Is It Too Easy For An Asset Manager To Beat A Constant Weight Benchmark?

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Abstract

We determine a simple dynamic benchmark for asset allocation by solving an optimal stochastic control problem to outperform the traditional constant proportion benchmark. An objective function based on a time averaged quadratic deviation from an elevated benchmark is proposed. We argue that this objective function combines the best features of tracking error and tracking difference. Assuming parametric models of the stock and bond processes, a closed form solution for the optimal control is obtained. The closed form optimal control is then clipped to prevent use of excessive leverage, and to prevent trading if insolvent. Monte Carlo computations using this clipped control are presented which show that for modest levels of outperformance (i.e. 80-170 bps per year), this optimal strategy outperforms the traditional constant proportion benchmark with high probability. We advocate this clipped optimal strategy as a suitable benchmark for active asset allocation.

Keywords: optimal control, benchmark outperformance, asset allocation

JEL codes: G11, G22

AMS codes: 91G, 65N06, 65N12, 35Q93

1 Introduction

Many pension plans have a benchmark portfolio which is used to measure the efficiency of the realized investment strategy. These benchmark (or reference) portfolios are invariably based on publicly traded financial assets.

The Canadian Pension Plan (CPP) with CAD 540 billion assets under management has a base reference portfolio of 85% global equity and 15% Canadian government bonds (Canadian Pension Plan, 2022). The non-base CPP portfolio has a benchmark of 55% global equity and 45% government bonds. Note that the CPP also allows use of leverage.

Another example is the Norwegian government pension plan, which has USD 1.35 trillion assets under management. The Norwegian plan has a benchmark of 70% equities and 30% bonds (Government Pension Fund Global, 2022).

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Typically, these plans will report results relative to the constant proportion benchmark, in terms of performance and risk measures. These reports are used to justify active investment strategies and/or investment in alternative assets and use of leverage.

Investments in alternative assets are a major strategy in the *Endowment Model* for managing a portfolio. This model was based on the spectacular success of the Yale endowment over many years. However, a *post hoc* analysis of endowments and public pension plans which attempted to emulate the Yale model compared to a 70-30 (equity-bond) reference portfolio showed poorer returns compared to the benchmark [Ennis 2021], post 2008.

Given the widely adopted industry practice in evaluating performance relative to the constant proportion benchmark strategy, the quest for computing a strategy which outperforms this benchmark becomes immediately relevant and important. Furthermore, one may ask whether it is time to revisit the ubiquitous constant proportion reference strategy.

A better benchmark strategy needs to satisfy at least two criteria: (i) it can be easily constructed (ii) it robustly performs better than the existing constant proportion benchmark.

We aim to outperform the traditional benchmark and discover a more useful benchmark by solving a stochastic optimal control problem with suitable objective functions.

There is a large literature on techniques for constructing dynamic strategies for outperforming a benchmark. We refer the reader to [Browne 2000; Oderda 2015; Al-Aradi and Jaimungal 2018; Ni et al.] [2022] and the references cited therein.

In the context of measuring the efficiency of index exchange traded funds (ETFs), there are two common metrics: tracking *error* and tracking *difference*. The original use of the these metrics was in the context of an index ETF, where the objective is to track the index closely, not outperform it.

In this context, tracking error of a portfolio relative to a benchmark is defined as

\[
\text{Tracking Error} = \text{std} \left( R - \hat{R} \right),
\]

where \( \text{std} \) is the standard deviation, \( R \) is the return of the active portfolio, and \( \hat{R} \) is the return of the benchmark.

In fact, the Norwegian Pension plan (more properly referred to as Government Pension Plan Global) specifies a very tight tracking error of the plan portfolio relative to the 70-30 benchmark of publicly traded assets [Norges Bank 2021].

The motivation for metric (1.1) is described in [Wander 2000]. Briefly, this metric might make sense if the investor wants to hire a portfolio manager who will outperform an index, without taking on too much risk. However, the tracking error (also known as the volatility of relative returns) has some odd properties. Suppose that total wealth in the active investment portfolio is denoted by \( W \) and the total wealth in the benchmark is \( \hat{W} \). Assume that both portfolios follow geometric Brownian motion (GBM),

\[
\frac{d\hat{W}}{\hat{W}} = \hat{\mu} \, dt + \hat{\sigma} \, d\hat{Z} ; \quad \frac{dW}{W} = \mu \, dt + \sigma \, dZ ; \quad d\hat{Z} \cdot dZ = \rho \, dt,
\]

where \( dZ, d\hat{Z} \) are increments of Wiener processes. From equation (1.1) we can see that the instantaneous tracking error per unit time, assuming processes (1.2) is

\[
\left( \text{Tracking Error} \right)^2 = \hat{\sigma}^2 + \sigma^2 - 2\rho\hat{\sigma}\sigma.
\]

