If your objective is to "Die with zero" should you invest in equities?

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Abstract

Suppose a retiree has earmarked a fixed bucket of wealth to be decumulated during the early years of retirement. The retiree's basic expenses are already covered by cash flows from pensions, annuities and other investments. Investing in stocks during rapid decumulation exposes the retiree to sequence of return risk. It is tempting to invest the decumulation bucket in low risk assets, i.e. short term bills. Does it make sense for the retiree to invest some of the decumulation bucket in equities?

Keywords: optimal control, benchmark outperformance, asset allocation
JEL codes: G11, G22
AMS codes: 91G, 65N06, 65N12, 35Q93

1 Introduction

This paper is motivated by several discussions I have had recently with newly retired investors. I will characterize these retirees in terms of a single, hypothetical investor, named Sam.

Sam is newly retired, and his basic cash flow requirements are met by a combination of pensions, annuities, and income producing investments. Sam also has mortgage free real estate, which he plans on leaving to his children as a bequest. Sam has looked at the cost of assisted living retirement homes, and thinks that when the day comes that he and his spouse have to move out of his house and into a home, the cost can be covered from his pensions and annuities. In the event of an unexpected medical expense (while he is in the assisted living home), his real estate is a hedge of last resort (too bad kids).

However, Sam also has an additional pot of cash. Sam has read *"Die with zero: getting all you can from your money and your life,"*(Perkins, 2020), and realizes that the point having extra cash when you retire is to spend it.[1] Sam is also aware of the fact that he sees many of his older friends, who have health problems, and are no longer able to spend the wealth they have accumulated working all those years.

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[1]In the Canadian context, retirees are shielded from ruinous health expenses, so, there is not an overwhelming need to self-insure health costs.

Sam has a plan. He is going to take his extra cash, put it in short term deposits, such as T-bills.[2]. He will then draw down half this pot of cash in equal annual amounts during his initial retirement years, when he can still enjoy it. As Sam says,

> *"What is the point of investing in equities with this cash? Short term real interest rates are now positive. Why do I want to take any equity risk? I am supposed to be spending this cash and having fun."*

In fact, in terms of decumulation, Sam has the right idea. There is a lot of literature which suggests that retirees are very poor at decumulating their wealth (De Nardi et al., 2009; Smith et al., 2009; Poterba et al., 2009; Browning et al., 2016; Ackerly et al., 2021). This is confirmed in Bannerje (2021) where surveys show that for most retirees, the ratio of non-discretionary income to guaranteed income drops sharply to one after retirement, and remains very close to unity after the age of 70. In other words, retirees adjust their lifestyle so that spending is covered by guaranteed cash flows, even if they have significant other wealth.

Sam has also carried out the typical *mental bucketing* of different assets, e.g. his real estate is earmarked as a bequest, and is not thought of in combination with his other assets. Similarly, Sam has earmarked a pot of cash for early retirement spending. We will refer to this *fun and games* wealth as Sam's decumulation account. This is consistent with the behavioral lifecycle model (Shefrin and Thaler, 1988; Thaler, 1990) whereby different asset classes are not fungible.

So, Sam is to be congratulated on planning to spend down some his wealth and enjoying it while he can. However, is it really a good idea to invest his decumulation account in low risk, low return assets?

The objective of this paper is to make the case that Sam should hold some equities in his decumulation account.

2 Decumulation Scenario

Sam is 65 years old. A Canadian 65-year old male has about a 0.13 chance of living to be 95.[3] Sam's median life expectancy is about 87. The chances of Sam seeing his 100'th birthday is about 0.02.

Based on these statistics, Sam has the following plan. He will withdraw five percent of the initial value in the decumulation account annually, adjusted for inflation, for the first ten years.[4] Sam wants to target having about half of his original decumulation account (real) left by year ten.

At the end of ten years, Sam will re-evaluate his spending plan, in light of his health. The objective being to spend the entire remaining decumulation account while he is still healthy. This may require increasing the withdrawal amount after ten years. Or perhaps he wants to give a living bequest to his children or a charity.

Sam is reluctant to invest in equities in his decumulation account. He has heard about sequence of return risk(Gooderham, 2021). Basically, if you are withdrawing cash at regular intervals, and the market crashes, it is possible you may never make up your losses. So, he thinks bonds are the way to go.

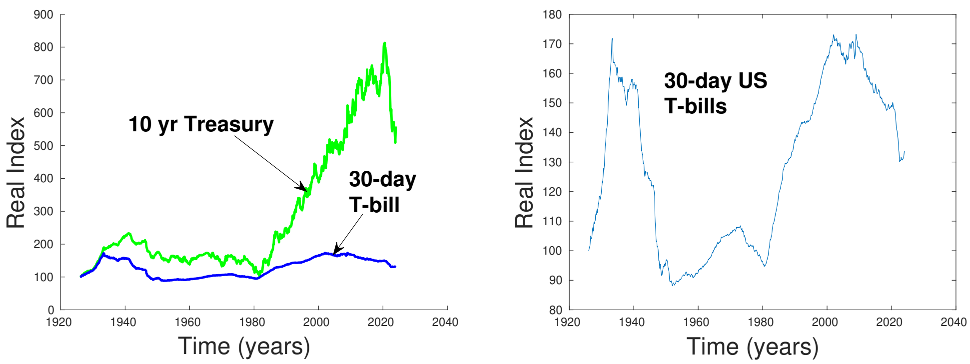
[2]An alternative, in Canada, could be Guaranteed Investment Certificates (GICs). These are guaranteed up to $100,000 by the Canadian government. These guarantees are for each institution. There are over 80 institutions covered by the Canadian Deposit Insurance Corporation (CDIC). In principle, an individual could insure over $8 million CAD in GICs

