Valuing Guarantees on Spending Funded by Endowments[*]

Y. Huang[†] P.A. Forsyth[‡] K.R. Vetzal[§]

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Abstract

Spending commitments by institutions such as colleges and universities or hospitals are frequently funded by endowments which are invested in risky assets. Many institutions use a simple endowment spending policy based on a maximum payout of a fixed fraction of a rolling average of the value of the endowment. However, periods of low investment returns on the endowment will reduce the amount available for disbursement. If this amount is less than the committed level of spending, the institution may be forced to make up the difference from other sources. For example, an endowed professorship at a university contains an implicit guarantee of a certain level of spending. If returns on invested capital are insufficient, the university must cover the deficit. To reduce the risk involved, some institutions have adopted a policy of setting aside surplus funds from periods of high returns in a reserve account which can be drawn upon in the event of a shortfall. We investigate the performance of this type of strategy. In particular, we determine the no-arbitrage value of guaranteeing a level of spending funded by an endowment that is invested in risky assets and which has a reserve account. Our results show that the reserve is not a panacea. For typical parameter values, the implied value of the guarantee is quite large.

Keywords: Endowment cash flows, valuation of guarantees, path-dependent contingent claims, spending rules

1 Introduction

The sustainability of long term spending commitments which are funded by risky investments has been the subject of considerable attention in the financial press over the past few years. The most commonly cited example is that of retirement programs such as defined benefit pension plans, but endowments and foundations are also part of the general picture.[1] The basic issue arises because

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[†]Department of Electrical and Computer Engineering, University of Waterloo, Waterloo ON, Canada N2L 3G1, `yqhuang@ecemail.uwaterloo.ca`

[‡]School of Computer Science, University of Waterloo, Waterloo ON, Canada N2L 3G1, `paforsyt@uwaterloo.ca`

[§]Centre for Advanced Studies in Finance, University of Waterloo, Waterloo ON, Canada N2L 3G1, `kvetzal@uwaterloo.ca`

[1]Examples of recent articles discussing problems with pensions include Lowenstein (2005), Levitt (2005), Arnott (2005), and Ezra (2005). With regard to issues facing endowments, see for instance Golden and Forelle (2002), Williams (2003), Piazza (2003), Kessler (2003), Hannon and Hammond (2003), and Chang (2005).

institutions (and individual investors) frequently have long term annual commitments which require spending at a rate in excess of that available on risk-free government debt instruments. This leads to funds being placed in risky assets such as equities, on the grounds that the higher risk premia available on such investments will support a higher level of spending. However, even if risky investments do earn relatively high returns on average over the long run, it is highly unlikely to be true each and every year. The very nature of risk is that low and even negative returns are quite possible. As pointed out by Dybvig (1999), when an endowment is invested in risky assets, there is

> ...a significant probability of a shortfall. To assert otherwise is to state that the fund is certain that stocks will go up and that going long stocks and short in the riskless asset is, in effect, a riskless arbitrage. Such cheerful optimism may be an appealing personality trait, but it is not a healthy attitude for an investment manager (p. 55).

In other words, despite this "cheerful optimism", periodic shortfalls are apt to arise because returns on risky investments are quite likely to occasionally fall below threshold rates necessary to support desired spending levels.

Despite the obvious importance of the topic (reflected in part by the number of related recent articles in the popular press), there has not been a great deal of academic research in this area. Arnott (2004) describes the attention given to the concept of a sustainable spending rate as "scant" (p. 6), whereas Milevsky and Robinson (2005) portray it as "sporadic" (p. 89). In any case, most of the extant research focuses on a somewhat different issue than is the case in this article. In particular, the existing literature has largely concentrated on the question of the rate at which funds can safely be withdrawn from an invested account to support present and planned future spending. This is clearly a critical issue, particularly in retirement planning. However, our focus is on the following related question. Given a plan to spend at a specified rate, what is the value of a guarantee to support that rate? In other words, what is the fair price of an insurance policy to protect a desired spending rate?

We investigate this issue in a particular context, that of an academic institution which has received a donation to fund an endowed research chair. The chairholder's salary is a commitment made by the institution: if returns on the invested donation fall short of the amount needed to cover this salary, the institution must make up the difference. We focus on this particular setting for a couple of reasons. First, as we will discuss in more detail below, some universities have adopted a policy of establishing a reserve account to act as a buffer to preserve spending in the event of poor investment returns. This practice allows us to study a particular and specific example of the vague but widely held notion that high investment returns in good years can be used to offset poor returns in bear markets. Second, we believe there is some value in considering a scenario that is not in the pension setting, simply to draw attention to the fact that the economics of this issue are more general than the widely publicized case of pensions. We emphasize, however, that although we cast our investigations in a particular academic setting, our results apply more broadly to the general question of the fair value of insurance on a spending rate.

It is worth noting that in the area of pensions in the United States, such insurance is provided by the Pension Benefit Guaranty Corporation (PBGC). According to Arnott (2005), unfunded liabilities guaranteed by the PBGC presently exceed $1 trillion (using the risk free yield curve for discounting). Moreover, the PBGC is not empowered to set premiums for this insurance, nor can it charge higher premia or deny coverage to the fiscally irresponsible. As a result, such insurance is highly likely to be mispriced and abused, with potentially disastrous financial consequences.

Let us now provide more background information about our particular institutional context of an endowed chair at a university. We begin by noting that many colleges and universities use a simple rule to target endowment spending. A typical spending rule is to allow a maximum disbursement of about five per cent of the twelve quarter average of total endowment funds. Any return in excess of the amount disbursed is added to the endowment principal. Originally suggested by the Ford Foundation in 1969, this spending rule generally works well in periods of healthy market returns. However, many endowment agreements require the university to maintain the real value of the principal. In this case, spending is typically reduced when the return on the endowment is less than five per cent plus an inflation adjustment.

As pointed out by Mehrling (2004), this approach is unsatisfactory for several reasons. During the era of high stock market returns of the 1990s, actual endowment returns were greatly in excess of five per cent. The five per cent rule then had the effect of ratcheting up the endowment principal, with an effective transfer of wealth to future generations. In recent times, with negative investment returns, spending from endowments has often been suspended, ostensibly to preserve the endowment capital. However, the capital may be artificially large, due to the ratcheting effect of the spending rule in times of high returns. This spending rule has also been criticized by Sedlacek and Clark (2003) on the basis that spending is too large during periods of high returns and too low in periods of low returns.

Williams (2003) and Mehrling (2004) suggest the use of a stabilization or reserve account which receives cash flows during periods of high returns, and can be drawn down in periods of low or negative investment returns. An alternative approach is advocated by Dybvig (1999), who suggests an investment strategy similar to constant proportions portfolio insurance. The total portfolio is split between risk free and risky assets, with the fraction placed at risk determined by the value of the total portfolio (more is invested in risk free assets if the overall portfolio value declines).

Although it appears that the idea of using a reserve account to smooth spending has gained some popularity, most universities seem unwilling to adopt the approach of Dybvig (1999). This may be because Dybvig's strategy would require universities to acknowledge that, in many cases, their committed endowment spending exceeds the amount which can be obtained from risk free investments. Another possible reason is that Dybvig's policy rule is designed to ensure that spending will never be cut, but this is achieved through lower initial spending. Such lower levels of spending may not be appealing to donors establishing endowments, and institutions often compete to attract donations by promising to spend a higher percentage of endowments.

In this paper, we concentrate on the reserve account idea. In particular, we focus on the class of spending rules which have the following two characteristics:

1. An attempt is made to ensure that the real value of the original endowment is maintained; and

2. A reserve fund is established to smooth out fluctuations in endowment returns, so that academic units can expect reliable cash flows.

The main idea here is that returns in excess of the inflation rate are either allocated to current spending or to a reserve fund. In this way, the transfer of wealth to future generations is avoided.

The policies of some institutions specify that if the endowment is underwater (i.e. the real value of the endowment is less than the original capital), then spending is halted until the real value of the endowment is restored. However, this is relatively rare. Most policies do not require a catch-up on returns to restore the real endowment value. Instead, in years when the endowment does not grow in real terms and the reserve fund is exhausted, spending stops. If the real return is positive in the following year, spending resumes. Virtually all universities following these types of spending rules have a cap on the size of the reserve fund. When the reserve fund reaches the cap, excess returns are then used to increase the principal of the endowment. It appears that the logic behind this approach is based on the idea that high return years (when the reserve fund is at its maximum) will restore the endowment capital.

Although in many situations university endowments are used to fund general expenses, it is also the case that donations are frequently used for specific purposes such as scholarships, construction and maintenance of buildings, or chaired professorships. Since we wish to focus on the effects of shortfalls from invested capital to fund committed spending, we will only consider the latter type of scenario. In particular, we take the point of view of an academic unit in a university. Consider the following situation. A donor has agreed to fund an endowed chair. The academic unit is informed that the expected real cash flow from the endowment is, for example, five per cent per year. On this basis, the unit hires a prominent professor and agrees to cover costs of salary and research support. These salary and research costs can be expected to increase at a known academic inflation rate. This represents a deterministic yearly liability over a lengthy time horizon (e.g. twenty years). However, this liability is funded by investing in risky assets, since it is not possible to obtain a real return of five per cent in risk free assets.[2] A reserve account is used to provide a cushion against poor investment returns. However, if the reserve is exhausted by years of low returns, then the academic unit must make up for the shortfall.