Note the unusual aspect of equation (1.3): tracking error decreases as correlation increases. This obviously rewards a manager whose active portfolio has a high positive correlation to the benchmark.
Tracking error (1.1) might be a valid criteria if we desire to track the benchmark as closely as possible, but this metric has been criticized (Johnson et al. 2013; Hougan 2015; Charteris and McCullough 2020; Boyde 2021). In fact, these authors suggest that, in measuring the post hoc performance of ETFs, the simple tracking difference is a more appropriate metric. Tracking difference is simply the difference between the cumulative returns of the investment portfolio and the benchmark.\footnote{In practice, this is often reported in an annualized fashion.}

Suppose that the amount invested in the benchmark at time $t$ is $\hat{W}(t)$, and the amount in the active portfolio is $W(t)$, with the same amounts invested at time zero, i.e. $W(0) = \hat{W}(0)$. We measure the performance of the active portfolio, relative to the benchmark, over the time horizon $[0,T]$. Then, following the spirit of the tracking difference metric, van Staden et al. (2022a) suggests the following control problem for outperformance relative to the benchmark

$$\min_{\mathbb{P}(\cdot)} \mathbb{E} \left[ (W(T) - e^{\beta T} \hat{W}(T))^2 \right]. \quad (1.4)$$

where $\mathbb{E}[\cdot]$ is the expectation and $\mathbb{P}(\cdot)$ is the dynamic control strategy (i.e. the asset allocation), and $W(T)$ and $\hat{W}(T)$ are the terminal wealth associated with strategy $\mathbb{P}(\cdot)$ and the benchmark respectively.

The intuition behind objective function (1.4) is clear. We desire to outperform the benchmark cumulatively over the period $[0,T]$ by a factor of $e^{\beta T}$ (i.e. continuously compounded at a rate of $\beta$ per year). We also desire to minimize the volatility relative to the elevated benchmark. The performance metric (1.4) thus directly targets an outperformance of $\beta$ per year, and attempts to minimize the uncertainty (risk) associated with meeting this target. In a sense, this performance metric combines the desirable features of tracking error and tracking difference.

We can vary the amount of risk we are willing to take, relative to the benchmark, by adjusting $\beta$. As $\beta \to 0$, then the optimal solution to problem (1.4) is to simply invest in the benchmark portfolio. However, as $\beta$ becomes large, we can expect to have to take on more risk than the benchmark, in order to increase outperformance.

A criticism of the objective function (1.4) is that it is symmetric with respect to the upside and the downside. This is, of course, a common problem with volatility-type performance criteria. In Ni et al. (2022), this objective function was modified to be

$$\min_{\mathbb{P}(\cdot)} \mathbb{E} \left[ \left( \max(0, e^{\beta T} \hat{W}(T) - W(T)) \right)^2 + \max(0, W(T) - e^{\beta T} \hat{W}(T)) \right]. \quad (1.5)$$

The objective function (1.5) has a quadratic penalty for underperformance, and a linear penalty for outperformance. However, use of objective function (1.5) does not permit closed form solutions, and requires use of machine learning techniques (Ni et al., 2022) in order to determine the optimal policy $\mathbb{P}(\cdot)$.

Another possible criticism of objective function (1.4) is that deviation from the elevated benchmark is only considered at the terminal time $T$. However, investment managers are usually required to report performance at regular intervals, perhaps quarterly or monthly. Therefore, deviations from the performance target throughout the investment horizon $[0,T]$ are also of concern.

To address this concern, the following objective function has been suggested in van Staden et al. (2022b)

$$\min_{\mathbb{P}(\cdot)} \mathbb{E} \left[ \int_0^T (W(t) - e^{\beta t} \hat{W}(t))^2 \, dt \right], \quad (1.6)$$
which is the time averaged quadratic deviation from the elevated benchmark over the investment horizon. The main focus in [van Staden et al. (2022b)] is on the use of machine learning methods, coupled with a data driven approach, to solve for optimal portfolios using objective function (1.6).

In contrast to [van Staden et al. (2022b)], the objective of this note is to study properties of closed form optimal control solution to problem (1.6) for a simple two asset (stock index and bond) portfolio. We also provide a short, intuitive derivation of the optimal control. We use extensive Monte Carlo simulations to examine the properties of this closed form solution. Since the closed form solution permits infinite leverage, and trading can continue if bankrupt, we apply the clipping technique to the closed form optimal control, as in [Vigna (2014)], to approximate the solution to the constrained optimal control problem. We find that imposition of a no-trading if bankrupt rule has almost no effect on the controlled solution, for reasonable values of $\beta$. This is in contrast to the situation for the closed form solution for multi-period mean-variance control ([Wang and Forsyth, 2010]).

In addition, if we permit a modest amount of leverage, i.e., borrowing up to 30% of net wealth, then this level of leverage constraint also appears to have a modest effect on the control. This suggests that the clipped optimal control is a good approximation of the true constrained control. This simple closed form approximation can then be used to obtain an intuitive understanding for the control produced by the objective function (1.6) with realistic investment constraints.

Our main conclusion is that the outperformance criteria (1.6), for modest values of $\beta$, i.e. $\beta < 200$ bps per year, results in fairly conservative controls, which have a high probability of outperforming the benchmark, without requiring unreasonable amounts of leverage at any time in $[0, T]$.

