems are generally nonlinear and require numerical solution. This article has described some of these problems, along

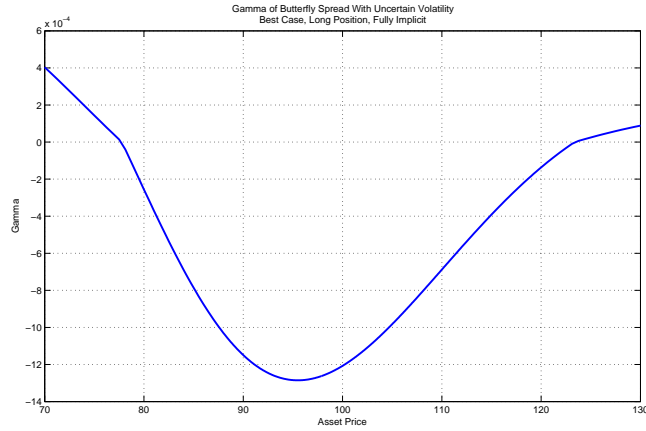


Fig. 7. Gamma of butterfly spread with uncertain volatility. Fully implicit time stepping, long position, best case.

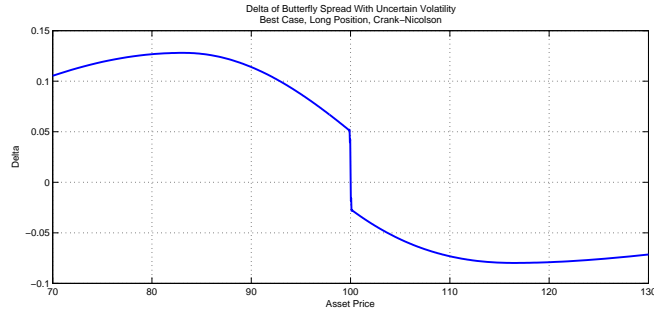


Fig. 8. Delta of butterfly spread with uncertain volatility. Crank-Nicolson time stepping, long position, best case.

with a general approach that can be taken to solve them numerically. This approach stresses the importance of using a positive coefficient discretization and fully implicit time stepping. This guarantees convergence to the viscosity solution, and has the important feature that the discrete solutions are arbitrage-free. Apparently reasonable discretizations such as Crank-Nicolson methods are not guaranteed to converge to the viscosity solution, nor can we be sure that they do not lead to free lunches. Moreover, the use of such methods can lead to serious errors in the estimation of hedging parameters.

A Discrete Equation Coefficients

Let Q_i^n denote the optimal control at node i and time level n , and set

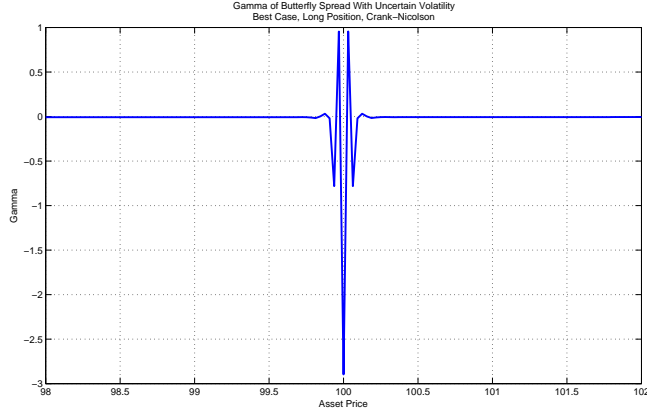


Fig. 9. Gamma of butterfly spread with uncertain volatility. Crank-Nicolson time stepping, long position, best case.

$$a_i^{n+1} = a(S_i, \tau^n, Q_i^n), \quad b_i^{n+1} = b(S_i, \tau^n, Q_i^n), \quad c_i^{n+1} = c(S_i, \tau^n, Q_i^n). \quad (55)$$

Then we can use central, forward or backward differencing at any node. For central differencing:

$$\alpha_{i,central}^n = \left[\frac{2a_i^n}{(S_i - S_{i-1})(S_{i+1} - S_{i-1})} - \frac{b_i^n}{S_{i+1} - S_{i-1}} \right]$$

$$\beta_{i,central}^n = \left[\frac{2a_i^n}{(S_{i+1} - S_i)(S_{i+1} - S_{i-1})} + \frac{b_i^n}{S_{i+1} - S_{i-1}} \right]. \quad (56)$$

For forward/backward differencing: ($b_i^n > 0/b_i^n < 0$)

$$\alpha_{i,forward/backward}^n = \left[\frac{2a_i^n}{(S_i - S_{i-1})(S_{i+1} - S_{i-1})} + \max \left(0, \frac{-b_i^n}{S_i - S_{i-1}} \right) \right]$$

$$\beta_{i,forward/backward}^n = \left[\frac{2a_i^n}{(S_{i+1} - S_i)(S_{i+1} - S_{i-1})} + \max \left(0, \frac{b_i^n}{S_{i+1} - S_i} \right) \right]. \quad (57)$$

References

1. L. Andersen, J. Andreasen, and R. Brotherton-Ratcliffe. The passport option. *Journal of Computational Finance*, 1(3):15–36, Spring 1998.
2. M. Avellaneda, A. Levy, and A. Parás. Pricing and hedging derivative securities in markets with uncertain volatilities. *Applied Mathematical Finance*, 2:73–88, 1995.
3. G. Barles. Convergence of numerical schemes for degenerate parabolic equations arising in finance. In L. C. G. Rogers and D. Talay, editors, *Numerical Methods in Finance*, pages 1–21. Cambridge University Press, Cambridge, 1997.