[3]From the CPM2014 table from the Canadian Institute of Actuaries `www.cia-ica.ca/docs/default-source/2014/214013e.pdf`.

[4]The probability that Sam sees his 75th birthday is 89%.

Sam looks at Figure 2.1, and decides that 10-year treasuries are too risky. Even 30-day T-bills do not keep up with inflation at times. Nevertheless, Sam will put this cash into 30-day T-bills.

We will assume that Sam's decumulation funds are in a tax-advantaged account such as an RRSP in Canada, or a 401(K) in the US. This means that we can ignore taxes on gains before withdrawals.[5] The historical mean real annual return of 30-day T-bills is 0.003 with a volatility of about .02.[6] Sam's hope is that the cash in his decumulation account retains its real value, although Figure 2.1(b) suggests that this is far from certain.



(a) *Comparison of 10yr Treasury vs T-bills.* (b) *T-bill only.*

FIGURE 2.1: *Ten year treasury bond index and 30 day T-bill index, inflation adjusted. CRSP data, 1926:1-2023:12.*

We can summarize the basic investment/decumulation scenario, for the first 10 years, in Table 2.1. Although Sam's initial thought is to invest 100% in 30-day T-bills, we will also consider the possibility of investing in a stock index (the Center for Research in Securities Pricing (CRSP) index).

So, basically Sam is going to start with 1000 (units arbitrary) in the decumulation account, and withdraw a total of 500 (real) over ten years. Sam hopes to have 500 (real) left in the account after ten years. Sam wants to minimize the probability of having less than 500 after withdrawals at the ten year mark. Of course, Sam would not mind having more than 500 at year ten.

We will consider two strategies involving a mix of stocks and T-bills.

- Rebalance to a constant weight in stocks
- Buy and hold

2.1 Rebalance to Constant Weight

Given a yearly withdrawal amount q per year, with Δt being the rebalancing interval, then, at each rebalancing date, we withdraw $q\Delta t$ from the portfolio, and then rebalance. If the portfolio value at time t is denoted by $W(t)$ (after withdrawals), then at rebalancing time t, $\big(p\ W(t)\big)$ is invested in the stock index, and $\big((1-p)\ W(t)\big)$ is invested in the bond index. The rebalancing weight p is constant.

Note that, trivially, if set $(p = 0)$, then we have Sam's benchmark portfolio of 100% T-bills.

[5]Withdrawals are normally taxed as regular income in Canada.

[6]Based on the Center for Research in Securities Prices (CRSP) data, 1926:1-2023:12.

Investment horizon T (years)	10.0
Equity market index	CRSP Cap-weighted index (real)
Bond index	30-day T-bill (US) (real)
Initial portfolio value W_0	1000
Withdrawal (per year)	50
Total withdrawals (over 10 years)	500
Target Amount at $t = 10$ years	500
Base case rebalancing interval	quarterly
Cash withdrawal/rebalancing times	$t = 0,.025, 0.5, \ldots ,9.75$
Rebalancing interval (years)	0.25

TABLE 2.1: *Input data for examples. All amounts and indexes are inflation adjusted. Data range: 1926:1 - 2023:12.*

2.2 Buy and Hold

For the buy and hold case, we initially invest $\big(p_{init}\, W(0)\big)$ in the stock index and $\big((1 - p_{init})\, W(0)\big)$ in the bond index, and never rebalance.

In the buy and hold case, at quarterly intervals, we withdraw $(q\Delta t)$ from the bond component of the portfolio (if possible), and as a last resort withdraw from the stock portfolio.

2.3 Insolvency

In principle, insolvency can occur, since we have fixed withdrawals from a risky portfolio. Note that the T-bill portfolio is inflation adjusted, hence is not risk-free in real terms. In the event that insolvency occurs, we set the amount in stocks to zero, and withdraw from the bond component (i.e. we are borrowing). In all our simulations, insolvency never occurs.

3 Data

We use data from the Center for Research in Security Prices (CRSP) on a monthly basis over the 1926:1-2023:12 period.[7] Our portfolios consist of the CRSP US 30 day T-bill for the bond asset and the CRSP value-weighted total return index for the stock asset. This latter index includes all distributions for all domestic stocks trading on major U.S. exchanges. All of these various indexes are in nominal terms, so we adjust them for inflation by using the U.S. CPI index, also supplied by CRSP. We use real indexes since investors should be focused on real (not nominal) wealth goals.