We argue that the bar for outperforming should be raised for asset managers. A dynamic strategy using publicly traded stock and bond indexes, which can be easily computed based on historical data, is a more appropriate benchmark for asset managers. Based on computational assessment, we propose the clipped optimal control of a stochastic optimal allocation problem as such a dynamic benchmark. We believe that this new dynamic benchmark would allow investors (and the taxpayers paying into public pension plans) to discern true investment skill (or the lack thereof) of the asset managers.

There is more room for success if we apply criteria (1.6) to cases where the investment portfolio has additional assets compared to the benchmark, which would normally be the case. In addition, it might be desirable to avoid postulating a parametric form for the portfolio constituents and generate market scenarios by bootstrapping historical returns ([Ni et al. 2022]). This requires use of numerical techniques (such as machine learning), which is the main topic of [van Staden et al. (2022b)].

2 Investment Market

We assume that the investor has access to two funds: a broad market stock index fund and a bond index fund and the investment horizon is $T$. Let $S_t$ and $B_t$ respectively denote the real (inflation adjusted) amounts invested in the stock index and the bond index respectively. In general, these amounts will depend on the investor’s strategy over time, as well as changes in the real unit prices of the assets. In the absence of an investor determined control (i.e. cash injections or rebalancing), all changes in $S_t$ and $B_t$ result from changes in asset prices.

We model the stock index as following a jump diffusion process. Let $S_t^\epsilon = S(t - \epsilon), \epsilon \to 0^+$, i.e. $t^\epsilon$ denotes immediately before time $t$, and let $\xi$ be a random jump multiplier. When a jump occurs, $S_t = \xi S_t^{\epsilon}$. We assume that $\log(\xi)$ follows a double exponential distribution ([Kou 2002; Kou and Wang 2004]) with parameters $\eta_1$ and $\eta_2$, respectively. The probability of an upward jump
is $\mathcal{P}_u$, while $1 - \mathcal{P}_u$ is the probability of a downward jump. The density function for $y = \log(\xi)$ is
\[ f(y) = \mathcal{P}_u \eta_1 e^{-\eta_1 y} 1_{y \geq 0} + (1 - \mathcal{P}_u) \eta_2 e^{\eta_2 y} 1_{y < 0}. \] (2.1)

Note that the density of $\xi$ has the form $g(\xi) = f(\log \xi) / \xi$. Define $\kappa = E[\xi - 1]$, which is given precisely in Appendix A. Assuming constant units of stock index holding,
\[ \frac{dS_t}{S_t} = (\mu - \lambda \kappa) \ dt + \sigma \ dZ_t + d \left( \sum_{i=1}^{\pi_t} (\xi_i - 1) \right), \] (2.2)
where $\mu$ is the (uncompensated) drift rate, $\sigma$ is the diffusive volatility, $Z_t$ is a Brownian motion, $\pi_t$ is a Poisson process with positive intensity parameter $\lambda$, and $\xi_i$ are i.i.d. positive random variables having distribution (2.1). Moreover, $\xi_i$, $\pi_t$, and $Z_t$ are assumed to all be mutually independent.

We assume the bond index follows
\[ dB_t = rB_t \ dt . \] (2.3)

Let $p_t$ be the fraction of total wealth $W_t$ invested in stock index at $t$. Assuming continuous rebalancing, the total wealth in the investment portfolio follows the process
\[ dW_t = p_t W_t \left( \frac{dS_t}{S_t} \right) + (1 - p_t) W_t \left( \frac{dB_t}{B_t} \right) + q \ dt , \] (2.4)
where $q$ is continuous constant rate of cash injection into the portfolio. Similarly, let $\hat{W}_t$ be the total wealth invested in the benchmark portfolio, with $\hat{p}$ being the fraction of total wealth invested in the benchmark portfolio. Then, analogously to equation (2.4), the process followed by the benchmark wealth is
\[ d\hat{W}_t = \hat{p} \hat{W}_t \left( \frac{dS_t}{S_t} \right) + (1 - \hat{p}) \hat{W}_t \left( \frac{dB_t}{B_t} \right) + q \ dt . \] (2.5)

**Remark 2.1** (Stochastic Bond Returns). Here we have assumed that bond process is non-stochastic, which is arguably a reasonable approximation for short term, low volatility bond indexes. However, it is possible to directly model real returns of a constant maturity bond index fund by a jump diffusion process [Lin et al., 2015; Forsyth, 2022].

### 3 Cumulative Tracking Difference

We will now proceed to formally define the investment problem based on the objective function (1.6), which we will refer to as the cumulative tracking difference (CD) in the following.