In this article, we will determine the no-arbitrage value of this implied guarantee. Using realistic parameters, it appears that the value of this guarantee is a substantial portion of the original endowed capital. The motivation for no-arbitrage valuation is as follows. Imagine a setting where the endowment is passively invested in an exchange-traded market index fund. If the academic institution were to approach a financial institution to purchase insurance for the implicit guarantees, standard practice would dictate that the cost of this insurance would be determined by the estimated cost for the financial institution of hedging the risk involved.[3] In principle, this would be the no-arbitrage value of the guarantee. Of course, this assumes that the underlying asset is known and tradeable. Other situations might well arise where this is not the case. For example, the endowment might be actively managed with high turnover of securities, in which case the financial institution would not be able to directly hedge its risk exposure with liquid market instruments. However, this would likely imply even larger guarantee values. In other words, our simple passive management with an indexed investing scenario can be seen as establishing a ballpark estimate, but one which is likely to be towards the lower end of the range of the cost of providing these guarantees in many practical situations.

[2]Even though inflation-indexed government bonds may be available, they would not offer a sufficiently high real return. Moreover, the academic inflation rate may be different from the general consumer price index used to determine cash flows on inflation-protected government debt instruments.

[3]Note that this is consistent with the idea expressed in Arnott (2005) that "the economic value of a liability ... should be calculated in such a way that an insurer would actually be willing to assume it".

2 Spending Rules

As discussed above, we focus on spending rules which use a reserve fund to smooth out disbursements. A few hours searching the internet with the keywords *university endowment, spending rule, stabilization* makes for informative reading. Many institutions have very *ad hoc* approaches. The spending rules clearly assume that the return on the endowment will be in excess of five per cent plus inflation, with perhaps some minor adjustments which can be handled by a small reserve fund (which may be limited to be as little as five per cent of the endowed capital). Once the reserve fund reaches its maximum size, excess returns are capitalized. It is interesting to observe that two or more years of negative endowment returns is regarded as extraordinary, and usually requires special intervention of the Board of Governors. Some institutions are quite specific about their spending rules. These institutions are usually very clear that no disbursements from the endowment can be expected if the endowment return is negative and the reserve fund is exhausted.

The spending rules of various universities have some or all of the characteristics described above.[4] We consider the following prototypical spending rule, which should not be considered as the exact rule used by any of these institutions, but rather is a model example.

The spending rule specifies a set of valuation dates $\{t_i\}$ (typically yearly). We assume that the valuation interval $\Delta t = t_{i+1} - t_i$ is constant. Let $S_i = S(t_i)$ be the level of endowed capital at $t = t_i$. Denote the value of the reserve fund at t_i by $R_i = R(t_i)$. Also let

$$\begin{aligned} I_i &= \text{ inflation factor in period } [t_i, t_{i+1}] = \exp[I_i^{rate}\Delta t], \\ I_i^{rate} &= \text{ inflation rate}, \\ C_r &= \text{ percentage cap on reserve fund}, \\ F_{sp} &= \text{ spending factor in period } [t_i, t_{i+1}] = F_{sp}^{rate}\Delta t, \\ F_{sp}^{rate} &= \text{ spending factor rate}. \end{aligned} \tag{2.1}$$

At valuation date t_{i+1} the real gain of the endowment over $[t_i, t_{i+1}]$, denoted by RG_{i+1}, is given by

$$RG_{i+1} = S_{i+1} - S_i I_i. \tag{2.2}$$

We assume either of two possibilities for the reserve fund. If the reserve fund is invested in risk free assets, then

$$R_{i+1} = R_i e^{r\Delta t} \tag{2.3}$$

where r is the continuously compounded nominal risk free rate of return, or the reserve fund can be invested in the same risky assets as the endowment fund

$$R_{i+1} = R_i \left(1 + \frac{S_{i+1} - S_i}{S_i}\right). \tag{2.4}$$

[4]See Williams (2003) for a description of the use of a reserve account at Wake Forest University. Among other universities to have adopted this general type of approach are California Polytechnic State University, North Carolina State University, Wilfrid Laurier University, Simon Fraser University, Ryerson University, Lakehead University, and the University of Waterloo. Details about the endowment spending policies of these various institutions may be found in California Polytechnic State University (2003); North Carolina State University (2004); Wilfrid Laurier University (2003); Simon Fraser University (1998); Ryerson University (2002); Lakehead University (2003); University of Waterloo (2005).

If RG_{i+1} is negative, the reserve fund R_{i+1} is drawn down to ensure that the real value of the endowment is preserved. If the reserve fund is exhausted, and $RG_{i+1} < 0$, then no disbursements are made.

If $RG_{i+1} > 0$, or $R_{i+1} > 0$ after transfers to the endowment principal account, then an attempt is made to disburse an amount $F_{sp}S_i$.[5]. This amount is first obtained by applying any positive real investment gain RG_{i+1}. If the real gain is insufficient to provide a cash flow of $F_{sp}S_i$, then the reserve fund R_{i+1} can be used to make up for the shortfall. However, the reserve fund is not allowed to go into a deficit, which means that the disbursement may be less than $F_{sp}S_i$.[6].

In any cases where the maximum amount of $F_{sp}S_i$ is disbursed, any remaining excess return is applied first to increase the reserve fund, to a maximum size of C_rS_i. Any excess return that cannot be used to increase the size of the reserve fund is then added to the endowment capital account.

There are many possible permutations of the above spending rule. As noted above, some institutions specify that an underwater endowment account (real value less than initial capital) has disbursements suspended until the real value is restored. Spending priorities (e.g. disbursement to the units or preservation of endowed capital) seem to vary considerably across various institutions.

In our example scenario, we consider the case of an endowed chair. In this case, we assume that the expenses associated with the chair simply increase with a known academic inflation factor. This liability is not directly tied to the size of the endowment capital. We will assume that the expense rate associated with this chair $E(t_i)^{rate} = E_i^{rate}$ has the following form

$$E_i^{rate} = E_0^{rate} \exp[A_f^{rate} t_i], \tag{2.5}$$

where E_0^{rate} is the initial expense and A_f^{rate} is the academic inflation factor. We make the simplifying assumption that the expense for the period $[t_{i-1}, t_i]$ is withdrawn from the endowment at t_i (there is no withdrawal at $t_0 = 0$). We denote this actual expense by E_i, which is given by

$$E_i = E_i^{rate} \Delta t. \tag{2.6}$$

In the following, we will refer to E_i as the promised cash flows of the endowment. Note that some institutions specify start-up rules, e.g. no spending is allowed until the endowment builds up several years of the promised cash flow in a reserve.

Let D_i be the disbursement associated with the spending rules described above. If $D_i > E_i$, then the excess amount is applied first to the reserve fund, and then to the endowment capital. If $D_i < E_i$, then this shortfall must be made up by the academic unit. Let $G(t_i) = G_i$ be the cash flow from the academic unit that must occur if there is a shortfall at t_i, so that

$$G_i = \max(0, E_i - D_i). \tag{2.7}$$

In the following, we will determine the no-arbitrage value of these cash flow guarantees.

Of course, the ultimate guarantor of these cash flows is the university as a whole. Although a given unit could simply consider that the university will fund the guarantee if there are a series of bad investment years, these cash flows must come from the base operating budget of the university, and hence represent a real cost.

[5]Note that we have simplified this rule. Often in practice the disbursement rule is based on an average value of the endowment over several valuation periods

[6]In practice, some institutions allow for borrowing, but we ignore this case

3 Terminal and Valuation Date Conditions

The endowment guarantee can be viewed as a path-dependent contingent claim. At any time t, the claim depends on $S(t)$, the current market value of the endowment; $P(t) = S(t_{i-1})$, $t_{i-1} < t < t_i$, the value of the endowment at the previous valuation date; and $R(t) = R(t_{i-1})$, $t_{i-1} < t < t_i$, the value of the reserve fund at the previous valuation date. Let $\hat{V}(S, P, R, t)$ be the value of the endowment guarantee. One of the objectives of the spending rules is to ensure that the real value of the endowment is preserved, so that the endowment pays out promised cash flows in perpetuity. However, we will consider the value of the guarantee over a specific time horizon T. For example, if the endowment is used to fund a chaired professorship, then T would be the expected time to retirement of the current chairholder. We then have

$$\hat{V}(S, P, R, t = T) = 0. \tag{3.1}$$

Note that this says that the value of the guarantee must be zero after the last cash flows are paid out. It does not preclude a cash payment arising from the guarantee immediately before the end of the time horizon T.

Let t_i^- and t_i^+ respectively denote the times the instant before and after valuation times t_i. Let (S_i, P_i, R_i) be the values of (S, P, R) at t_i^-. Let $(S_i^{sp}, P_i^{sp}, R_i^{sp})$ be the values obtained after applying the spending rules to (S, P, R) (i.e. at t_i^+). Then, the no-arbitrage value of the guarantee must satisfy

$$\hat{V}(S_i^{sp}, P_i^{sp}, R_i^{sp}, t_i^+) = \hat{V}(S_i, P_i, R_i, t_i^-) + G_i, \tag{3.2}$$

where $G_i = G(t_i)$ is the non-negative cash flow required to make up for any endowment disbursement shortfall, as in equation (2.7). (S_i^{sp}, R_i^{sp}) are functions of (S_i, P_i, R_i), as set out in the spending rules. Note that by definition $P_i^{sp} = S_i^{sp}$.

4 Model Formulation

In general, we assume that the value of the underlying endowment S follows a Poisson jump-diffusion process as in Merton (1976). Allowing for possible discontinuous jumps permits us to explore the effects of severe market crashes on the values of these guarantees. Note, however, that in most of the examples we consider, S will simply be assumed to follow geometric Brownian motion as in the standard Black-Scholes model (i.e. we will suppress any possible jumps). In particular, we assume that the risk neutral potential paths followed by S can be modeled by a stochastic differential equation given by

$$\frac{dS}{S} = (r - \lambda\kappa)dt + \sigma dz + (\eta - 1)dq, \tag{4.1}$$

where r is the risk free rate, dz is the increment of a standard Gauss-Wiener process, σ is the volatility associated with dz, dq is an independent Poisson process with mean arrival rate λ (i.e. $dq = 1$ with probability $\lambda\, dt$ and $dq = 0$ with probability $1 - \lambda\, dt$), $\eta - 1$ is an impulse function producing a jump from S to $S\eta$, and κ is the mean relative jump size (i.e. $\kappa = \int_0^\infty (\eta - 1)g(\eta)d\eta$ where $g(\eta)$ is the probability density function of the jump amplitude η.