4 Stationary Block Bootstrap resampling

We will examine the performance of our portfolios using a pure data driven approach. We will use block bootstrap resampling of the inflation adjusted CRSP capitalization weighted index, and the inflation adjusted 30-day T-bill index (see Section 3). The data set covers the historical range 1926:1-2023:12.

[7] More specifically, results presented here were calculated based on data from Historical Indexes, ©2023 Center for Research in Security Prices (CRSP), The University of Chicago Booth School of Business. Wharton Research Data Services (WRDS) was used in preparing this article. This service and the data available thereon constitute valuable intellectual property and trade secrets of WRDS and/or its third-party suppliers.

An overview of block bootstrap resampling is given in Appendix A.

5 Pathwise performance

We will examine the pathwise performance of a stock-bond portfolio versus a 100% T-bill portfolio by examining the pathwise outperformance ratio

$$\begin{aligned} R_t &= \frac{W_t^{sb}}{W_t^{bond}} \\ & W_t^{sb} = \text{ Portfolio at time } t \text{ under stock-bond strategy} \\ & W_t^{bond} = \text{ Benchmark portfolio at time } t, \text{ 100 \% T-bills } . \end{aligned} \tag{5.1}$$

To be clear, R_t is evaluated along each stochastic path. A value of $R_T > 1$ indicates that Sam is better off with the stock-bond portfolio compared to the 100% bond portfolio at the ten year mark. We would like to see that $R_t > 1$, with high probability, during the entire ten year investing cycle.

To summarize, along each stochastic path, with a fixed withdrawal schedule, we compare the performance of W_t^{sb} and W_t^{bond}, using the ratio R_t. This is a much stricter criteria than examining the statistics of W_t^{sb} and W_t^{bond} separately.

5.1 Performance Metrics

As well as examining the percentiles of R_t through time, we will also consider tail risk measures at time T. Some typical tail risk measures are Value at Risk, e.g. the 5^{th} percentile of R_T, and the expected shortfall at the 5% level (ES(5%)). ES(5%) is the mean of the worst five per cent of the outcomes.

Another interesting measure of performance is the Omega ratio (Keating and Shadwick, 2002; Bertrand and Prigent, 2011; Kapsos et al., 2014; Bernard et al., 2019), which is a measure of upside versus downside.

Define the Omega ratio as

$$\begin{aligned} \text{Omega(L)} &= \frac{\int_L^\infty \Big(1 - F(R_T)\Big)\; dR_T}{\int_{-\infty}^L F(R_T)\; dR_T} \quad ; \quad F(X) \quad \text{CDF of } X \\ &= \frac{E[\max(R_T - L, 0)]}{E[\max(L - R_T, 0)]} \\ & R_T = \frac{W_T^{sb}}{W_T^{bond}} \; . \end{aligned} \tag{5.2}$$

If terms of the ratio R_T, it is natural to choose $L = 1$, i.e. we want maximize the upside of outperformance vs downside.

Omega(1) can be written as

$$\begin{aligned} \text{Omega(1)} &= \frac{E[\max(R_T - 1, 0)]}{E[\max(1 - R_T, 0)]} \\ &= 1 + \frac{E[R_T - 1]}{E[\max(1 - R_T, 0)]} \; . \end{aligned} \tag{5.3}$$

5.2 Bootstrap Results

We use stationary block bootstrap resampling, and simulate the results for the scenario described in Table 2.1.

5.2.1 Rebalancing

Table 5.1 shows the summary statistics for rebalancing to a constant weight in stocks. ES(5%) is the expected shortfall at the five per cent level, i.e. the mean of the worst 5% of the outcomes.

The case $p = 0.2$ in Table 5.1 is quite interesting. The probability, along each path, that this strategy will be superior to investing 100% in T-bills is 89%. The 5^{th} percentile of R_T is 0.92, indicating that with 95% probability, the rebalanced portfolio will achieve a result better than 0.92 of the all T-bill portfolio. As a more extreme measure of left tail risk, we have that $ES(5\%) = 0.84$, i.e. the mean of the worst 5% of the outcomes, along each path, is 84% of the all bond portfolio. This strategy has Median$[R_T] = 1.25$, meaning that the rebalanced portfolio achieves greater than 25% more wealth than the all bond portfolio, 50% of the time. The simulation with $p = 0.2$ also has the largest Omega(1) ratio for the constant weights tested.

Equity Fraction	$E[R_T]$	Median$[R_T]$	$R_T : 5^{th}$ percentile	$R_T : 95^{th}$ percentile	ES(5%)	$Prob[R_T < 1]$	Omega(1))
p = 0.1	1.13	1.13	0.96	1.29	0.92	0.10	26.5
p = 0.2	1.26	1.25	0.92	1.64	0.84	0.11	27.7
p = 0.3	1.42	1.39	0.87	2.03	0.75	0.11	26.5
p = 0.4	1.58	1.53	0.82	2.49	0.67	0.12	26.4

TABLE 5.1: *Statistics for $R_T = W_T^{rebal}/W_T^{bond}$. The rebalanced portfolio uses quarterly rebalancing with a fraction p in stocks. The benchmark portfolio is 30-day T-bills. The stock index is the CRSP capitalization weighted index. All indexes are real (inflation adjusted). Block bootstrap resampling, with expected blocksize: one year. 10^4 resamples. Data from CRSP: 1926:1 to 2023:12. ES(5%) is the mean of the worst 5% of the outcomes. Outperformance indicated if $R_T > 1$. Scenario in Table 2.1.*

Restricting attention to the rebalancing case, with $p = 0.2$, Table 5.2 compares the effect of different rebalancing frequencies, ranging from one month to one year. There is very little effect of rebalancing frequencies in this range. Even annual rebalancing is a reasonable strategy.