Begin with equations (2.4 - 2.5). Define the value function $\hat{V}(w, \hat{w}, t)$ as
\[ \hat{V}(w, \hat{w}, t) = \inf_{p} \left\{ E_p^{(w, \hat{w}, t)} \left[ \int_t^T (\hat{W}(s) e^{\hat{\rho} s} - W(s))^2 \ ds \left| W(t) = w, \hat{W}(t) = \hat{w} \right. \right] \right\} , \] (3.1)
where $E_p^{(w, \hat{w}, t)}[\cdot]$ denotes the expectation under the control $p(\cdot)$ as observed at $(w, \hat{w}, t)$. For notational simplicity, we subsequently omit the dependence in $p(\cdot)$, when there is no confusion.
For $t \in [0, T - \Delta t]$ the tower property gives
\[
\hat{V}(w, \hat{w}, t) = \inf_p \left\{ E_p^{(w, \hat{w}, t)} \left[ \int_t^{t+\Delta t} (\hat{W}(s)e^{\beta s} - W(s))^2 \, ds \right. \right.
\]
\[
\left. + \int_t^{T} (\hat{W}(s)e^{\beta s} - W(s))^2 \, ds \left| W(t) = w, \hat{W}(t) = \hat{w} \right. \right\}
\]
\[
= \inf_p \left\{ E_p^{(w, \hat{w}, t)} \left[ \int_t^{t+\Delta t} (\hat{W}(s)e^{\beta s} - W(s))^2 \, ds \right. \right.
\]
\[
+ \hat{V}(W(t + \Delta t), \hat{W}(t + \Delta t), t + \Delta t) \left| W(t) = w, \hat{W}(t) = \hat{w} \right. \right\}.
\]  
(3.2)

Let $\Delta t \to 0$ in equation (3.2), using equations (2.4 - 2.5), and Ito’s Lemma with jumps (Tankov and Cont, 2009), gives us
\[
V_\tau = \inf_p \left\{ ((r + (\mu - r - \lambda \kappa)wp) + c)V_w + ((r + (\mu - r - \lambda \kappa)\hat{w}\hat{p}) + c)V_{\hat{w}} \right.
\]
\[
+ \frac{\sigma^2}{2}V_{ww} + \frac{\sigma^2}{2}V_{\hat{w}\hat{w}} + (\hat{p}\hat{w}\sigma^2) V_{\hat{w}w}
\]
\[
+ \lambda \int_0^\infty V(w + pw(\xi - 1), \hat{w} + \hat{p}(\xi - 1), \tau)g(\xi) \, d\xi \right\} + (\hat{w}e^{\beta \tau} - w)^2 - \lambda V
\]
\[
\tau = T - t
\]
\[
V(w, \hat{w}, \tau) = \hat{V}(w, \hat{w}, T - \tau),
\]  
(3.3)

where $g(\xi)$ is the density of $\xi$ and subscript in $V$, e.g., $V_\tau$, denotes partial derivative. Since there are no investment constraints, the domain of PDE (3.3) is $(w, \hat{w}, \tau) \in (-\infty, +\infty) \times (-\infty, +\infty) \times [0, T]$, with
\[
V(w, \hat{w}, 0) = 0.
\]  
(3.4)

### 3.1 Closed form solution

We give a brief overview of the method used to derive the closed form solution here and in Appendix A. For a rigorous solution of problem 3.3, we refer the reader to van Staden et al. (2022b). Assume
\[
V = A(\tau)w^2 + B(\tau)w + C(\tau) + \hat{A}(\tau)\hat{w}^2 + \hat{B}(\tau)\hat{w} + D(\tau)w\hat{w}
\]
\[
V_\tau = A_\tau w^2 + B_\tau w + C_\tau + \hat{A}_\tau \hat{w}^2 + \hat{B}_\tau \hat{w} + D_\tau w\hat{w}
\]  
(3.5)

Here the subscript, e.g., $V_\tau$, denotes partial derivative. Substitute equation (3.5) into equation (3.3). This is a quadratic function of the control $p$. It is easily verified (after the fact) that, for the objective function of the optimization problem in (3.3), the coefficient of $p^2$ is positive. Applying the first order condition determines the optimal control. This yields a system of ODEs for the unknown $A, B, \ldots$, with initial conditions for the ODEs determined by matching equation (3.5) with equation (3.4). The final result for the optimal control is given by
\[
p = \frac{1}{\sigma^2} \left[ (\mu - r) + \hat{p}f(t) - w \right] + \hat{w}f(t)
\]  
(3.6)

where the expressions for $\sigma^2, h(t), f(t)$ are given in Appendix A. It turns out that $e^{\beta T} \geq f(t) \geq e^{\beta t}$ and $h(t) \geq 0$ (van Staden et al. 2022b).
3.1.1 Intuition from control (3.6)

Consider the simple case where there is no cash injection, i.e. \( q = 0 \). In this case, it then follows from equation (3.6) that

\[
p \begin{cases}
= \hat{p} & \text{if } w = \hat{w} f(t) \\
> \hat{p} & \text{if } w < \hat{w} f(t) \\
< \hat{p} & \text{if } w > \hat{w} f(t)
\end{cases}
\]

(3.7)

The strategy is fundamentally contrarian. If the active portfolio performs poorly relative to the benchmark, then the stock index weight is increased. On the other hand, if we are fortunate, and the active portfolio does well relative to the benchmark, then the stock index weight is decreased.

Remark 3.1 (Robustness of control to misspecification). In van Staden et al. (2021), it has been noted that optimal multi-period mean-variance strategies are robust to model misspecification errors, in contrast to the single period mean-variance case. This robustness can be traced to the nature of a contrarian control. We conjecture that, similarly to the multi-period mean-variance case, the optimal control (3.6) is also robust to model parameter misspecification.