Although we focus on no-arbitrage valuation (i.e. under the risk neutral probability measure, as in equation (4.1)), our methods may also be used to calculate the expected value of the guarantees under the real world probability measure. In cases where we are interested in this alternative measure, we assume that

$$\frac{dS}{S} = (\xi - \lambda^P \kappa^P)dt + \sigma dz + (\eta - 1)dq^P, \tag{4.2}$$

where ξ is the real world drift rate, λ^P is the real world mean arrival rate of the Poisson process, κ^P is the real world mean relative jump size (i.e. $\kappa^P = \int_0^\infty (\eta - 1) g^P(\eta) d\eta$, with $g^P(\eta)$ being the real world probability density function of the jump amplitude η), and dq^P is the independent Poisson process under the real world probability measure. We assume either process (4.1) or process (4.2) as appropriate for both the partial integro differential equation (PIDE) and Monte Carlo methods (as described below) for valuing the guarantee.

We now describe the PIDE formulation, beginning with the no-arbitrage value. Let $\hat{V}(S, R, P, t)$ be the value of a contingent claim that depends on the underlying endowment value S (and the auxiliary state variables R and P) and time t. Since we typically solve option pricing problems in terms of backwards time $\tau = T - t$, denote $V(S, R, P, \tau) = \hat{V}(S, R, P, T - t)$. For the moment, ignore any dependence on R and P. As these variables only change at observation times, for any particular values of R and P between observation times, the following backward PIDE determines the value of $V(S, R, P, \tau)$ (see,e.g. Merton, 1976; Wilmott, 1998; Andersen and Andreasen, 2000):

$$V_\tau = \frac{\sigma S^2}{2} V_{SS} + (r - \lambda\kappa) S V_S - rV + \left(\lambda \int_0^\infty V(S\eta) g(\eta) d\eta - \lambda V \right). \tag{4.3}$$

Although in principle it is possible to use any reasonable distribution for the jump amplitude η, in this paper we will restrict attention to the commonly used lognormal probability density function suggested by Merton (1976). If we denote the mean log jump size by μ and its standard deviation by γ, then the expected relative change in the stock price (conditional on a jump occurring) is given by $\kappa = E[\eta - 1] = \exp(\mu + \gamma^2/2) - 1$. Note that if we set suppress jumps by setting $\lambda = 0$ in (4.3), then the classical Black-Scholes partial differential equation for pricing European options is obtained.

In some cases it is also of interest to determine the expected value of the guarantee under the real world probability measure. Assuming that the real world process is (4.2), then the expected value is given by the solution of

$$V_\tau = \frac{\sigma S^2}{2} V_{SS} + (\xi - \lambda^P \kappa^P) S V_S - \rho V + \left(\lambda^P \int_0^\infty V(S\eta) g^P(\eta) d\eta - \lambda^P V \right), \tag{4.4}$$

where equation (4.4) contains the real world drift rate ξ, and the superscripts P refer to real world (i.e. P measure) quantities. For simplicity in the following, we will set the discount rate $\rho = r$. Hence we can interpret the solution to equation (4.4) as the negative of the amount which must be placed in a risk free investment to cover the expected loss of the guarantee. Note that from a computational standpoint, the PIDE for the no-arbitrage value and the expected value have the same form, so it is straightforward to calculate either value.

To complete the description of the PIDE formulation, we need to specify terminal and valuation date conditions. In terms of $V(S, P, R, \tau)$, the terminal condition (3.1) is

$$V(S, P, R, \tau = 0) = 0. \tag{4.5}$$

The valuation date conditions (3.2) become

$$V(S, P, R, \tau_i^+) = V(S^{sp}, P^{sp} = S^{sp}, R^{sp}, \tau_i^-) - G_i, \tag{4.6}$$

where τ_i^+ and τ_i^- are the instants before and after the valuation dates (with time running backwards). As before, we have that (S^{sp}, R^{sp}) are known functions of (S, P, R), as given by the spending rules, and $P^{sp} = S^{sp}$. Note that equation (4.6) differs from equation (3.2) since we solve the PIDE backwards in time.

Let $B(S, \tau)$ be the value of the guarantee assuming that the entire capital S is invested in a risk free asset. If we allow the holder of a short position in this guarantee (the academic unit) to optimally switch from investing in a risky asset (under the spending rules) or to simply invest the endowment in a risk free asset, then the valuation date condition (4.6) becomes

$$V(S, P, R, \tau_i^+) = \max\left[V(S^{sp}, P^{sp} = S^{sp}, R^{sp}, \tau_i^-), B(S^{sp} + R^{sp}, \tau_i^-)\right] - G_i. \tag{4.7}$$

We emphasize that equation (4.7) specifies the value if the investment is optimally switched once from being invested in risky assets to the risk free asset (i.e. we are not permitting multiple switches back and forth).

Note that PIDE (4.3) contains no derivatives with respect to (R, P). Hence equation (4.3) represents a set of one-dimensional PIDEs embedded in the three dimensional space (S, P, R). These one-dimensional PIDEs exchange information at valuation dates through the valuation date conditions (4.6) or (4.7).

As an independent check on the PIDE results, we also use a Monte Carlo method. This also allows us to examine the distribution of the guarantee value under both the risk neutral and the real world measure.

To determine the no-arbitrage value of the guarantee, the Monte Carlo method consists of the following three steps (see, e.g. Boyle et al., 1997):

1. Simulate sample paths of the underlying endowment according to equation (4.1).
2. Evaluate the discounted cash flows of the guarantee values on each sample path.
3. Average the discounted cash flows over all the sample paths.

If we are interested instead in the expected value under the real world probability measure, then we simply change step 1 above so that equation (4.2) is used instead. (This assumes we are interested in the expected value as of today; in some cases we are interested in it at T, and in these situations the cash flows are not discounted.)

As we have outlined two alternative formulations, an obvious question is which of them is more appropriate or efficient in what circumstances. In cases where the guarantee value over a range of initial capital values is of interest, then the PIDE method is suitable. As well, if optimal decision making is required, this is easily handled with a PIDE method. Alternatively, if the statistical

Parameter	Value
Promised cash flow rate E_0^{rate}	5
Initial reserve R_0	0
Reserve cap C_r	.15
Maximum spending rate F_{sp}^{rate}	.05
Time horizon (T)	20 years
Valuation frequency	yearly
Reserve investment	risk free
General inflation rate I_i^{rate}	.02
Academic inflation rate A_f^{rate}	.02
σ	.10
r	.04
λ	0

TABLE 5.1: *Base case parameters. Note that parameters such as the cash flow rate, spending rate, inflation rates, and the risk free interest rate are expressed in annual terms.*

properties of the guarantee value and the endowment capital are of interest, then the Monte Carlo technique is appropriate.

Appendix A provides technical implementation details for both of these numerical methods. We also outline some verification tests which show that both methods converge to the same solution. In the following, we show examples for various choices of the contract and market parameters, using either numerical PIDE or Monte Carlo methods as appropriate.

5 Numerical Examples

As a base case for our illustrative calculations, we use the parameter values given in Table 5.1. Note that we assume that $E_0^{rate} = 5$. We will examine the value of the implied guarantee for initial capital levels ranging from 0 to 250. To put this in perspective, this means that if we expect to be able to fund promised cash flows with a real return of 5%, this corresponds to an initial capital of 100. Note the implied shortfall—with an initial capital of 100, we are seeking nominal returns of 7% (5% real plus 2% inflation), but the nominal risk free rate is 4%.

Base case scenario. Figure 5.1 shows the no-arbitrage value of the guarantee, assuming the data in Table 5.1. The figure depicts the value of $V(S, S, R = 0, \tau = T)$, which is the initial value of the guarantee, assuming the reserve fund is zero at inception. We also show the value of the guarantee if the endowment is entirely invested in the risk free asset ($\sigma = 0.0$) as well as a high volatility case ($\sigma = .3$).

If we assume that the initial endowment is 100, which is just enough to fund the promised cash flows under the assumption of a 5% real return, then the value of the guarantee is -38.90 (assuming $\sigma = .10$). This clearly represents a substantial fraction of the initial endowment. In fact, if we double the size of the initial endowment, but keep the promised cash flows constant, then we only need a 2.5% real return to fund the cash flows. However, in this case, the value of

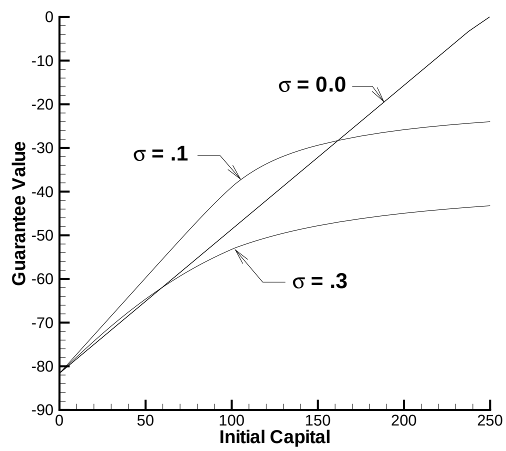

FIGURE 5.1: *The no-arbitrage value of the cash flows funded by the endowment. Base case parameters are given in Table 5.1. All examples used the base case parameters except where noted in the figure. The $\sigma = 0$ line represents investment of the endowment capital in a risk free asset.*

the guarantee is still substantial (-25.60). The value of the guarantee is also significantly higher if the endowment is invested in riskier assets ($\sigma = .30$). For low levels of the initial endowment, the guarantee is worth more if the funds are placed in the risk free asset. However, as the initial endowment increases, the guarantee eventually becomes worth more if the endowment is invested in risky assets (the crossover points being around 60 for the high volatility case and about 160 for the low volatility case). Note that all three cases intersect the vertical axis at around -81.60 when the initial endowment is zero. In this situation, the guarantee is simply a promise to pay out a real annuity over the next twenty years and its value can easily be calculated as follows. The annually compounded inflation rate is $e^{.02} - 1 = .020201$, and the annually compounded nominal discount rate is $e^{.04} - 1 = .040811$, implying a real discount rate of $1.040811/1.020201 - 1 = .020201$ (annually compounded). The present value of the annuity is then

$$5 \times \left[\frac{1 - 1.020201^{-20}}{.020201} \right] = 81.60.$$

To conclude our discussion of the base case scenario, we note that Figure 5.1 shows a feature which is counter-intuitive. Recall that the endowment is used to fund cash flows which increase by a known inflation factor. Consequently, we would expect that the value of the guarantee should approach zero as the initial capital becomes large. However, in Figure 5.1, the guarantee value increases (from a negative value) quite quickly at first (as a function of initial capital), but then rapidly levels off. In order to understand this phenomenon and to illustrate various other aspects of these guarantees, we will carry out a set of additional illustrative examples.