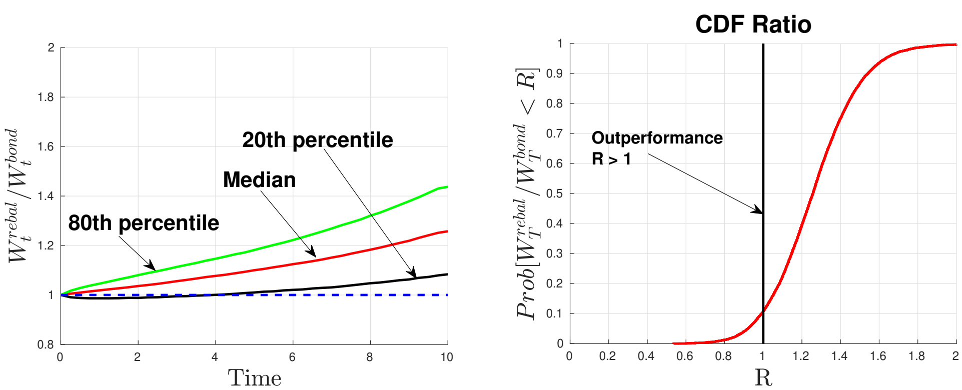
Figure 5.1(a) shows percentiles of R_t versus time. We can see that 80% of the time, the rebalanced portfolio wealth will be larger than the all T-bill portfolio, for all times greater than four years. Figure 5.1(b) indicates that there is much more upside (value of $R_T > 1$) compared to the downside, in terms of the Cumulative Distribution Function (CDF) of R_T. This is also reflected in the Omega ratio in Table 5.2.

5.2.2 Buy and Hold

Given that the rebalanced case with $p = 0.2$ seems to be a good tradeoff between risk and reward, we experimented with various values of the initial stock fraction p_{init}. Recall that for the buy and hold strategy, we allocate the amount $p_{init}W(0)$ to stocks and never rebalance. The value of $p_{init} = 0.14$ gave a value of Median$[R_T] = 1.26$ which is close to the median value of R_T for the rebalancing strategy with $p = 0.2$. Table 5.3 shows the summary statistics for the rebalancing strategy versus buy and hold. These strategies are quite comparable, with buy and hold having a larger Omega

Rebalancing Frequency	E[R_T]	Median[R_T]	R_T : 5^{th} percentile	R_T : 95^{th} percentile	ES(5%)	$Prob[R_T < 1]$	Omega(1))
Monthly	1.26	1.25	0.91	1.62	0.83	0.11	24.6
Quarterly	1.26	1.25	0.92	1.64	0.84	0.11	27.7
Yearly	1.26	1.25	0.92	1.64	0.85	0.11	27.7

TABLE 5.2: *Test of rebalancing frequencies, $p = 0.2$, scenario in Table 2.1.Statistics for $R_T = W_T^{rebal}/W_T^{bond}$. The rebalanced portfolio uses quarterly rebalancing with a fraction p in stocks. The benchmark portfolio is 30-day T-bills. The stock index is the CRSP capitalization weighted index. All indexes are real (inflation adjusted). Block bootstrap resampling, with expected blocksize: one year. 10^4 resamples. Data from CRSP: 1926:1 to 2023:12. ES(5%) is the mean of the worst 5% of the outcomes. Outperformance indicated if $R_T > 1$. Scenario in Table 2.1.*



(a) *Percentiles of R_t, rebalancing.*

(b) *CDF of R_T, rebalancing.*

FIGURE 5.1: *$R_t = W_t^{rebal}/W^{bond}$, where W_t^{rebal} is the wealth of the rebalanced strategy. The rebalanced portfolio uses quarterly rebalancing with a fraction $p = 0.2$ in stocks. The bond portfolio is 30-day T-bills. The stock index is the CRSP capitalization weighted index. All indexes are real (inflation adjusted). Block bootstrap resampling, with expected blocksize: one year. 10^4 resamples. Data from CRSP: 1926:1 to 2023:12. Outperformance indicated if $R_t > 1$. Scenario in Table 2.1.*

ratio, and a larger value of $ES(5\%)$, indicating less extreme left tail risk, and higher probabilities of extremely favourable outcomes.