3.2 Clipped control: Handling bankruptcy and bounded leverage

The closed form solution (3.6) is for the unconstrained optimal control problem, e.g. infinite leverage is allowed and trading can continue if bankrupt. This can produce unrealistically optimistic results for some closed form solutions of optimal control in financial applications, e.g., a closed form solution for multi-period Mean-Variance (MV) optimal strategies (Zhou and Li, 2000). In the MV case, if the MV control problem under a no-bankruptcy constraint (i.e. trading stops if bankrupt) is solved, then this constraint has a large effect on the solution, compared to the unconstrained solution (Wang and Forsyth, 2010).

We consider two simple adjustments to the optimal control (3.6) to obtain realistic investment strategies without solving (3.3) with additional constraints. First, since the control (3.6) is to be applied even in the case of bankruptcy (negative wealth), we revise the control so that trading stops if bankrupt. In other words, if \( W(t) \leq 0 \), the control (3.6) is no longer applied and wealth simply remains at this level.

Second, the closed form solution (3.6) permits unbounded leverage. A rigorous solution of the Hamilton-Jacobi-Bellman (HJB) problem (3.3) with bounds on leverage would require numerical solution techniques (Wang and Forsyth, 2010). A simple technique to approximate the optimal control under constraints is to carry out a clipping procedure.

In this case, we modify equation (3.6), so that our clipped optimal control \( \tilde{p}^*(w, \hat{w}, t) \) is given by

\[
p^* = \frac{(\mu - r)}{\sigma^2 w} \left( qh(t) + (\hat{w} f(t) - w) \right) + \frac{\hat{p} \hat{w} f(t)}{w}
\]

\[
\tilde{p}^* = \min (\max (0, p^*), p_{\text{max}})
\]

(3.8)

Equation (3.8) ensures that

\[
\tilde{p}^* \in [0, p_{\text{max}}]
\]

(3.9)

Imposing bounded leverage also ensures that there is no trading if bankrupt (Wang and Forsyth, 2010). A similar idea was exploited in Vigna (2014) in the context of closed form solutions for multi-period mean-variance asset allocation. Note that we require that \( p \geq 0 \) since, if \( w \gg \hat{w} f(t) \)
in equation (3.8), then, it is possible that $p < 0$. This problem can be attributed to the symmetric risk measure in equation (3.1), since extreme outperformance, i.e., $w > e^{\beta \hat{w}}$, is also penalized.

Some pension plans are required to undertake a policy of no-leverage, i.e. $p_{\text{max}} = 1$, while other plans allow limited leverage. In our model set-up, we have only two assets: a stock index and a bond index for both the benchmark and the optimal portfolio. Usually the benchmark is a stock index and a bond index. However, many pension plans are using alternative assets, such as private equity and private credit. Although controversial, some authors have suggested that returns on private equity can be replicated using a leveraged small cap stock index (see Phalippou, 2014; L’Her et al., 2016). To this end, we set $p_{\text{max}} = 1.3$ to approximate (very roughly) a portfolio with some exposure to alternative assets.

4 Numerical Results

We use data from the Center for Research in Security Prices (CRSP) on a monthly basis over the 1926:1-2021:12 period. Our base case tests use the CRSP 30 day T-bill for the bond asset and the CRSP value-weighted total return index for the stock index. This latter index includes all distributions for all domestic stocks trading on major U.S. exchanges. All of these various indexes are in nominal terms, so we adjust them for inflation by using the U.S. CPI index, also supplied by CRSP.

We use the threshold technique (Cont and Mancini, 2011; Dang and Forsyth, 2016) to estimate the parameters for the parametric stochastic process models. Table 4.1 shows the results of calibrating the models to the historical data.

<table>
<thead>
<tr>
<th>$\mu$</th>
<th>$\sigma$</th>
<th>$\lambda$</th>
<th>$P_u$</th>
<th>$\eta_1$</th>
<th>$\eta_2$</th>
<th>T-bill return $r$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0897</td>
<td>0.1464</td>
<td>0.3229</td>
<td>0.2258</td>
<td>4.3638</td>
<td>5.5316</td>
<td>0.0035</td>
</tr>
</tbody>
</table>

Table 4.1: Estimated annualized parameters for double exponential jump diffusion model. Value-weighted CRSP index, 30 day US T-bill index deflated by the CPI. Sample period 1926:1 to 2021:12. The mean return of the 30-day T-bill index is $r = 0.0035$.

4.1 Investment scenario

Table 4.2 shows our base case investment scenario. We consider $T = 10$ years, with an initial investment of 100. Cash injection occurs continuously at a rate of 10 per year. The target benchmark is $\hat{p} = 0.70$ in the stock index and 0.30 in bonds.

4.1.1 No trading if bankrupt (clipped optimal control)

We carry out Monte Carlo simulations assuming the processes (2.2-2.3). We use 1000 timesteps and $6.4 \times 10^5$ simulations. Figure 4.1 shows the cumulative distribution function (CDF) of the ratio $(W_T/\hat{W}_T)$ for both the unconstrained control (3.6) and the case where we use the same control, but trading stops if bankrupt. We consider $\beta = .01, .02$. A desirable strategy should achieve high probability of $(W_T/\hat{W}_T) > 1$.

