The Valuation of Convertible Bonds With Credit Risk

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Abstract

Convertible bonds can be difficult to value, given their hybrid nature of containing elements of both debt and equity. Further complications arise due to the frequent presence of additional options such as callability and puttability, and contractual complexities such as trigger prices and "soft call" provisions, in which the ability of the issuing firm to exercise its option to call is dependent upon the history of its stock price.

This paper explores the valuation of convertible bonds subject to credit risk using an approach based on the numerical solution of linear complementarity problems. We argue that many of the existing models, such as that of Tsiveriotis and Fernandes (1998), are unsatisfactory in that they do not explicitly specify what happens in the event of a default by the issuing firm. We show that this can lead to internal inconsistencies, such as cases where a call by the issuer just before expiry renders the convertible value independent of the credit risk of the issuer, or situations where the implied hedging strategy may not be self-financing. By contrast, we present a general and consistent framework for valuing convertible bonds assuming a Poisson default process. This framework allows various models for stock price behaviour, recovery, and action by holders of the bonds in the event of a default.

We also present a detailed description of our numerical algorithm, which uses a partially implicit method to decouple the system of linear complementarity problems at each timestep. Numerical examples illustrating the convergence properties of the algorithm are provided.

Keywords: Convertible bonds, credit risk, linear complementarity, hedging simulations

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1 Introduction

The market for convertible bonds has been expanding rapidly. In the U.S., over $105 billion of new convertibles were issued in 2001, as compared with just over $60 billion in 2000. As of early in 2002, there were about $270 billion of convertibles outstanding, more than double the level of five years previously, and the global market for convertibles exceeded $500 billion.[1] Moreover, in the past couple of decades there has been considerable innovation in the contractual features of convertibles. Examples include liquid yield option notes (McConnell and Schwartz, 1986), mandatory convertibles (Arzac, 1997), "death spiral" convertibles (Hillion and Vermaelen, 2001), and cross-currency convertibles (Yigitbasioglu, 2001). It is now common for convertibles to feature exotic and complicated features, such as trigger prices and "soft call" provisions. These preclude the issuer from exercising its call option unless the firm's stock price is either above some specified level, has remained above a level for a specified period of time (e.g. 30 days), or has been above a level for some specified fraction of time (e.g. 20 out of the last 30 days).

The modern academic literature on the valuation of convertibles began with the papers of Ingersoll (1977) and Brennan and Schwartz (1977, 1980). These authors build on the "structural" approach for valuing risky non-convertible debt (e.g. Merton, 1974; Black and Cox, 1976; Longstaff and Schwartz, 1995). In this approach, the basic underlying state variable is the value of the issuing firm. The firm's debt and equity are claims contingent on the firm's value, and options on its debt and equity are compound options on this variable. In general terms, default occurs when the firm's value becomes sufficiently low that it is unable to meet its financial obligations.[2] An overview of this type of model is provided in Nyborg (1996). While in principle this is an attractive framework, it is subject to the same criticisms that have been applied to the valuation of risky debt by Jarrow and Turnbull (1995). In particular, because the value of the firm is not a traded asset, parameter estimation is difficult. Also, any other liabilities which are more senior than the convertible must be simultaneously valued.

To circumvent these problems, some authors have proposed models of convertible bonds where the basic underlying factor is the issuing firm's stock price (augmented in some cases with additional random variables such as an interest rate). As this is a traded asset, parameter estimation is simplified (compared to the structural approach). Moreover, there is no need to estimate the values of all other more senior claims. An early example of this approach is McConnell and Schwartz (1986). The basic problem here is that the model ignores the possibility of bankruptcy. McConnell and Schwartz address this in an *ad hoc* manner by simply using a risky discount rate rather than the risk free rate in their valuation equation. More recent papers which similarly include a risky discount rate in a somewhat arbitrary fashion are those of Cheung and Nelken (1994) and Ho and Pfeffer (1996).