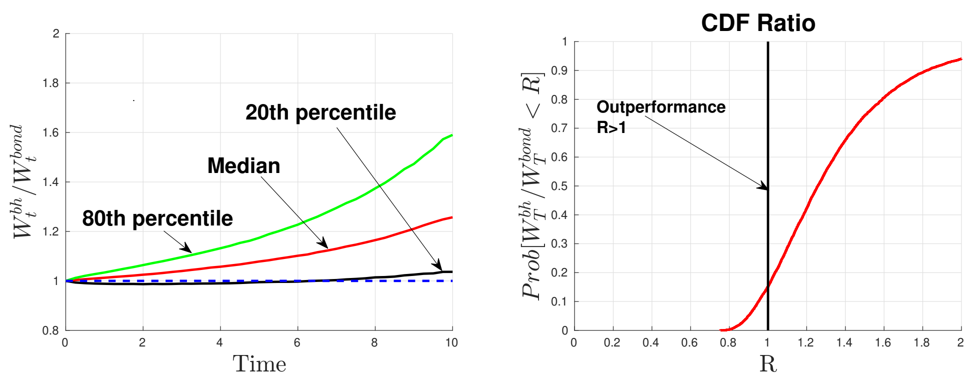
Figure 5.2(a) shows percentiles of R_t versus time. Compared to Figure 5.1(a), the buy and hold strategy spends more time underwater (relative to the all T-bill portfolio) at the 20th percentile, compared to the rebalanced case. However, Figure 5.2(b) shows a very rapid decrease for $R_T < 1$, which results in smaller left tail risk compared to rebalancing.

Figure 5.3 shows the percentiles of the fraction in stocks versus time, for buy and hold. Note that initially, the fraction in stocks is 0.14. Withdrawals are taken from the bond holdings, so that the stock fraction increases simply due to withdrawals, as well as stock appreciation.

Initially, $W_0 = 1000$, so that the amount in stocks is 140. If the bond and stock returns are flat, then after withdrawals from the bonds, the fraction in equities would be about $(140/500) = 0.28$. The median path in Figure 5.3 ends up with $p \simeq 0.40$, indicating increase in stock value. The downside for the buy and hold strategy, from a behavioral point of view, is the large equity fraction as $t \to T$, which of course will increase portfolio volatility. Note that the 80th percentile equity fraction at ten years is about 50%. This might appear cause concern, however, this large equity fraction means that the value in stocks (and hence total portfolio value) has increased substantially.

Strategy	$E[R_T]$	Median$[R_T]$	$R_T : 5^{th}$ percentile	$R_T : 95^{th}$ percentile	ES(5%)	$Prob[R_T < 1]$	Omega(1))
Buy and hold	1.34	1.26	0.90	2.06	0.86	0.15	29.1
Rebalancing	1.26	1.25	0.92	1.64	0.84	0.11	27.7

TABLE 5.3: *Scenario in Table 2.1. Statistics for* $R_T = W_T^{rebal}/W_T^{bond}$*, where* W_T^{rebal} *is the final wealth of the rebalanced strategy, and* $R_T = W_T^{bh}/W^{bond}$*, where* W_T^{bh} *is the final wealth of the buy and hold strategy. The rebalanced portfolio uses quarterly rebalancing with a fraction* $p = 0.2$ *in stocks. The buy and hold strategy has* $p_{init} = 0.14$ *fraction in stocks. The benchmark portfolio is 30-day T-bills. The stock index is the CRSP capitalization weighted index. All indexes are real (inflation adjusted). Block bootstrap resampling, with expected blocksize: one year.* 10^4 *resamples. Data from CRSP: 1926:1 to 2023:12. ES(5%) is the mean of the worst 5% of the outcomes. Outperformance indicated if* $R_T > 1$*. Scenario in Table 2.1.*



(a) *Percentiles of* R_t*, buy and hold.* (b) *CDF of* R_T*, buy and hold.*

FIGURE 5.2: $R_t = W_t^{bh}/W^{bond}$*, where* W_t^{bh} *is the wealth of the buy and hold strategy. The buy and hold portfolio has* $p - 0.14$ *fraction in stocks at* $t = 0$ *and never rebalances. The bond portfolio is 30-day T-bills. The stock index is the CRSP capitalization weighted index. All indexes are real (inflation adjusted). Block bootstrap resampling, with expected blocksize: one year.* 10^4 *resamples. Data from CRSP: 1926:1 to 2023:12. Outperformance indicated if* $R_t > 1$*. Scenario in Table 2.1.*

6 Alternative Performance Criteria: Stochastic Dominance

The pathwise criteria used in the last section is very strict. Suppose $R_t = W_t^{sb}/W_t^{bond}$. In fact, it is not possible to find a strategy such that $Prob[R_T < 1] = 0$, since this would imply the existence of an arbitrage opportunity, i.e. short the bond portfolio, go long the stock-bond portfolio. We could make billions.

We will consider the less strict comparison based on the idea of partial stochastic dominance. For those of you who are not familiar with first order stochastic dominance, you can find some intuition in Appendix B

Again, we will simulate the basic strategies using block bootstrap resampling. Figure 6.1 compares the CDFs for the 100% T-bill portfolio, and the rebalanced and the buy and hold strategies. Consider Figure 6.1(a) (rebalanced portfolio). The probability that the final wealth will be below the target 500 for the rebalanced strategy is about 14%. In contrast, the 100% T-bill portfolio has a 40% probability of being below 500. In this sense the T-bill portfolio is more risky than the portfolio which rebalances to 20% stocks.