Figure 4.1 shows that the CDFs for the unconstrained control and the no-bankruptcy control virtually overlap, for the values of $\beta$ shown. This indicates that for this control, there is very little probability of continuing to trade if bankrupt, in contrast to the mean-variance case. This implies
| Investment horizon $T$ (years) | 10.0 |
| Equity market index | CRSP Cap-weighted index (real) |
| Bond index | 30-day T-bill (US) (real) |
| Initial portfolio value $W_0$ | 100 |
| Cash Injection per year $q$ | 10 |
| Rebalancing times | Continuous |
| Outperformance target (per year) $\beta$ | $\{.01, .02\}$ |
| Benchmark fraction in stock index $\hat{p}$ | .70 |
| Market parameters | See Table 4.1 |

Table 4.2: Input data for examples.

That the cumulative difference control (3.1) is inherently more conservative than the mean-variance control.

\[
\text{Figure 4.1: CDF of the ratio } R_T = \frac{W_T}{\hat{W}_T} \text{ for the scenario in Table 4.2. Unconstrained control, equation (A.6), compared to the same control, but trading stops if bankrupt. Outperformance is indicated if } R > 1.\]

4.1.2 Bounded leverage (clipped optimal control)

We carry out Monte Carlo simulations assuming the processes (2.2, 2.3). Figure 4.2 shows the cumulative distribution function (CDF) of the ratio $W_T/\hat{W}_T$, for both the unconstrained control (3.6) and the clipped control (3.8), for $\beta = .01, .02$. A desirable outcome is that $(W_T/\hat{W}_T) > 1$ (the active portfolio has outperformed the benchmark).

Of course, the solution to the constrained control problem analogous to (3.3) but with $p \in [0, p_{\text{max}}]$ will differ from the clipped control solution. The clipped optimal control $\hat{p}^*$ in (3.8) only approximates the solution to the constrained optimal control problem. Consequently we would expect the true constrained solution CDF of $W/\hat{W}$ to differ from the clipped control solution. However, Figure 4.2(a) shows that, for $\beta = .01$, the CDFs from the clipped control and the unconstrained control overlap. This indicates that the clipped control (3.8) is almost exact in this case, since the constraints do not appear to be binding for this value of $\beta$. For the case of $\beta = .02$, Figure 4.2(b) shows that the CDFs for the clipped control approximation and the exact unconstrained control overlap, except for a small difference near $(W/\hat{W}) = 1$. 

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The implication is that the clipped optimal control is a reasonable approximation to the exact optimal control under constraints, at least for moderate levels of the outperformance target $\beta \leq 200$ bps per year.

From now on, we will show results using only the clipped approximate control $\tilde{p}^*$ (3.8). We will refer the clipped optimal control to distinguish this strategy from the benchmark. It will be understood that $\tilde{p}^*$ is in fact only an approximation to the optimal control under constraints.

Figure 4.2: $CDF$ of the ratio $R_T = W_T / \hat{W}_T$, scenario in Table 4.2. Clipped optimal control $\tilde{p}^*$ with $p_{\text{max}} = 1.3$ in (3.8). Unconstrained optimal control, $p^*$ from equation (A.6). Outperformance is indicated if $R > 1$. 1000 timesteps and $6.4 \times 10^5$ Monte Carlo simulations.

4.1.3 Wealth ratio

Figure 4.3 shows the time evolution of the wealth ratio $(W_t / \hat{W}_t)$, assuming the clipped control (3.8). Recall that outperformance at $t$ is indicated when $(W_t / \hat{W}_t) > 1$. Observe that for $\beta = .01$, there is an 80% probability that the clipped control strategy generates wealth greater than 0.99 of the benchmark wealth, at all times during the ten year investment horizon. There is an 80% probability of outperforming the benchmark at all times greater than about 2.5 years, for both values of $\beta$. In addition, from Figure 4.2, we observe that the clipped control solution has a 90% probability of outperforming the benchmark at $t = T$, for both values of $\beta$. For $\beta = .01$, there is clearly a smaller spread of the wealth ratio around the median value (over time) compared with $\beta = .02$, in Figure 4.3. This corresponds to our intuition: as the outperformance target $\beta$ is increased, it is necessary to take on more risk.

4.1.4 Fraction in stocks

Figure 4.4 shows the percentiles of the time evolution of the fraction in the stock index. In this case, there is a striking difference between Figure 4.4(a) ($\beta = .01$) and Figure 4.4(b) ($\beta = .02$). For $\beta = .01$, the median equity fraction starts off at about 0.83 and decreases as time goes on. The upper and lower percentiles are tightly clustered about the median. The 80th percentile fraction in equities never exceeds .90 (recall that the benchmark equity fraction is .70). In contrast, the $\beta = .02$ case shows a much wider variation about the median. At the 80th percentile level, the clipped optimal control in this case shows a modest amount of leverage ($p \leq 1.05$).