4. G. Barles and J. Burdeau. The Dirichlet problem for semilinear second-order degenerate elliptic equations and applications to stochastic exit time control problems. *Communications in Partial Differential Equations*, 20:129–178, 1995.
5. G. Barles, CH. Daher, and M. Romano. Convergence of numerical schemes for parabolic equations arising in finance theory. *Mathematical Models and Methods in Applied Sciences*, 5:125–143, 1995.
6. G. Barles and E. Rouy. A strong comparison result for the Bellman equation arising in stochastic exit time control problems and applications. *Communications in Partial Differential Equations*, 23:1995–2033, 1998.
7. G. Barles and P.E. Souganidis. Convergence of approximation schemes for fully nonlinear equations. *Asymptotic Analysis*, 4:271–283, 1991.
8. S. Basak and G. Chabakauri. Dynamic mean-variance asset allocation. Working Paper, London Business School, 2007.
9. Y. Bergman. Option pricing with differential interest rates. *Review of Financial Studies*, 8:475–500, 1995.
10. S. Chaumont. A strong comparison result for viscosity solutions to Hamilton-Jacobi-Bellman equations with Dirichlet conditions on a non-smooth boundary. Working paper, Institute Elie Cartan, Université Nancy I, 2004.
11. Z. Chen and P. A. Forsyth. A numerical scheme for the impulse control formulation for pricing variable annuities with a guaranteed minimum withdrawal benefit (GMWB). *Numerische Mathematik*, 109:535–569, 2008.
12. M. G. Crandall, H. Ishii, and P. L. Lions. User’s guide to viscosity solutions of second order partial differential equations. *Bulletin of the American Mathematical Society*, 27:1–67, 1992.
13. M. Dai, Y. K. Kwok, and J. Zong. Guaranteed minimum withdrawal benefit in variable annuities. *Mathematical Finance*, 18:595–611, 2008.
14. E. Derman and I. Kani. The ins and outs of barrier options: Part 1. *Derivatives Quarterly*, 3(Winter):55–67, 1996.
15. P. A. Forsyth and G. Labahn. Numerical methods for controlled Hamilton-Jacobi-Bellman PDEs in finance. *Journal of Computational Finance*, 11 (Winter):1–44, 2008.
16. H. J. Kushner and P. G. Dupuis. *Numerical Methods for Stochastic Control Problems in Continuous Time*. Springer-Verlag, New York, 1991.
17. H. E. Leland. Option pricing and replication with transaction costs. *Journal of Finance*, 40:1283–1301, 1985.
18. D. Li and W.-L. Ng. Optimal dynamic portfolio selection: Multiperiod mean variance formulation. *Mathematical Finance*, 10:387–406, 2000.
19. J. Lorenz. Optimal trading algorithms: Portfolio transactions, multiperiod portfolio selection, and competitive online search. PhD Thesis, ETH Zurich, 2008.
20. J. Lorenz and R. Almgren. Adaptive arrival price. In *Algorithmic Trading III: Precision, Control, Execution*, Brian R. Bruce, editor, Institutional Investor Journals, 2007.
21. T. Lyons. Uncertain volatility and the risk free synthesis of derivatives. *Applied Mathematical Finance*, 2:117–133, 1995.
22. M. A. Milevsky and T. S. Salisbury. Financial valuation of guaranteed minimum withdrawal benefits. *Insurance: Mathematics and Economics*, 38:21–38, 2006.
23. M. Mnif and A. Sulem. Optimal risk control under excess of loss reinsurance. Working paper, Université Paris 6, 2001.
24. H. Pham. On some recent aspects of stochastic control and their applications. *Probability Surveys*, 2:506–549, 2005.

25. D.M. Pooley, P.A. Forsyth, and K.R. Vetzal. Numerical convergence properties of option pricing PDEs with uncertain volatility. *IMA Journal of Numerical Analysis*, 23:241–267, 2003.
26. S. Shreve and J. Vecer. Options on a traded account: Vacation calls, vacation puts, and passport options. *Finance and Stochastics*, 4:255–274, Spring 2000.
27. J. Wang and P.A. Forsyth. Maximal use of central differencing for Hamilton-Jacobi-Bellman PDEs in finance. *SIAM Journal on Numerical Analysis*, 46:1580–1601, 2007.
28. X.Y. Zhou and D. Li. Continuous time mean variance portfolio selection: A stochastic LQ framework. *Applied Mathematics and Optimization*, 42:19–33, 2000.