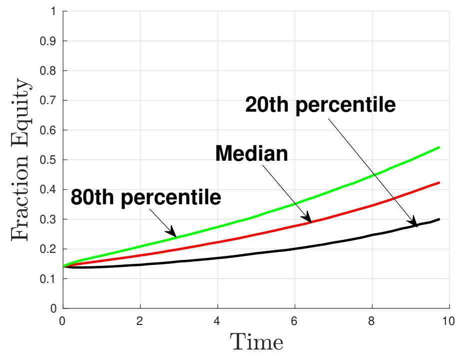
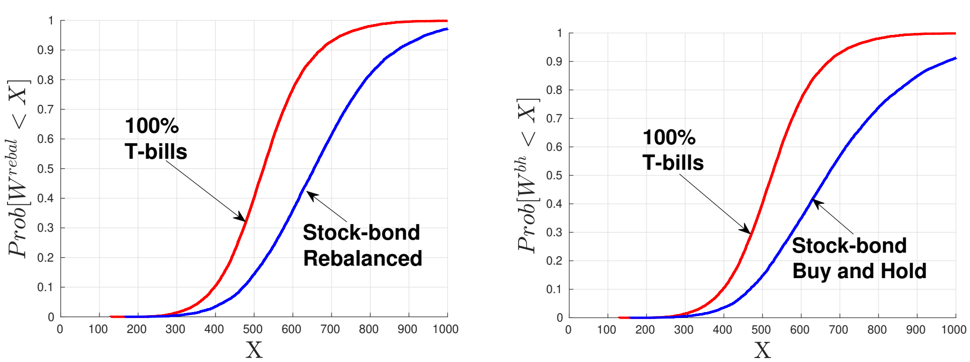


FIGURE 5.3: *Percentiles equity fraction, buy and hold. The buy and hold portfolio has* $p = 0.14$ *fraction in stocks at* $t = 0$ *and never rebalances. The bond portfolio is 30-day T-bills. The stock index is the CRSP capitalization weighted index. All indexes are real (inflation adjusted). Block bootstrap resampling, with expected blocksize: one year.* 10^4 *resamples. Data from CRSP: 1926:1 to 2023:12. Scenario in Table 2.1.*

In fact, this same analysis applies to every point along the X-axis. Given an level of final wealth X, the probability of getting less than X is larger for the 100% T-bill portfolio compared to the rebalanced stock-bond portfolio.

In this case, we can say that the rebalanced strategy *stochastically dominates* (to first order) the 100% T-bill policy. Independent of any particular utility function, any investor who prefers more rather than less will prefer the rebalanced portfolio over the 100% T-bill portfolio. More discussion about stochastic dominance can be found in Appendix B.

Similar results can be seen for the buy-and-hold policy in Figure 6.1(b).



(a) *CDF rebalancing.*

(b) *CDF of buy and hold.*

FIGURE 6.1: *The rebalanced portfolio uses quarterly rebalancing with a fraction* $p = 0.2$ *in stocks. The (5th,10th,50th) percentiles of the rebalanced portfolio are* $(423, 470, 654)$*, with* $Prob[W^{rebal} < 500] = .14$*. The buy and hold portfolio sets the initial fraction of stocks as* $p_{init} = 0.14$*. The (5th,10th,50th) percentiles of buy and hold are* $(422, 467, 667)$*, with* $Prob[W^{bh} < 500] = .15$*. The bond index is 30-day T-bills. The stock index is the CRSP capitalization weighted index. All indexes are real (inflation adjusted). Block bootstrap resampling, with expected blocksize: one year.* 10^4 *resamples. Data from CRSP: 1926:1 to 2023:12. Scenario in Table 2.1.*

Remark 6.1 (Stochastic Dominance for Higher Stock Allocations)**.** *If we repeat the above tests, using the ubiquitous 60-40 portfolio (rebalance to 60% stocks), then the rebalanced portfolio no longer* *stochastically dominates the 100% T-bill portfolio. See Appendix C.*

7 Optimal Policies

We have proposed several simple strategies, i.e. rebalancing to constant weight or buy and hold, for investing a decumulation bucket. However, given a specification of risk and reward, it is possible to treat decumulation as a problem in optimal stochastic control, see for example (Forsyth, 2022; 2021; van Staden et al., 2023; Forsyth et al., 2024). In this case, the optimal control is usually dynamic, i.e. the fraction in stocks responds to investment experience. We can expect improved performance, in terms of risk and reward, using an optimal control, but at the expense of a more complicated strategy.

8 Conclusions

Many studies have shown that retirees are inefficient at decumulating their wealth(Browning et al., 2016; Ackerly et al., 2021). A reasonable strategy is therefore to set aside cash in a decumulation account, with a fixed withdrawal schedule, during the early retirement years, while the retiree is still healthy.

Consider a retiree who wants to spend about half of this decumulation account during the first ten years of retirement. Investing in stocks would seem to expose the retiree to sequence of return risk(Gooderham, 2021). It is tempting to invest the decumulation account in short term, low risk assets (e.g. T-bills).