The reader should note that for any given stochastic path, the control does not stay at the percentile bounds, but responds to actual investment experience. For example, in the $\beta = .01$ case,
Figure 4.3: Time evolution of the wealth ratio $W_t/\hat{W}_t$, clipped optimal strategy (3.8). Scenario in Table 4.2. Outperformance is indicated if $R_t = (W_t/\hat{W}_t) > 1$. 1000 timesteps and $6.4 \times 10^5$ Monte Carlo simulations.

Figure 4.4(a) can be interpreted as indicating that the fraction in equities never exceeds 0.88 (at the 80th percentile) and is never less than 0.75 (at the 20th percentile) over the ten year horizon.

Figure 4.4: Time evolution of the equity fraction, clipped optimal strategy equation (3.8). Scenario in Table 4.2. 1000 timesteps and $6.4 \times 10^5$ Monte Carlo simulations.

4.1.5 Internal rate of return

Another way of examining the results is to compute the annualized pathwise internal rate of return (IRR), for both the clipped optimal strategy (3.8) and the benchmark, over the entire 10 year period. CDFs of the pathwise IRR difference are given in Appendix B.

4.1.6 Summary statistics

Table 4.3 gives some summary statistics for the clipped optimal control and the constant proportion benchmark. We can see directly from this table that the median IRR for the clipped control for the aggressive case of $\beta = .02$ is about 170 bps higher than the benchmark. However, there is no free lunch here, the 5th percentile for the clipped control is 147 compared to the 5th percentile for the benchmark of 169. In Table 4.3 we include the expected shortfall at the 5% level, which is simply
the mean of the worst five per cent of the terminal wealth values $W_T$. We denote this tail measure by $ES(5\%)$. For the $\beta = .02$ case, the $ES(5\%)$ for the benchmark is 145 compared to 110 for the clipped optimal control.

On the other hand, for the (relatively) conservative case of $\beta = .01$, (Table 4.3) the median for the clipped optimal strategy outperforms the benchmark by 80 bps per year, and has about the same result at the 5th percentile. The $ES(5\%)$ is, in this case, only slightly worse than the benchmark. In this case, the results using the clipped optimal strategy are quite impressive. If we target an outperformance of 100 bps per year, then the actual median outperformance is about 80 bps per year, with very little increase in the downside tail risk. This is almost a free lunch.

We remind the reader that the total real amount invested over 10 years is 200, hence these tail outcomes (at the 5th percentile) are very poor, for both the constant proportion benchmark and clipped optimal strategy. Pension plan holders would be very disappointed in these results for either strategy.

<table>
<thead>
<tr>
<th>Mean $W_T$</th>
<th>Median $W_T$</th>
<th>5th percentile</th>
<th>95th percentile</th>
<th>$ES(5%)$</th>
<th>Median IRR</th>
</tr>
</thead>
<tbody>
<tr>
<td>Benchmark $\hat{p} = .70$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>329.38</td>
<td>303.66</td>
<td>168.6</td>
<td>570.35</td>
<td>144.97</td>
<td>0.054</td>
</tr>
<tr>
<td>Clipped optimal control $\beta = .01$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>352.17</td>
<td>325.43</td>
<td>164.43</td>
<td>623.26</td>
<td>129.27</td>
<td>0.062</td>
</tr>
<tr>
<td>Clipped optimal control $\beta = .02$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>375.61</td>
<td>348.70</td>
<td>147.08</td>
<td>681.12</td>
<td>110.33</td>
<td>0.071</td>
</tr>
</tbody>
</table>

Table 4.3: Statistics of $W_T$ for the clipped optimal strategy, and the constant proportion benchmark. Scenario in Table 4.2. Clipped control (3.8) used for the stock index weight. $ES(5\%)$ is the mean of the worst 5\% of the outcomes for $W_T$. We use 1000 timesteps and $6.4 \times 10^5$ simulations. The target outperformance $\beta$ is as shown. Scenario in Table 4.2.

5 Conclusions

In this paper, we have shown that the clipped form of the closed form control for the cumulative difference objective function can achieve a high probability (90\%) of outperforming a benchmark, with an outperformance of 80-170 bps per year. The clipped form of the control prevents the desirable property that (i) leverage is bounded and (ii) no trading if bankrupt. Technically, the clipped control is suboptimal, but our Monte Carlo simulations indicate that the degree of sub-optimality is small. This property can be traced to the inherent conservative policy of the cumulative difference objective function.

This shows that a dynamic trading strategy can beat a fixed weight benchmark by 80-170 bps per year with little risk. This is, of course, not surprising, since the admissible control set for a dynamic trading strategy is clearly larger than the singleton fixed weight control.

The optimal control solution reminds us of a very important fact. Any attempt to outperform a benchmark has some risk of underperforming the benchmark. To assert otherwise is to postulate an arbitrage opportunity. Hence, it is important to quantify this risk-reward tradeoff.

Consequently, we advocate the use of the clipped control from the cumulative difference objective function as a dynamic benchmark strategy. Since a closed form control is readily available, it would
be straightforward to apply this clipped optimal control to historical return data of publicly traded assets. This would then differentiate true investment skill from the easy gains due to dynamic trading.