An additional complication which arises in the case of a convertible bond (as opposed to risky debt) is that different components of the instrument are subject to different default risks. This is noted by Tsiveriotis and Fernandes (1998), who argue that "the equity upside has zero default risk since the issuer can always deliver its own stock [whereas] coupon and principal payments and any put provisions . . . depend on the issuer's timely access to the required cash amounts, and thus introduce credit risk" (p. 95). To handle this, Tsiveriotis and Fernandes propose splitting convertible bonds into two components: a "cash-only" part, which is subject to credit risk, and an equity part, which is not. This leads to a pair of coupled partial differential equations that can be solved to value convertibles. A simple description of this model in the binomial context may be found in Hull (2003). Yigitbasioglu (2001) extends this framework by adding an interest rate factor and, in the case of cross-currency convertibles, a foreign exchange risk factor.

Recently, an alternative to the structural approach has emerged. This is known as the "reduced-form" approach. It is based on developments in the literature on the pricing of risky debt (see, e.g. Jarrow and Turnbull, 1995; Duffie and Singleton, 1999; Madan and Unal, 2000). In contrast to the structural approach, in this setting default is exogenous, the "consequence of a single jump loss event that drives the equity value to zero and requires cash outlays that cannot be externally financed" (Madan and Unal, 2000, p. 44). The probability of default over the next short time interval is determined by a specified hazard rate. When default occurs, some portion of the bond (either its market value immediately prior to default, or its par value, or the market value of a default-free bond with the same terms) is assumed to be recovered. Authors who have used this approach in the convertible bond context include Davis and Lischka (1999), Takahashi et al. (2001), Hung and Wang (2002), and Andersen and Buffum (2003). As in models such as that of Tsiveriotis and Fernandes (1998), the basic underlying state variable is the firm's stock price (though some of the authors of these papers also consider additional factors such as stochastic interest rates or hazard rates).

[1] See A. Schultz, "In These Convertibles, a Smoother Route to Stocks", *The New York Times*, April 7, 2002.

[2] There are some variations across these models in terms of the precise specification of default. For example, Merton (1974) considers zero-coupon debt and assumes that default occurs if the value of the firm is lower than the face value of the debt at its maturity. On the other hand, Longstaff and Schwartz (1995) assume that default occurs when the firm value first reaches a specified default level, much like a barrier option.

While this approach is quite appealing, the assumption that the stock price instantly jumps to zero in the event of a default is highly questionable. While it may be a reasonable approximation in some circumstances, it is clearly not in others. For instance, Clark and Weinstein (1983) report that shares in firms filing for bankruptcy in the U.S. had average cumulative abnormal returns of -65% during the three years prior to a bankruptcy announcement, and had abnormal returns of about -30% around the announcement. Beneish and Press (1995) find average cumulative abnormal returns of -62% for the three hundred trading days prior to a Chapter 11 filing, and a drop of 30% upon the filing announcement. The corresponding figures for a debt service default are -39% leading up to the announcement and -10% at the announcement. This clearly indicates that the assumption of an instantaneous jump to zero is extreme. In most cases, default is better characterized as involving a gradual erosion of the stock price prior to the event, followed by a significant (but much less than 100%) decline upon the announcement, even in the most severe case of a bankruptcy filing.

However, as we shall see below, in some models it is at least implicitly assumed that a default has no impact on the firm's stock price. This may also be viewed as unsatisfactory. To address this, we propose a model where the firm's stock price drops by a specified percentage (between 0% and 100%) upon a default. This effectively extends the reduced-form approach which, in the case of risky debt, specifies a fractional loss in market value for a bond, to the case of convertibles by similarly specifying a fractional decline in the issuing firm's stock price.

The main contributions of this work are as follows.

- We provide a general single factor framework for valuing risky convertible bonds, assuming a Poisson type default process.
- We consider precisely what happens on default, assuming optimal action by the holder of the convertible. Our framework permits a wide variety of assumptions concerning the behaviour of the stock of the issuing company on default, and also allows various assumptions concerning recovery on default.
- We demonstrate that the widely used convertible bond model of Tsiveriotis and Fernandes (1998) is internally inconsistent.
- We develop numerical methods for determining prices and hedge parameters for convertible bonds under the framework developed here.