We use a data-driven approach, based on block bootstrap resampling of historical data, to generate the probability distributions of various investment strategies. All withdrawal amounts and cash balances are inflation adjusted.

A strict pathwise risk analysis of a portfolio of 80% T-bills and 20% stocks, rebalanced quarterly, has the following characteristics:

- the median value of the stock-bond portfolio is 25% greater than the all bond portfolio at the ten year mark;
- along each stochastic path, the rebalanced stock-bond portfolio has 89% probability of a larger terminal wealth than a 100% T-bill portfolio;
- the expected shortfall at the 5% level[8] of the rebalanced stock-bond portfolio, relative to the 100% T-bill portfolio, is 84%;

In addition, the rebalanced the stock-bond portfolio stochastically dominates the all T-bill portfolio. In other words, independent of any utility function, any retiree who prefers more rather than less should choose the stock-bond portfolio over the all bond strategy.

We obtain similar results if we rebalance annually (compared to quarterly) during the 10 year decumulation period. We also obtain similar results for a buy and hold portfolio, whereby we invest about 14% in stocks at the initial time, and never rebalance. The buy and hold strategy would be useful if the decumulation wealth is held in a taxable account.

Consequently, there seems to be overwhelming evidence that for retirees who set up a decumulation account, with high spending during the early years of retirement, that their decumulation account should have at least 20% invested in stocks (if rebalancing) or about 14% in stocks initially (if buy and hold).

[8] Expected shortfall at the five per cent level is the mean of the worst five per cent of the outcomes.

Appendices

A Overview of Block Bootstrap Resampling

A ten year investment scenario consists of 120 consecutive one month returns. A single scenario is constructed as follows. We select a month at random from the historical data, and use this as our first month's return. Then, we select another month at random (with replacement) which is the second month's return in our ten year scenario. We keep doing this until we have a set of 120 returns (one 10-year path). We then repeat this procedure many times, to produce many 10-year return paths.

However, this bootstrapping approach does not take into account possible serial correlation in the returns. This is just another way of saying that next month's returns may be affected by the returns of the past few months or years.

To take this into account, we select an initial month at random, but use b consecutive monthly returns (starting at the initial random month). We repeat this (120/b) times to generate a single 10 year path. We call b the blocksize.

But we are not done yet. It turns out that a better approach is to not use a fixed blocksize, but to specify an average blocksize b, and randomly vary the blocksize within each ten year path. This is called the stationary block bootstrap method.

For more details about this method, see (Politis and Romano, 1994; Politis and White, 2004; Patton et al., 2009; Dichtl et al., 2016; Forsyth and Vetzal, 2019; Anarkulova et al., 2022). Detailed pseudo-code for block bootstrap resampling is given in Ni et al. (2022).

We will block bootstrap the returns for both the CRSP capitalization index, and for the CRSP 30 day T-bill index (both inflation adjusted), based on the historical data over the period 1926:1-2023:12. We will simultaneously draw returns from both the stock index and the bond index (preserving any possible correlations. We use an expected blocksize of one year. Experiments with expected blocksizes ranging from 3 months to two years do not change the results significantly. The basic scenario is shown in Table 2.1.

B Partial Stochastic Dominance: Intuition

How do we compare the performance of various investment strategies, or indexes? Simplistic measures are things like mean, variance and Sharpe Ratio. However, these are just a few summary statistics of investment performance. We would like to look at the entire probability distribution function for each strategy.

Suppose we have an investment strategy, which, starting at time zero, generates wealth W at time T. Of course, in the real world, W is a random variable, with probability density $p(W)$. The cumulative distribution function $F(W)$ is given by

$$F(W) \;\; = \;\; \int_{-\infty}^{W} p(W')\; dW' \; . \tag{B.1}$$

If W_T is a possible value of wealth at time T, then we can interpret the CDF as

$$Prob(W_T < W) \;\; = \;\; F(W), \tag{B.2}$$

that is, $F(W)$ is the probability that we end up with less than W dollars. Suppose we would like to obtain a final wealth of W^* dollars. Then, we would like $F(W^*)$ to be as small as possible, i.e. the probability of ending up with less than W^* is very small, which is what we want.

Figure B.1 shows the cumulative distribution functions of the final wealth W, for two investment strategies, A and B, which we denote by $F_A(W)$ and $F_B(W)$. Consider the point on the x-axis $W = 10{,}000$. We can see that for strategy A, the probability of obtaining less than 10,000 is 0.6, while for strategy B, the probability of obtaining less than 10,000 is 0.86. Hence, if we want to have a strategy which minimizes the probability of obtaining less than 10,000, we would prefer strategy A. However, note that for Figure B.1 we have that

$$F_A(W) \leq F_B(W) \; ; \; -\infty \leq W \leq \infty \tag{B.3}$$

and there is at least one point $\widetilde{W}$ such that $F_A(\widetilde{W}) < F_B(\widetilde{W})$, i.e. a point where equation (B.3) holds with strict inequality. In fact, in Figure B.1, the strict inequality holds for many points. So, we can repeat the argument we went through for $W = 10{,}000$ for every value of W along the x-axis. In other words, for every value of W, strategy A has a smaller probability of ending up with less than W compared to strategy B.