Of course, most of these pension plans employ a large universe of possible assets, including private equity and private credit. It is arguable than many of these alternative assets can be replicated using publicly traded factor portfolios (Ang (2014)). Hence, a better outperforming strategy would be an optimal dynamic strategy comprised of standard indexes and factor portfolios. We intend to report on this in our future work (van Staden et al. (2022b)).

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The CRSP data were calculated based on data from Historical Indexes, ©2022 Center for Research in Security Prices (CRSP), the University of Chicago Booth School of Business. Wharton Research Data Services was used in preparing this article. This service and the data available thereon constitute valuable intellectual property and trade secrets of WRDS and/or its third-party suppliers.

7 Conflicts of interest

The authors have no conflicts of interest to report.

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Appendices

A Closed Form Solution: Details

From equation (2.1), recalling that \( y = \log(\xi) \), we have

\[
\begin{align*}
\kappa &= E[\xi - 1] = \frac{\mathcal{P}_u \eta_1}{\eta_1 - 1} + \frac{(1 - \mathcal{P}_u) \eta_2}{\eta_2 + 1} - 1, \\
\kappa_2 &= E[(\xi - 1)^2] = \frac{\mathcal{P}_u \eta_1}{\eta_1 - 2} + \frac{(1 - \mathcal{P}_u) \eta_2}{\eta_2 + 2} - 2\kappa - 1. 
\end{align*}
\]

(A.1)

It turns out that the control \( p(w, \hat{w}, \tau) \), \( \tau = T - t \), depends only on \( A, B, D \), which are given by

\[
\begin{align*}
A_\tau &= (2r - \eta)A + 1; \quad A(0) = 0 \\
D_\tau &= (2r - \eta)D - 2e^{\beta \tau}; \quad D(0) = 0 \\
B_\tau &= (r - \eta)B + 2qA + qD; \quad B(0) = 0 \\
\eta &= \frac{(\mu - r)^2}{\sigma_e^2}; \quad \sigma_e^2 = \sigma^2 + \lambda \kappa_2; \quad \kappa_2 = E[(\xi - 1)^2] \quad \text{(A.2)}
\end{align*}
\]

The solutions are

\[
A = \frac{e^{(2r-\eta)\tau} - 1}{(2r - \eta)}; \quad D = 2e^{\beta T} \left( \frac{e^{-\beta \tau} - e^{(2r-\eta)\tau}}{2r - \eta + \beta} \right) \quad \text{(A.3)}
\]

and

\[
B = \frac{2q}{2r - \eta} \left( \frac{e^{(2r-\eta)\tau} - e^{(r-\eta)\tau}}{r} - \frac{(e^{(r-\eta)\tau} - 1)}{r - \eta} \right) \\
+ \frac{2qe^{\beta T}}{2r - \eta + \beta} \left( \frac{e^{(r-\eta)\tau} - e^{-\beta \tau}}{r - \eta + \beta} - \frac{(e^{(2r-\eta)\tau} - e^{(r-\eta)\tau})}{r} \right). \quad \text{(A.4)}
\]

The optimal control \( p^* \) (from the first order condition) is given by

\[
\begin{align*}
p^* &= \frac{(\mu - r)}{w \sigma_e^2} \left( qh(t) + (\hat{w} f(t) - w) \right) + \hat{p} \frac{\hat{w}}{w} f(t) \\
h(t) &= -\frac{B}{2Aq}; \quad f(t) = -\frac{D}{2A} \quad \text{(A.5)}
\end{align*}
\]

(A.6)

Some algebra (van Staden et al., 2022b) shows that

\[
e^{\beta T} \geq f(t) \geq e^{\beta t}; \quad h(t) \geq 0. \quad \text{(A.7)}
\]

B Internal Rate of Return (IRR) Results

Denote the the IRR of the clipped optimal strategy by \( IRR_{co} \) and the IRR of the benchmark by \( IRR_{bench} \). The pathwise difference \( IRR_{diff} \) is then determined by

\[
IRR_{diff} = IRR_{co} - IRR_{bench}. \quad \text{(B.1)}
\]
Figure B.1 shows the CDF of $\text{IRR}_{\text{diff}}$. For the aggressive target outperformance $\beta = 0.02$ (Figure B.1(b)), observe that there is an 80% chance that the IRR of the clipped optimal strategy beats the benchmark by more than 100 bps per annum. The median outperformance is about 170 bps per annum. As expected, the less aggressive case of $\beta = 0.01$ (Figure B.1(a)), has about a 92% probability of beating the benchmark at ten years, with a median pathwise outperformance of about 85 bps per year.

**Figure B.1**: CDF of the pathwise difference in terminal IRR (clipped optimal strategy compared to the benchmark), over $[0,T]$, see equation (B.1). Scenario in Table 4.2. Clipped control, $p_{\text{max}} = 1.3$ in equation (3.8). Unconstrained control, equation (A.6). 1000 timesteps and $6.4 \times 10^5$ Monte Carlo simulations. Outperformance indicated by $\text{IRR}_{\text{diff}} > 0$. 