The outline of the article is as follows. Section 2 outlines the convertible bond valuation problem in the absence of credit risk. Section 3 reviews credit risk in the case of a simple coupon bearing bond. Section 4 presents our framework for convertible bonds, which is valid for any assumed recovery process. Section 5 then describes some aspects of previous models, with particular emphasis on why the Tsiveriotis and Fernandes (1998) model has some undesirable features. We provide some examples of numerical results in Section 6, and in Section 7, we present some Monte Carlo hedging simulations. These simulations reinforce our contention that the Tsiveriotis and Fernandes (1998) model is inconsistent. Appendix A describes our numerical methods. In some cases a system of coupled linear complementarity problems must be solved. We discuss various numerical approaches for timestepping so that the problems become decoupled. Section 8 presents conclusions.

Since our main interest in this article is the modelling of default risk, we will restrict attention to models where the interest rate is assumed to be a known function of time, and the stock price is stochastic. We can easily extend the models in this paper to handle the case where either or both of the risk free rate and the hazard rate are stochastic. However, this would detract us from our prime goal of determining how to incorporate the hazard rate into a basic convertible pricing model. We also note that practitioners often regard a convertible bond primarily as an equity instrument, where the main risk factor is the stock price, and the random nature of the risk free rate is of second order importance.[3] For ease of exposition, we also ignore various contractual complications such as call notice periods, soft call provisions, trigger prices, dilution, etc.

[3]This is consistent with the results of Brennan and Schwartz (1980), who conclude that "for a reasonable range of interest rates the errors from the [non-stochastic] interest rate model are likely to be slight" (p. 926).

2 Convertible Bonds: No Credit Risk

We begin by reviewing the valuation of convertible bonds under the assumption that there is no default risk. We assume that interest rates are known functions of time, and that the stock price is stochastic. We assume that

$$dS = \mu S dt + \sigma S dz, \tag{2.1}$$

where S is the stock price, μ is its drift rate, σ is its volatility, and dz is the increment of a Wiener process. Following the usual arguments, the no-arbitrage value $V(S,t)$ of any claim contingent on S is given by

$$V_t + \left(\frac{\sigma^2}{2} S^2 V_{SS} + (r(t) - q) S V_S - r(t) V \right) = 0, \tag{2.2}$$

where $r(t)$ is the known interest rate and q is the dividend rate.

We assume that a convertible bond has the following contractual features:

- A continuous (time-dependent) put provision (with an exercise price of B_p).
- A continuous (time-dependent) conversion provision. At any time, the bond can be converted to κ shares.
- A continuous (time-dependent) call provision. At any time, the issuer can call the bond for price $B_c > B_p$. However, the holder can convert the bond if it is called.

Note that option features which are only exercisable at certain times (rather than continuously) can easily be handled by simply enforcing the relevant constraints at those times.

Let

$$\mathcal{L}V \equiv -V_t - \left(\frac{\sigma^2}{2} S^2 V_{SS} + (r(t) - q) S V_S - r(t) V \right). \tag{2.3}$$

We will consider the points in the solution domain where $\kappa S \geq B_c$ and $\kappa S < B_c$ separately:

- $B_c > \kappa S$. In this case, we can write the convertible bond pricing problem as a linear complementarity problem

$$\begin{pmatrix} \mathcal{L}V = 0 \\ (V - \max(B_p, \kappa S)) \geq 0 \\ (V - B_c) \leq 0 \end{pmatrix} \vee \begin{pmatrix} \mathcal{L}V \geq 0 \\ (V - \max(B_p, \kappa S)) = 0 \\ (V - B_c) \leq 0 \end{pmatrix} \vee \begin{pmatrix} \mathcal{L}V \leq 0 \\ (V - \max(B_p, \kappa S)) \geq 0 \\ (V - B_c) = 0 \end{pmatrix} \tag{2.4}$$

  where the notation $(x = 0) \vee (y = 0) \vee (z = 0)$ is to be interpreted as at least one of $x = 0$, $y = 0$, $z = 0$ holds at each point in the solution domain.

- $B_c \leq \kappa S$. In this case, the convertible value is simply

$$V = \kappa S \tag{2.5}$$

  since the holder would choose to convert immediately.