Reserve fund investment. We begin by considering the effect of assuming that the reserve fund is invested in the same risky assets as the endowment itself, as opposed to the base case situation where the reserve is placed in a risk free account. Figure 5.2 shows the results. For both volatilities

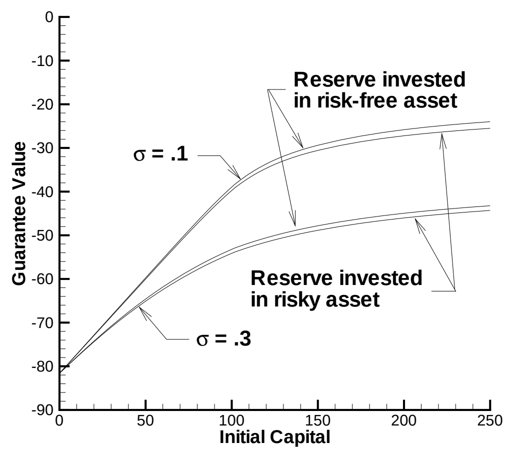

FIGURE 5.2: *Comparison of the effect on the no-arbitrage value of the guarantee if the reserve fund is invested in risk free assets or in the same risky assets as the endowment itself. Base case values from Table 5.1 are used unless otherwise indicated.*

considered, it appears to be slightly more advantageous to invest the reserve in risk free assets, but the effect is not very large. Moreover, the general pattern of the magnitude of the guarantee as a function of initial endowment capital falling rapidly for low levels of the endowment but falling at a much lower rate as the endowment level is increased persists, independent of how the reserve account is invested.

Reserve fund cap. We now explore the effect of varying the reserve fund cap C_r on the no-arbitrage guarantee value. The results are illustrated in Figure 5.3. Recall that C_r is the maximum percentage of endowed capital permitted to be in the reserve fund. Clearly, when $C_r = 0$ (no reserve fund), the academic unit is very exposed to risk. Increasing the size of the reserve cap is initially very beneficial, particularly for larger values of initial capital, but the effect rapidly tapers off as C_r is increased.

Optimal switching. What happens if we allow the academic unit to switch from being invested in risky assets to a risk free asset at each anniversary date of the inception of the endowment? Note that we only allow one actual switch, but we check whether it is optimal to switch at each anniversary date. In other words, the option to switch is of the Bermudan type. Figure 5.4 shows the guarantee value assuming that the academic unit makes the optimal choice (risky or risk free investment) in order to minimize the value of the guarantee. The figure indicates that it is optimal to switch to a risk free investment (from a no-arbitrage perspective) if the endowment is sufficiently large.

Jump diffusion. Thus far, we have explored situations where there are no discontinuous jumps in the model for the underlying risky asset. The effect of allowing such jumps is examined in Figure 5.5. This example uses the base case parameters and the jump parameters given in Table 5.2.[7] The

[7] The jump distribution parameters μ and γ indicate that the expected value of a jump is severely negative, but also that jumps have a large standard deviation. The parameter values used here correspond closely with those

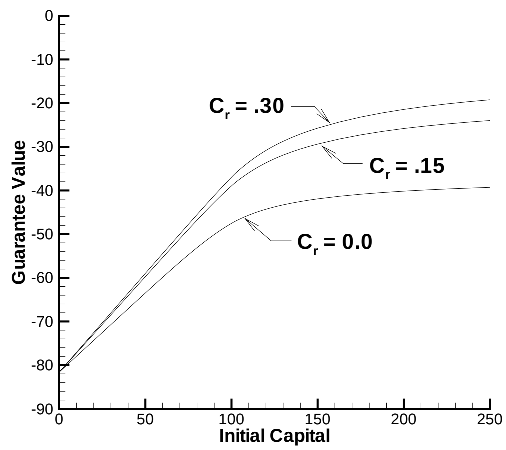

FIGURE 5.3: *The effect of the reserve fund cap C_r on the no-arbitrage guarantee value. The base case has $C_r = .15$. Base case parameters are provided in Table 5.1. All examples use the base case parameters except as noted in the figure.*

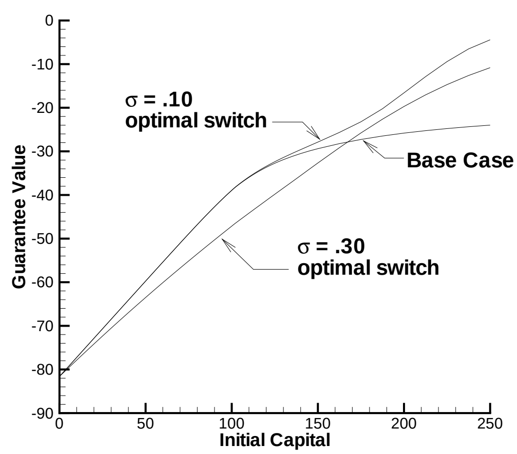

FIGURE 5.4: *The effect of adding the option to switch to a risk free investment on each anniversary date after the inception of the endowment. Base case parameters are provided in Table 5.1. All examples use the base case parameters except as noted in the figure.*

Parameter	Value
γ	.45
μ	-.9
λ	.1

TABLE 5.2: *Jump parameter values. The implied volatility which matches the price of a 20 year European call option at the strike priced under a jump diffusion model with the above parameters is* $\sigma_{imp} = .2811$.

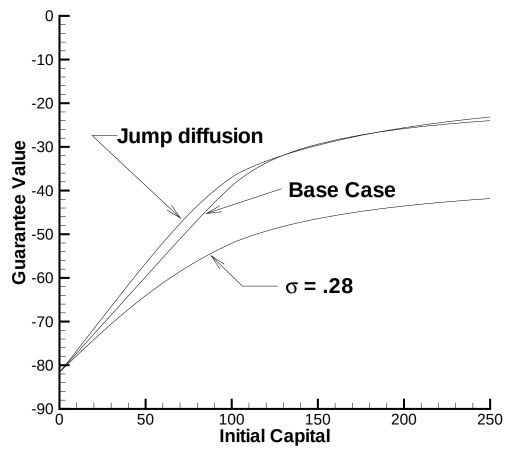

FIGURE 5.5: *The effect on the no-arbitrage guarantee value of assuming the underlying process follows a jump diffusion with parameters given in Table 5.2. All other parameters are base case parameters as in Table 5.1. Also shown is the base case result, as well as the guarantee value with a constant volatility model (no jumps) with* $\sigma_{imp} = .2811$. *This is the implied volatility which, in a no-jump model, matches the price of a 20 year at-the-money European call option under a jump diffusion model.*

guarantee under jump diffusion is very close to the base case solution. It is important to note that the jump diffusion case uses the same diffusive volatility σ as the base case. In contrast, we also show the value given by a no-jump model using the implied volatility which gives that same price as the jump model for a twenty year at-the-money vanilla European call option. Clearly, for this example, a diffusion model gives completely different results compared to the jump-diffusion model. Essentially, the spending rules are such that jumps do not have much effect on the guarantee, compared to an increase in volatility. In other words, as far as the endowment guarantee is concerned, jumps should not be modeled by using an effective volatility.

Underwater endowments. As noted above, some institutions enforce a no-spending policy for underwater endowments, i.e. endowments having a real value less than the initial capital. Figure 5.6

reported by Andersen and Andreasen (2000), which were found by calibrating the jump diffusion model to observed prices of S&P 500 index options.

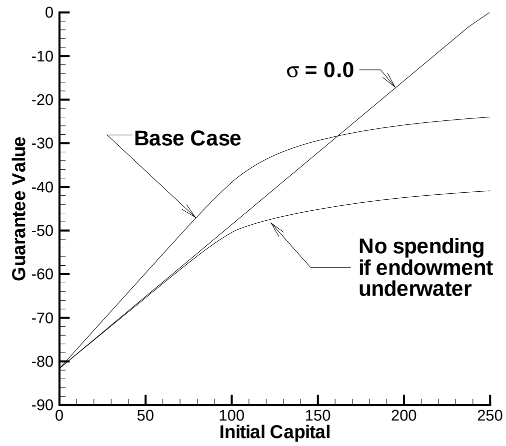

FIGURE 5.6: *The effect on the no-arbitrage guarantee value of not allowing spending if the real value of the endowment is less than the initial value. Base case parameters (Table 5.1 are used unless indicated otherwise.*

compares this policy with the base case spending rules. In this case, from the standpoint of the unit providing the guarantee, it is less costly (from a no-arbitrage point of view) to simply invest in a risk free asset with a certain loss.