In this case, we say that strategy A stochastically dominates (in the first order sense) strategy B. Any reasonable investor (i.e. with any reasonable utility function) would always prefer strategy A to strategy B. Note that this criteria is based on the entire distribution function, not just a few summary statistics. First order stochastic dominance is easy to spot from the CDFs. If the CDF of strategy A always plots at or below the CDF of strategy B, then A dominates B.

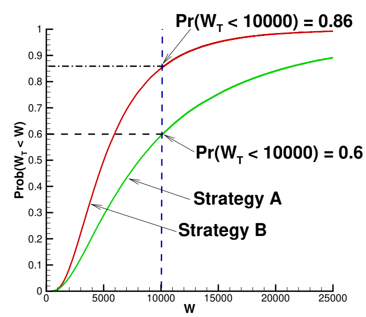


FIGURE B.1: *Illustration of first order stochastic dominance. Cumulative distribution functions for two strategies, in terms of final wealth W. Strategy A stochastically dominates strategy B (in the first order sense). Any reasonable investor would always prefer strategy A.*

However, in practice, given two reasonable strategies, it is rare to find that one strategy dominates another. Often the CDFs cross at various points. For example, Figure B.2 shows CDFs for various strategies. Each of these strategies is *reasonable*, yet no strategy strictly stochastically dominates another strategy, which would be typical.

This leads us to the definition of partial stochastic dominance(van Staden et al., 2021). We say that strategy A partially dominates strategy B if

$$F_A(W) \quad \leq \quad F_B(W) \; ; \; -\infty \leq W \leq W^* \; . \tag{B.4}$$

This a practical criteria: if W^* is quite large (i.e. we would be fabulously wealthy if $W_T > W^*$), then we really don't care if strategy A underperforms B for these large wealth values, as long as A outperforms B for all values of $W_T \leq W^*$. We are very happy with any amount larger than W^*.

We can generalize this a bit more. We can say that A partially dominates B if

$$F_A(W) \quad \leq \quad F_B(W) \; ; \; \hat{W} \leq W \leq W^* \; . \tag{B.5}$$

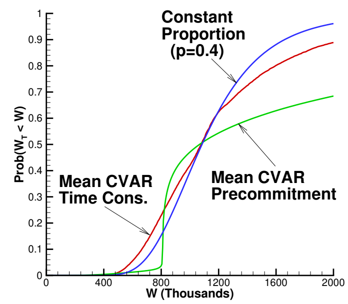


FIGURE B.2: *Comparison of CDFs for various strategies. Note that in general, the CDFs cross, and we do not observe strict stochastic dominance. See Forsyth (2020).*

We have just explained why the upper bound can be a reasonable criteria for partial stochastic dominance. However, at first sight it seems foolhardy to also apply a lower bound criteria $\hat{W}$. This means that we allow A to have a worse performance than B in the left tail, where results are bad.

However, sometimes this can be reasonable. Suppose we start off with an initial wealth of 1000, and that strategy A has $Med[W_T] = 10{,}000$, which looks quite good. Suppose condition (B.5) is satisfied with $\hat{W} = 10$ and $W^* = 50{,}000$. We don't care what happens if we start off with 1000 and end up with more than 50,000.

However, A underperforms B in the left tail where $W_T < 10$. These are the scenarios where essentially everything has turned bad. We started with 1000, and after years of investing, we are left with only 10. Basically, we are bankrupt. Under strategy A, perhaps our probability of having, say, five dollars or less, is twice the probability of strategy B having five dollars or less. So, strategy A has twice the probability of being in this extreme left tail compared to strategy B. This sounds bad. But this is all peanuts compared to our original stake of 1000. So, perhaps in this case, we don't care about the extreme left tail either. The fact that in these bad cases, we are more likely to end up with two cents in our pocket from strategy B compared to one cent from strategy A is cold comfort.

C Rebalance 60-40

Figure C.1 shows that with a more aggressive rebalancing, 60% stocks, the rebalanced portfolio no longer stochastically dominates the 100% bond portfolio.

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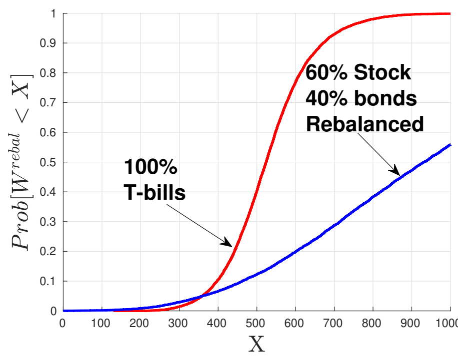

FIGURE C.1: *The rebalanced portfolio uses quarterly rebalancing with a fraction $p = 0.6$ in stocks. The bond index is 30-day T-bills. The stock index is the CRSP capitalization weighted index. All indexes are real (inflation adjusted). Block bootstrap resampling, with expected blocksize: one year. 10^4 resamples. Data from CRSP: 1926:1 to 2023:12. Note that the rebalanced CDF crosses the 100% bond CDF. Scenario in Table 2.1.*

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