Equation (2.4) is a precise mathematical formulation of the following intuition. The value of the convertible bond is given by the solution to $\mathcal{L}V = 0$, subject to the constraints

$$\begin{aligned} V &\geq \max(B_p, \kappa S) \\ V &\leq \max(B_c, \kappa S). \end{aligned} \tag{2.6}$$

More specifically, either we are in the continuation region where $\mathcal{L}V = 0$ and neither the call constraint nor the put constraint are binding (left side term in (2.4)), or the put constraint is binding (middle term in (2.4)), or the call constraint is binding (right side term in (2.4)).

As far as boundary conditions are concerned, we merely alter the operator $\mathcal{L}V$ at $S = 0$ and as $S \to \infty$. At $S = 0$, $\mathcal{L}V$ becomes

$$\mathcal{L}V \equiv -(V_t - r(t)V); \qquad S \to 0, \tag{2.7}$$

while as $S \to \infty$ we assume that the unconstrained solution is linear in S

$$\mathcal{L}V \equiv V_{SS}; \qquad S \to \infty. \tag{2.8}$$

The terminal condition is given by

$$V(S, t = T) = \max(F, \kappa S), \tag{2.9}$$

where F is the face value of the bond.

Equation (2.4) has been derived by many authors (though not using the precise linear complementarity formulation). However, in practice, corporate bonds are not risk free. To highlight the modelling issues, we will consider a simplified model of risky corporate debt in the next section.

3 A Risky Bond

To motivate our discussion of credit risk, consider the valuation of a simple coupon bearing bond which has been issued by a corporation having a non-zero default risk. The ideas are quite similar to some of those presented in Duffie and Singleton (1999). However, we rely only on simple hedging arguments, and we assume that the risk free rate is a known deterministic function. For ease of exposition, we will assume here (and generally throughout this article) that default risk is diversifiable, so that real world and risk neutral default probabilities will be equal.[4] With this is mind, let the probability of default in the time period t to $t + dt$, conditional on no-default in $[0, t]$, be $p(S, t)dt$, where $p(S, t)$ is a deterministic hazard rate.

Let $B(S, t)$ denote the price of a risky corporate bond. Construct the standard hedging portfolio

$$\Pi = B - \beta S. \tag{3.1}$$

In the absence of default, if we choose $\beta = B_S$, the usual arguments give

$$d\Pi = \left[B_t + \frac{\sigma^2 S^2}{2} B_{SS} \right] dt + o(dt), \tag{3.2}$$

where $o(dt)$ denotes terms that go to zero faster than dt. Assume that:

- The probability of default in $t \to t + dt$ is $p\,dt$.
- The value of the bond immediately after default is RX where $0 \le R \le 1$ is the recovery factor. It is possible to make various assumptions about X. For example, for coupon bearing bonds, it is often assumed that X is the face value. For zero coupon bonds, X can be the accreted value of the issue price, or we could assume that $X = B$, the pre-default value.
- The stock price S is unchanged on default.

Then equation (3.2) becomes

$$\begin{aligned} d\Pi &= (1 - p\,dt) \left[B_t + \frac{\sigma^2 S^2}{2} B_{SS} \right] dt - p\,dt(B - RX) + o(dt) \\ &= \left[B_t + \frac{\sigma^2 S^2}{2} B_{SS} \right] dt - p\,dt(B - RX) + o(dt). \end{aligned} \tag{3.3}$$

The assumption that default risk is diversifiable implies

$$E(d\Pi) = r(t)\Pi\,dt, \tag{3.4}$$

where E is the expectation operator. Combining (3.3) and (3.4) gives

$$B_t + r(t)SB_S + \frac{\sigma^2 S^2}{2} B_{SS} - (r(t) + p)B + pRX = 0. \tag{3.5}$$

[4] Of course, in practice this is not the case (see, for instance, the discussion in Chapter 26 of Hull, 2003). More complex economic equilibrium arguments can be made, but these lead to pricing equations of the same form as we obtain here, albeit with risk-adjusted parameters.