Allocation of initial capital in reserve fund. Figure 5.7 shows the effect on the no-arbitrage guarantee value of different initial allocations between the risky investment and the reserve fund. The reserve cap $C_r = .20$ for these examples. All other parameters are base case values, as in Table 5.1. Increasing the allocation to the reserve decreases the risk, but the effect tapers off when the fraction of the initial capital invested in the reserve is above 15%.

Modified spending rule. While the examples presented thus far provide many interesting insights into the qualitative nature of various features of the value of the guarantee, none of them explain why the guarantee value tends to zero only very slowly as the initial capital increases. By re-examining the spending rule in our model, we see that the first priority is to maintain the real value of the endowment capital (many institutions specify this goal as part of their endowment policy). The reserve fund cannot be used to pay the promised cash flows unless the real value of the endowment capital has been preserved over the valuation interval.

To be more precise, let S_{targ_i} denote the target real value of the endowment capital at time t_i, where $S_{targ_i} = P_i I_{i-1}$. If $S_i - S_{targ_i}$ is negative, then the reserve fund R_i is drawn down first in an attempt to top-up the endowment to S_{targ_i}. The reserve fund is only used to pay the promised cash flow after top-up of the endowment fund to S_{targ_i}. Essentially, $(S_i - S_{targ_i} + R_i)$ rather than R_i alone is available to the academic unit to fund the promised cash flow.

To gain some insight into this effect, we examined many of the paths generated by Monte Carlo simulation. Figure 5.8 shows one typical sample path of $S - S_{targ_i}$, where S_{targ_i} is the target value of the endowment required to preserve the real capital during $[t_{i-1}, t_i]$.[8] We also show the R and

[8]The actual path shown was generated under the risk neutral probability measure, but any such path could also

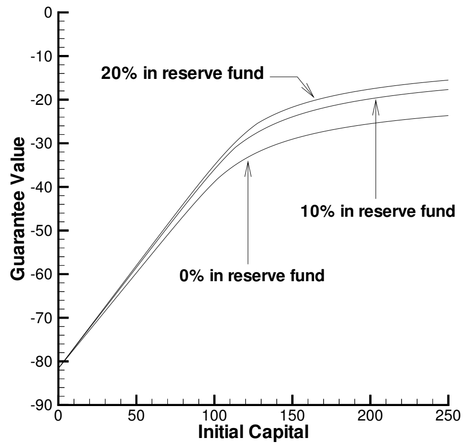

FIGURE 5.7: *The effect on the no-arbitrage guarantee value of different initial capital allocation to the reserve fund. The reserve cap* $C_r = .20$*. All other parameters are base case parameters, as in Table 5.1.*

$-G$ values. The base case parameters were used with initial capital of 200. More detailed data for this realized path can be found in Table A.2 in the appendix. Figure 5.8 clearly shows that most of the time the reserve fund is used to maintain the real value of the endowment fund. Although the reserve occasionally becomes fairly large (i.e. several times the yearly promised cash flow), this does not guarantee that the academic unit will be protected from deficit. This is essentially because the reserve fund is reduced to zero in a year when the endowment capital suffers a large loss. Although the reserve fund may hold several times the promised cash flows, a small relative loss of a large endowment capital can quickly eliminate the reserve.

Based on the above analysis, we modify the spending rule so that there is no obligation to maintain the endowment's real value on a year by year basis. To be more specific, when there is a year such that the endowment capital is less than the previous year's inflated value, we allow the reserve fund to be used to fund the promised cash flow. In reality, some universities do have such spending rules (see, e.g. Wilfrid Laurier University, 2003). Under this new spending rule, the absolute no-arbitrage guarantee value becomes significantly smaller when the initial capital is large. Figure 5.9 shows the effect of the modified spending rule on the no-arbitrage guarantee value. The figure clearly indicates that a spending rule which attempts to preserve the endowment capital (on a year by year basis) transfers risk to the academic unit. Under the original spending rule, this risk reduces only very slowly as the initial capital becomes large.

Table 5.3 shows the mean values of the endowment fund at the end of the time horizon (twenty years) under both the original and the modified spending rules. We can see that when the initial capital is small, both rules are unsuccessful in maintaining the real value of the initial endowment.

be obtained under the real world probability measure. In other words, because the risk neutral measure and the real world measure are equivalent, the set of possible paths under each measure are identical. The effect of the measure change is merely to alter the probabilities of the paths.

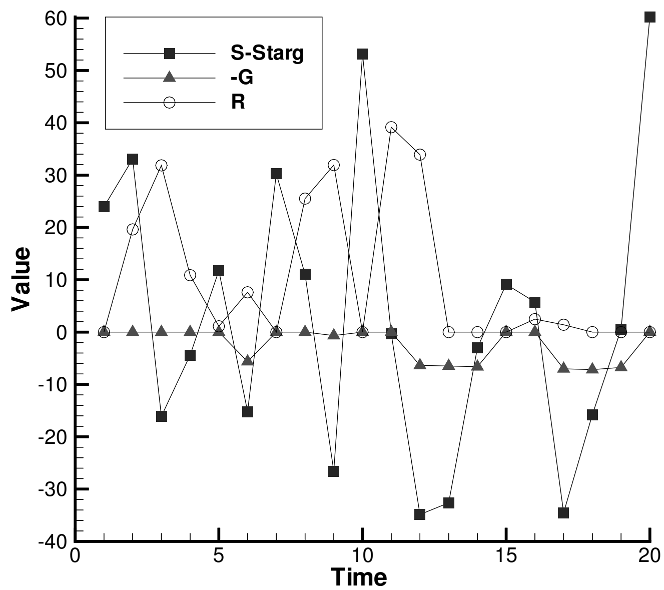

FIGURE 5.8: *A sample path with initial capital value of* 200*. Remaining parameters are base case (Table 5.1) unless otherwise indicated.* $S - S_{targ}$ *is the value of the capital account which exceeds the value required to preserve the real value of the endowment.*

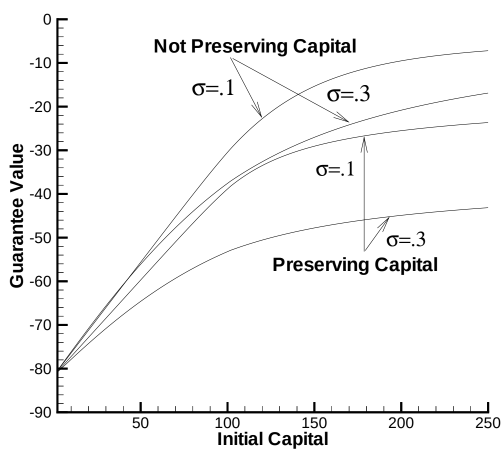

FIGURE 5.9: *The effect of not preserving the endowment real value on the no-arbitrage guarantee value. When the return on the endowment is less than inflation in any given year, spending is allowed from reserve fund (denoted by* not preserving capital*), compared to the base case spending rule (denoted by* preserving capital*). Remaining parameters are base case (Table 5.1) unless otherwise indicated.*

S_{init}	Inflated S_{init}	Mean
Preserving the initial capital		
100	149.18	121.24
200	298.36	299.73
Not preserving the initial capital		
100	149.18	101.14
200	298.36	255.84

TABLE 5.3: *Mean values of the endowment fund at $T = 20$ years under the risk-neutral probability measure. Parameters are base case (Table 5.1) unless otherwise indicated. S_{init} stands for the initial capital value. Inflated S_{init} refers to S_{init} increased by the continuously compounded inflation rate I^{rate} after $T = 20$ years.*

However, when the initial capital is large, the original spending rule is more successful at maintaining the real value of the initial capital, whereas the modified spending rule still leads to a loss in the real value of the initial capital.

Expected value of guarantee. All of the results reported thus far have been in terms of the no-arbitrage value of the guarantee. As explained earlier, this type of analysis is applicable when the portfolio of risky assets that the endowment is invested in is known and tradeable, for example, a passive investment in an exchange-traded market index. Alternatively, if the endowment is invested in an actively managed fashion, the composition of the portfolio may be unknown (to anyone except the fund managers), and the no-arbitrage value would not be an appropriate measure. As an alternative, we can calculate the expected value of the guarantee. As described above, the expected value of the guarantee is simply obtained under the real-world probability measure by using the real drift rate ξ (and the other P measure quantities such as $\lambda^P, \gamma^P, \mu^P$ in equation (4.4)) in either the PIDE or the Monte Carlo methods (where we also assume that the guarantee is discounted by the risk free rate). Recall that we can interpret the expected value as the negative of the amount which should be invested in a risk free account to cover the expected value of the guarantee. In the following, we will restrict attention to the no-jump case ($\lambda^P = 0$). Consequently, the only parameter which must be estimated is the real world drift ξ in equation (4.4).

If the commonly used real return target of 5% is assumed, this implies a drift rate of 7% assuming constant inflation of 2%. If we assume that the CAPM equation

$$\xi = r + \lambda_R \sigma \tag{5.1}$$

holds, where λ_R is the market price of risk, then for a volatility of $\sigma = .10$ and a risk free rate of $r = .04$, we have a market price of risk of $\lambda_R = .3$. Figure 5.10 shows the expected value of the endowment guarantee, where the underlying endowment has the volatilities $\sigma = \{0, .1, .3\}$ and the drift rates are given by equation (5.1) with $\lambda_R = .3$. These results are obtained using the PIDE approach. Comparing Figures 5.10 and 5.1, we see that the expected absolute values of the guarantee are somewhat lower than the no-arbitrage values (provided $\sigma \neq 0$). This reflects the stronger upward drift of the underlying asset under the real world probability measure.

Alternatively, the Monte Carlo method can be used to give an idea of the distribution of the expected guarantee values. Table 5.4 summarizes the statistical properties of the distribution of

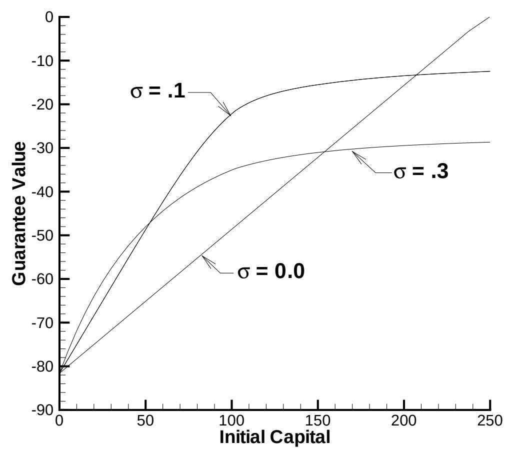

FIGURE 5.10: *The expected value of the guarantee. All cases shown use the base case parameters (Table 5.1) except where indicated. The drift rate is given by equation (5.1) with* $\lambda_R = .3$.

the expected guarantee value for the base case with parameters in Table 5.1. The drift rate is given in equation (5.1) with $\lambda_R = .3$ and $\sigma = \{.1, .3\}$. 1,000,000 Monte Carlo simulations were used. Table 5.4 also shows the 95% conditional value-at-risk (CVaR), i.e. the mean of the 5% worst outcomes. Here the 95% CVaR value turns out to be -52.22 when $\sigma = .1$, which is highly significant compared to the initial capital of 100. If the funds invested are placed in riskier assets ($\sigma = .3$), the 95% CVaR increases to almost 70% of the initial capital.

Recall that by changing the spending rule so that there is no obligation to preserve the endowment fund real value on a year by year basis, the no-arbitrage guarantee value becomes significantly closer to zero. Table 5.5 presents the statistical properties of the expected guarantee value under this modified spending rule. Comparing Tables 5.5 and 5.4, we can see that although the magnitude of the expected guarantee values have been significantly reduced (i.e. become much closer to zero) under the new rule, the absolute value of the 95% CVaR remains very large.

Table 5.6 shows the statistical properties of the endowment fund capital after 20 years under the real world probability measure. With the modified spending rule, on average the real value of the endowment fund is increased from its original level at the inception of the endowment. However, the real increase is less than that observed under the original spending rule. Consequently, even

σ	Mean	Std. Dev.	95% CVaR
0.1	-22.04	13.26	-52.22
0.3	-35.04	15.46	-69.52

TABLE 5.4: *Statistical properties of the distribution of the expected guarantee value for the base case with parameters in Table 5.1. The drift rate is given by equation (5.1) with* $\lambda_R = .3$ *and the initial capital* $S_{init} = 100$.

σ	Mean	Std. Dev.	95% CVaR
0.1	-14.92	12.66	-46.01
0.3	-17.13	17.87	-62.05

TABLE 5.5: *The effect of the modified spending rule of not preserving the real value of the capital on a year by year basis. Statistical properties of the distribution of the expected guarantee value are given for the base case with parameters in Table 5.1. The drift rate is given by equation (5.1) with* $\lambda_R = .3$ *and the initial capital* $S_{init} = 100$.

S_{init}	Inflated S_{init}	Mean	Std. Dev.
Preserving the initial capital			
100	149.18	196.91	84.49
200	298.36	521.20	231.07
Not preserving the initial capital			
100	149.18	171.32	82.14
200	298.36	475.68	231.11

TABLE 5.6: *Statistical properties of the endowment fund capital at the end of twenty years under the real world probability measure. Parameters are base case (Table 5.1), unless otherwise noted.* S_{init} *stands for the initial capital value.*

if we examine the expected value of the guarantee under the real world measure, the effect of the original spending rule is to transfer wealth to future generations and to transfer risk to the academic units. The modified spending rule also results in a real increase in the endowment capital and subsequent transfer of risk. Note that in all cases the standard deviation of the value of the endowment principal after 20 years is about 50% of the expected value of the principal.

6 Summary and Conclusion

The examples presented above illustrate the fact that the no-arbitrage value of the endowment guarantee is surprisingly large. To put this in context, consider the following scenario. A donation of $3 million is received for an endowed chair. Assuming an expected real return of 5%, an endowment of this amount is projected to generate $150,000 per year (in constant dollars). Based on this, an academic unit recruits a prominent 45 year old professor to fill the endowed chair. The salary and benefits associated with this position are currently about $150,000 per year. This amount will, of course, increase at an academic inflation rate.

Now, suppose the endowment invests in low risk assets and follows the spending rules according to the parameter values in Table 5.1. From Figure 5.1, we see that for every $100 of initial capital the guarantee value of meeting an initial $5 expense per year (which increases with inflation) is about $40. In other words, we see that the no-arbitrage value of the implied salary and benefit guarantee over the 20 years until the retirement of the professor is about 40% of the original capital.

More generally, we conclude with the following observations. The typical university endowment spending rule (a cap of 5% of the 12-quarter average endowment capital, with reduced spending during periods of low market returns), has been under criticism due to intergenerational transfers of wealth and cyclical spending (Mehrling, 2004; Sedlacek and Clark, 2003). An alternative approach, which uses a reserve fund, attempts to achieve the following goals:

- preservation of the real value of the endowment;
- participation in the higher returns of risky investments (equities); and
- risk reduction by using the reserve fund to smooth out possible short-falls in promised cash flows.

Based on an analysis of the spending rules at several institutions, we have developed a model spending rule which captures many of the features of this alternative approach. We consider a typical situation where an academic unit uses the endowment returns to fund fixed cash flows which increase at a (relatively) predictable academic inflation rate. An example of this situation would be the funding of the salary and benefits associated with an endowed chair.

Investing in risky assets means that there is some probability of a shortfall in meeting the annual promised cash flows. The institutional guarantee being provided to ensure that the promised cash flows are met can be valued using a no-arbitrage approach. It is common to assume that the endowment will achieve a 5% real return. Based on typical assumptions we find that in the situation where this expected return is just sufficient to fund the yearly promised cash flows, the no-arbitrage value of the guarantee is about 40% of the original endowed capital.

This result is perhaps not altogether surprising. As suggested by Dybvig (1999), it is somewhat foolhardy to attempt to meet promised cash flows using an endowment which cannot fund these cash flows via risk free investments. In the above example, the initial endowment is such that the real risk free return is insufficient to fund the promised cash flows. The higher expected returns of investment in equities can only be achieved by taking on risk. The cost of this risk is given by the no-arbitrage value of the guarantee.

To make another analogy, as beginning students of finance are taught, leverage (whether of the financial or the operating variety) increases risk. A firm which is highly levered has large amounts of fixed spending, and funding these commitments from variable operating revenues is very risky. The situation of such a firm is quite similar in certain respects to an institution attempting to meet fixed commitments while depending on variable revenues derived from risky financial investments. It would not seem to be very surprising to find out that the cost of avoiding this risk through an insurance policy could be quite high.

It is interesting to observe that the size of the guarantee value decreases very slowly as the initial endowment capital is increased. This is due to the priority given to preserving the real value of the endowment on a yearly basis. If this priority is removed, then the guarantee value more rapidly approaches zero as the initial endowment capital is increased.

If we examine the expected value of the guarantee, as opposed to the no-arbitrage value, then the situation appears somewhat less gloomy. Although the expected value of the size of the guarantee is smaller, especially if we remove the priority on yearly capital preservation, the risk, as measured by the 95% CVaR, remains large.

We have only considered the effects of investment in risky assets, assuming that interest rates and inflation rates are known constants. In practice, stochastic interest rates and inflation rates will introduce additional sources of risk in managing the promised cash flows from the endowment. Moreover, all of our illustrative calculations are for the case of a fixed time horizon, but in most situations the endowment is expected to provide funds in perpetuity. These effects (extra risk factors and infinite time horizon) will both imply that the cost of guaranteeing that a given level of spending can be maintained is likely to be somewhat higher than our estimates.

In summary, these illustrative computations show that apparently reasonable endowment spending rules can have unexpected side effects. These side effects include intergenerational transfer of wealth, and institutional assumption of a significant level of risk due to the implied guarantee on the promised cash flows. Currently, it appears that most universities are unaware of the cost of this risk, and are perhaps being overly optimistic in their assumptions about endowment cash flows.

Technical Appendix

A Numerical Methods

This Appendix provides technical details about the numerical methods used to value the endowment guarantee. We use both a numerical PIDE method and a Monte Carlo method.

A.1 Numerical Solution of the PIDE

Recall that the PIDE for the value of the guarantee $V(S, R, P, \tau)$ is given by equation (4.3). For computational purposes, we consider the finite domain

$$\begin{aligned} 0 \leq & P \leq P_{\max} \\ 0 \leq & S \leq S_{\max} \\ 0 \leq & R \leq R_{\max}. \end{aligned} \tag{A.1}$$

The domain (A.1) assumes that the reserve is not allowed to become negative. In cases where no spending is allowed if the real value of the endowment is less than the initial value we extend the range of R values so that $-R_{\max} \leq R \leq R_{\max}$. Negative R values can be used to track the deficit in the capital account. However, for ease of exposition, we will focus on the typical case where the reserve fund is nonnegative.

Note that the PIDE (4.3) is independent of the new state variables (P, R). Consequently, we can discretize the state variables as $\{P_1, \ldots, P_j, \ldots, P_{j_{\max}}\}$ and $\{R_1, \ldots, R_k, \ldots, R_{k_{\max}}\}$. For each discrete value of (P_j, R_k), we can solve the one dimensional PIDE (4.3) at times between the valuation dates. To move the solution across an observation date, we use the valuation date conditions (4.6) or (4.7).

For fixed (P_j, R_k), each one dimensional PIDE (4.3) is a function of (S, t) only. Our numerical calculations utilize Crank-Nicolson timestepping with the modification suggested by Rannacher (1984). Other details of the discretization can be found in d'Halluin et al. (2005). In situations where a jump diffusion model was used, the discrete algebraic equations are solved using a fixed point iteration combined with a fast Fourier transform evaluation of the integral term in the PIDE (4.3). This is described in detail in d'Halluin et al. (2005). The tolerances for all iterative

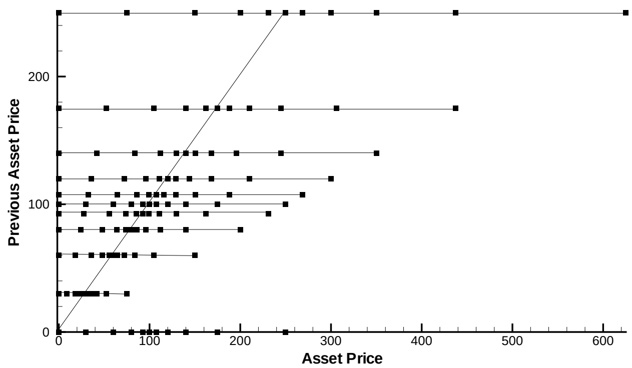

FIGURE A.1: *Locally refined grid near $S = P$.*

methods (within each timestep) were set to ensure that the error in the solution of the discretized equations did not affect the first six significant digits of the solution.

With regard to the mesh for the R variable, there are no particularly noteworthy issues. Suppose that we use an S grid with $\mathcal{S}_g = \{S_1, \ldots, S_i, \ldots, S_{i_{\max}}\}$ and a P grid, $\mathcal{P}_g = \{P_1, \ldots, P_j, \ldots, P_{j_{\max}}\}$, with $\mathcal{P}_g = \mathcal{S}_g$ (i.e. a Cartesian product $S \times P$ grid, with the same node spacing in the S and P directions). In this case, no interpolation in the S or P directions is required during the application of the state variable updating rule

$$P^{sp} = S^{sp}. \tag{A.2}$$

However, Windcliff et al. (2001) show that this type of grid results in poor convergence for shout options. Normally, we choose a fine node spacing near $S = S_{init}$, since this is the region of most interest. However, since the nodes $P = S$ for all values of S are required during the application of the jump condition (A.2), these values may have poor accuracy in areas where the S node spacing is large. It is therefore desirable to have a fine node spacing in the S direction for all nodes near $S_i \simeq P_j$, the diagonal of the (S, P) grid. Such a grid is shown in Figure A.1.

For this type of grid, application of the jump conditions (4.6) will normally require interpolation of the discrete solution values. We use linear diagonal interpolation in the (S, P) plane, as described in Windcliff et al. (2001), and linear interpolation in the R direction. Since observation times are independent of the timestep size of the PIDE solve, this will result in global second order convergence (Forsyth et al., 2002).

Away from valuation dates, each one dimensional PIDE is independent of the other PIDEs, hence we need only consider the boundary conditions at $S = 0$ and $S = S_{\max}$. No boundary condition is required at $S = 0$, since the PIDE becomes an ordinary differential equation at that point. At $S = S_{\max}$, we apply the condition $V_{SS} = 0$. See d'Halluin et al. (2005) for details on applying these boundary conditions to PIDE (4.3). Note that our grid construction method (as in Figure A.1) ensures that for each fixed (P_j, R_k), $S_{\max} \gg P_j$, so that it is reasonable to apply the condition $V_{SS} = 0$ at $S = S_{\max}$.

At observation dates, we note that no information is required for $P, R < 0$. However, for a fixed choice of $S_{\max}, R_{\max}, P_{\max}$, there will be some cases (particularly when $S > P_{\max}$) when the

$P_{\max} = 10000, R_{\max} = 5000$			$P_{\max} = 100000, R_{\max} = 50000$		
Nodes ($n_x \times n_y \times n_z$)	Timesteps	Guarantee value at $S = 100$	Nodes ($n_x \times n_y \times n_z$)	Timesteps	Guarantee value at $S = 100$
$59 \times 59 \times 18$	240	-39.15	$63 \times 63 \times 22$	240	-39.15
$117 \times 117 \times 35$	480	-38.96	$126 \times 126 \times 43$	480	-38.96
$233 \times 233 \times 69$	960	-38.90	$251 \times 251 \times 85$	960	-38.90

TABLE A.1: *Convergence test of PIDE method. Base case parameters as in Table 5.1. n_x, n_y, n_z refer to the number of nodes in the S, P and R directions respectively.*

spending rules will require evaluation of the solution outside the computational domain. In these cases, we modify the valuation date conditions (4.6) to be

$$V(S, P, R, \tau_i^+) = V(\min(S^{sp}, S_{\max}), \min(P^{sp} = S^{sp}, P_{\max}), \min(R^{sp}, R_{\max}), \tau_i^-) - G_i. \quad \text{(A.3)}$$

Effectively, we can think of the modified valuation date conditions (A.3) as altering the spending rules so that there is a cap on the absolute value of the endowment principal and the reserve fund. If $S_{\max}, P_{\max}, R_{\max}$ are sufficiently large, these states will have a very low probability and hence will have little impact on the solution in states of interest. This will be verified in some numerical tests below.

We conclude our discussion of the PIDE pricing algorithm by providing some convergence tests. These tests use the base case parameters from Table 5.1. Note that $\lambda = 0$, indicating that the base case has no jumps. Table A.1 shows the results for a sequence of grids. Each fine grid is constructed by inserting nodes halfway between each coarse grid node. The timestep size on each fine grid is one half the timestep size on the preceding coarse grid. Table A.1 also shows the effect of applying valuation date condition (A.3) when the spending rule requires data outside the computational domain. The effect of finite $P_{\max}, R_{\max}$ when $P_{\max} = 10000, R_{\max} = 5000$ on the solution at S values of interest is clearly very small (the solution is unaffected to four digits by increasing $P_{\max}, R_{\max}$ by a factor of ten). We will use $P_{\max} = 10000, R_{\max} = 5000$ for all subsequent computations.

A.2 Monte Carlo Methods

We now turn to describing the Monte Carlo algorithm. To simulate a sample path of a jump diffusion, we take the approach of simulating the jump times $0 < t_1^* < t_2^* < \dots$ explicitly (Glasserman, 2004), where $\{t_j^*, j = 0, 1, 2, \dots\}$ is the j^{th} random arrival time of the jump. For ease of notation, assume that we are using process (4.1) to determine the no-arbitrage value of the guarantee. Since we assume z and q in the process (4.1) are independent of each other, $S(t)$ evolves as an ordinary geometric Brownian motion from one jump time to the next. When a jump occurs, $S(t)$ has a jump amplitude of $\eta(t)$. To be more precise, equation (4.1) can be re-written in the following form:

$$\begin{aligned} \frac{dS}{S} &= (\xi - \lambda\kappa)dt + \sigma dz, \text{ if a jump does not occur },\\ &= (\xi - \lambda\kappa)dt + \sigma dz + (\eta - 1), \text{ if a jump occurs.} \end{aligned}$$

Let t_j^{*-} denote the instant before the j^{th} jump event occurs, and let $\Delta t_{j+1}^* = t_{j+1}^* - t_j^*$ denote the time interval between the $(j+1)^{th}$ and j^{th} jump event. Then, we have the following exact solution of $S(t)$:

$$S(t_{j+1}^{*-}) = S(t_j) \exp\left[\left(\xi - \lambda\kappa - \frac{\sigma^2}{2}\right)\Delta t_{j+1}^* + \sigma\phi\sqrt{\Delta t_{j+1}^*}\right], \tag{A.4}$$

and

$$S(t_{j+1}^*) = S(t_{j+1}^{*-})\eta(t_{j+1}), \tag{A.5}$$

where ϕ is a random variable drawn from a standard normal distribution. With the assumption that dq is a Poisson process having the mean arrival rate of λ, the arrival times $\{\Delta t_j^*, j = 1, 2, 3, \dots\}$ are i.i.d. exponential random variables having mean of $1/\lambda$. By generating exponentially distributed random numbers, we can determine the jump times.

Recall that we also assume that $\eta \sim \text{lognormal}(\mu, \gamma)$ and η is independent of dq and dz. Consequently, the jump amplitude is generated by drawing a random number that is lognormally distributed.

As before, let the observation times be denoted by t_i, with $\Delta t = t_{i+1} - t_i$. Algorithm 1 describes the steps to generate one sample path of jump diffusion in the observation interval of (t_i, t_{i+1}).

Algorithm 1 Generating one sample path of jump diffusion from t_i to t_{i+1}^-

Require: $\Delta t^* + t_i$ is the next jump time after t_i
$S \Leftarrow S_i^{sp}$, $T_{elapsed} \Leftarrow \Delta t^*$
if $T_{elapsed} > \Delta t$ **then**
  generate standard normally distributed random number ϕ
  $S \Leftarrow S\exp[(\xi - \lambda\kappa - \frac{\sigma^2}{2})\Delta t + \sigma\phi\sqrt{\Delta t}]$
  $\Delta t^* \Leftarrow \Delta t^* - \Delta t$
else
  while $T_{elapsed} < \Delta t$ **do**
    generate standard normally distributed random number ϕ
    $S \Leftarrow S\exp[(\xi - \lambda\kappa - \frac{\sigma^2}{2})\Delta t^* + \sigma\phi\sqrt{\Delta t^*}]$
    generate lognormally distributed random number R, where $R \sim LN(\mu, \gamma)$
    $S \Leftarrow SR$
    generate Δt^* from the exponential distribution with mean $1/\lambda$
    $T_{elapsed} \Leftarrow T_{elapsed} + \Delta t^*$
  end while
  $\Delta t_{left} \Leftarrow \Delta t - (T_{elapsed} - \Delta t^*)$
  generate standard normally distributed random number ϕ
  $S \Leftarrow S\exp[(\xi - \lambda\kappa - \frac{\sigma^2}{2})\Delta t_{left} + \sigma\phi\sqrt{\Delta t_{left}}]$
  $\Delta t^* \Leftarrow T_{elapsed} - \Delta t$
end if
$S_{i+1} \Leftarrow S$
return Δt^* $\{\Delta t^* + t_{i+1}$ is now the next jump time after $t_{i+1}\}$

For the special case of $\lambda = 0$, Algorithm 1 can be simplified as a simulation of an ordinary geometric Brownian motion from t_i to t_{i+1}^-. Because there are no jumps involved, we do not need to compute or return Δt^*. In this simplified setting we can simply calculate S_{i+1} using

$$S_{i+1} = S_i^{sp} \exp[(\xi - \frac{\sigma^2}{2})\Delta t + \sigma\phi\sqrt{\Delta t}]. \tag{A.6}$$

Algorithm 1 generates a sample path of jump diffusion from the instant after the previous valuation to the instant right before the current observation. Then we apply the spending rule in equation (2.7) to compute G_{i+1} and to update $(S_{i+1}^{sp}, P_{i+1}^{sp}, R_{i+1}^{sp})$ based on the values of (S_i, P_i, R_i).

Let G_i^m denote the non-negative cash flow required to make up for any endowment disbursement shortfall at time t_i on the m^{th} simulated path, with $N = T/\Delta t$ being the number of valuation dates. Then the discounted cash flow V^m on m^{th} simulation can be computed using

$$V^m = -\sum_{i=1}^{i=N} G_i^m e^{-rt_i}. \tag{A.7}$$

Suppose there are M simulated sample paths. Setting $\xi = r$ gives the no-arbitrage value of the endowment guarantee by averaging V_m over all the sample paths

$$\hat{V}(S, S, R, t = 0) = \frac{1}{M} \sum_{m=1}^{m=M} V^m. \tag{A.8}$$

Alternatively, the expected value of the guarantee is given by using the real world drift ξ in equation (A.6). As for the PIDE case, we use r for the discount rate (recall that we are computing the amount which should be placed in a risk free reserve to fund the expected value of the guarantee).

Table A.2 shows a representative sample path for the base case scenario with an initial capital value of 200. This is the same realized path shown in Figure 5.8.

We conclude our discussion of the Monte Carlo technique with some convergence tests. Table A.3 shows the results for a sequence of tests performed with an increasing number of sample paths (without jumps). The base case no-arbitrage value was computed using the data in Table 5.1. The results in Table A.3 should be compared to the PIDE results in Table A.1. Assuming that the PIDE method is converging quadratically, then extrapolation of the results in Table A.1 gives a value of -\$38.88. This is in close agreement with the value of -\$38.89 from Table A.3.

Figure A.2 shows the base case no-arbitrage guarantee value versus number of simulation paths by Monte Carlo methods.

References

Andersen, L. and J. Andreasen (2000). Jump-diffusion processes: Volatility smile fitting and numerical methods for option pricing. *Review of Derivatives Research* 4, 231–262.

Arnott, R. D. (2004). Sustainable spending in a lower-return world. *Financial Analysts Journal* 60(5), 6–9.

Arnott, R. D. (2005). The pension problem: On demographic time bombs and odious debt. *Financial Analysts Journal* 61(6), 12–17.

t_i	S_i	S_{targ_i}	S_i^+	$S_i - S_{targ_i}$	R_i	R_i^+	$-G_i$	E_i
0	200							
1	227.99	204.04	204.04	23.95	0	18.85	0	5.10
2	241.24	208.16	225.02	33.08	19.62	30,61	0	5.20
3	213.51	229.60	229.60	-16.09	31.86	10.46	0	5.31
4	229.84	234.24	234.23	-4.40	10.88	1.07	0	5.42
5	250.69	238.96	238.97	11.73	1.11	7.31	0	5.53
6	228.6	243.80	236.21	-15.20	7.61	0	-5.64	5.64
7	271.25	240.98	240.99	30.27	0	24.51	0	5.75
8	256.89	245.86	245.85	11.03	25.51	30.68	0	5.87
9	224.21	250.82	250.82	-26.61	31.93	0	-0.66	5.99
10	309.05	255.89	265.32	53.16	0	37.62	0	6.11
11	270.33	270.68	270.68	-0.35	39.16	32.58	0	6.23
12	241.29	276.15	275.20	-34.86	33.91	0	-6.36	6.36
13	248.13	280.76	248.13	-32.63	0	0	-6.48	6.48
14	250.16	253.14	250.16	-2.98	0	0	-6.62	6.62
15	264.35	255.21	255.22	9.14	0	2.38	0	6.75
16	266.13	260.38	260.37	5.75	2.48	1.35	0	6.89
17	231.11	265.63	232.52	-34.52	1.40	0	-7.02	7.02
18	221.45	237.22	221.45	-15.77	0	0	-7.17	7.17
19	226.49	225.92	225.93	0.57	0	0	-6.75	7.31
20	290.72	230.49	249.37	60.23	0	33.89	0	7.46

Table A.2: *A sample path with initial capital value of 200. Remaining parameters are base case (Table 5.1) unless otherwise indicated. $S - S_{targ}$ is the value of the capital account which exceeds the value required to preserve the real value of the endowment. This is the same path as shown in Figure 5.8.*

Boyle, P., M. Broadie, and P. Glasserman (1997). Monte Carlo methods for security pricing. *Journal of Economic Dynamics and Control* 21, 1267–1321.

California Polytechnic State University (2003). The university program endowment fund investment goals, objectives and policies.

Chang, H. (2005). Standard & Poor's reports higher education endowments showing gains. *Bond Buyer*, May 10.

d'Halluin, Y., P. A. Forsyth, and K. R. Vetzal (2005). Robust numerical methods for contingent claims under jump diffusion processes. *IMA Journal of Numerical Analysis* 25, 87–112.

Dybvig, P. H. (1999). Using asset allocation to protect spending. *Financial Analysts Journal* 55(1), 49–62.

Ezra, D. (2005). Retirement income guarantees are expensive. *Financial Analysts Journal* 61(6), 74–77.

Number of paths	Guarantee Value $S = 100$
1000	-39.11
5000	-38.96
10000	-38.88
25000	-38.90
100000	-38.89
1000000	-38.89

TABLE A.3: *Convergence test of Monte Carlo method. No-arbitrage value, using the base case parameters as in Table 5.1.*

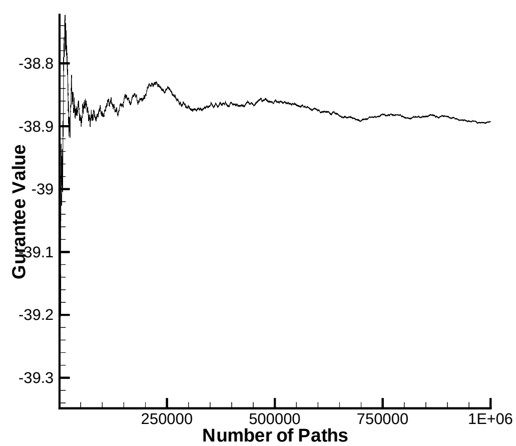

FIGURE A.2: *The convergence of the base case no-arbitrage value of the cash flows funded by the endowment by using a Monte Carlo method. Base case parameters are given in Table 5.1. Initial capital* $S = 100$.

Forsyth, P. A., K. R. Vetzal, and R. Zvan (2002). Convergence of lattice and PDE methods for valuing path dependent options using interpolation. *Review of Derivatives Research* 5, 273–314.

Glasserman, P. (2004). *Monte Carlo Methods in Financial Engineering.* Springer-Verlag, New York.

Golden, D. and C. Forelle (2002). Colleges feel pinch as endowments shrink. *The Wall Street Journal*, July 19.

Hannon, D. and D. Hammond (2003). The looming crisis in endowment spending. *Journal of Investing* 12(3), 9–20.

Kessler, S. L. (2003). Financial modeling of withdrawals from an endowment portfolio. *The Journal of Investment Consulting* 6(1), 26–27.

Lakehead University (2003). Trust and Endowment Management Policy.

Levitt, A. (2005). Pensions unplugged. *The Wall Street Journal*, November 10.

Lowenstein, R. (2005). The end of pensions. *The New York Times*, October 30.

Mehrling, P. (2004). Endowment spending policy: An economist's perspective. *Futures Forum 2004*, Forum for the Future of Higher Education.

Merton, R. C. (1976). Option pricing when underlying stock returns are discontinuous. *Journal of Financial Economics* 3, 125–144.

Milevsky, M. A. and C. Robinson (2005). A sustainable spending rate without simulation. *Financial Analysts Journal* 61(6), 89–100.

North Carolina State University (2004). The endowment fund spending procedures.

Piazza, J. (2003). Fitch: Higher ed sector seeing sinking endowment returns. *Bond Buyer*, July 28.

Rannacher, R. (1984). Finite element solution of diffusion problems with irregular data. *Numerische Mathematik* 43, 309–327.

Ryerson University (2002). Financial Services - Endowment Funds Policy.

Sedlacek, V. O. and S. E. Clark (2003). Why do we feel so poor? how the overspending of the nineties created a crisis in higher education. Whitepaper, Commonfund Institute.

Simon Fraser University (1998). Endowment Management Policy (GP 20).

University of Waterloo (2005). Private communication, Dennis Huber, Vice-President of Administration and Finance.

Wilfrid Laurier University (2003). Policy 13.4: Guidelines for the Financial Administration of Endowment Accounts.

Williams, F. (2003). Official proposes reserve fund to dampen an endowment's volatility. *Pensions & Investments* 31(6), 2–3.

Wilmott, P. (1998). *Derivatives: The Theory and Practice of Financial Engineering.* Wiley, West Sussex, England.

Windcliff, H., P. A. Forsyth, and K. R. Vetzal (2001). Shout options: A framework for pricing contracts which can be modified by the investor. *Journal of Computational and Applied Mathematics* 134, 213–241.