Note that if $p = p(t)$, and we assume that $X = B$, then the solution to equation (3.5) for a zero coupon bond with face value F payable at $t = T$ is

$$B = F \exp\left[-\int_t^T (r(u) + p(u)(1-R))\, du\right] \tag{3.6}$$

which corresponds to the intuitive idea of a spread $s = p(1-R)$.[5]

We can change the above assumptions about the stock price in the event of default. If we assume that the stock price S jumps to zero in the case of default, then equation (3.3) becomes

$$\begin{aligned} d\Pi &= (1 - p\,dt)\left[B_t + \frac{\sigma^2 S^2}{2} B_{SS}\right] dt - p\,dt(B - RX - \beta S) + o(dt) \\ &= \left[B_t + \frac{\sigma^2 S^2}{2} B_{SS}\right] dt - p\,dt(B - RX - \beta S) + o(dt). \end{aligned} \tag{3.7}$$

Following the same steps as above with $\beta = B_S$, we obtain

$$B_t + (r(t) + p) S B_S + \frac{\sigma^2 S^2}{2} B_{SS} - (r(t) + p) B + pRX = 0. \tag{3.8}$$

Note that in this case p appears in the drift term as well as in the discounting term. Even in this relatively simple case of a risky corporate bond, different assumptions about the behavior of the stock price in the event of default will change our valuation. While this is perhaps an obvious point, it is worth remembering that in some popular existing models for convertible bonds no explicit assumptions are made regarding what happens to the stock price upon default.

4 Convertible Bonds With Credit Risk: The Hedge Model

We now consider adding credit risk to the convertible bond model described in Section 2, using the approach discussed in Section 3 for incorporating credit risk. We follow the same general line of reasoning described in Ayache et al. (2002). Let the value of the convertible bond be denoted by $V(S,t)$. To avoid complications at this stage, we assume that there are no put or call features and that conversion is only allowed at the terminal time or in the event of default. Let S^+ be the stock price immediately after default, and S^- be the stock price right before default. We will assume that

$$S^+ = S^-(1-\eta), \tag{4.1}$$

where $0 \leq \eta \leq 1$. We will refer to the case where $\eta = 1$ as the "total default" case (the stock price jumps to zero), and we will call the case where $\eta = 0$ the "partial default" case (the issuing firm defaults but the stock price does not jump anywhere).

As usual, we construct the hedging portfolio

$$\Pi = V - \beta S. \tag{4.2}$$

If there was no credit risk, i.e. $p = 0$, then choosing $\beta = V_S$ and applying standard arguments gives

$$d\Pi = \left[V_t + \frac{\sigma^2 S^2}{2} V_{SS}\right] dt + o(dt). \tag{4.3}$$

Now, consider the case where the hazard rate p is nonzero. We make the following assumptions:

- Upon default, the stock price jumps according to equation (4.1).
- Upon default, the convertible bond holders have the option of receiving
  (a) the amount RX, where $0 \leq R \leq 1$ is the recovery factor (as in the case of a simple risky bond, there are several possible assumptions that can be made about X (e.g. face value, pre-default value of bond portion of the convertible, etc.), but for now, we will not make any specific assumptions), or:

[5] This is analogous to the results of Duffie and Singleton (1999) in the stochastic interest rate context.

(b) shares worth $\kappa S(1-\eta)$.

Under these assumptions, the change in value of the hedging portfolio during $t \to t+dt$ is

$$\begin{aligned} d\Pi &= (1-p\,dt)\left[V_t + \frac{\sigma^2 S^2}{2} V_{SS}\right] dt - p\,dt(V - \beta S\eta)) + p\,dt \max(\kappa S(1-\eta), RX) + o(dt) \\ &= \left[V_t + \frac{\sigma^2 S^2}{2} V_{SS}\right] dt - p\,dt(V - V_S S\eta) + p\,dt \max(\kappa S(1-\eta), RX) + o(dt). \end{aligned} \tag{4.4}$$

Assuming the expected return on the portfolio is given by equation (3.4) and equating this with the expectation of equation (4.4), we obtain

$$r[V - SV_S]\,dt = \left[V_t + \frac{\sigma^2 S^2}{2} V_{SS}\right] dt - p[V - V_S S\eta]\,dt + p[\max(\kappa S(1-\eta), RX)]\,dt + o(dt). \tag{4.5}$$

This implies

$$V_t + (r(t) + p\eta) SV_S + \frac{\sigma^2 S^2}{2} V_{SS} - (r(t)+p)V + p\max(\kappa S(1-\eta), RX) = 0. \tag{4.6}$$

Note that $r(t)+p\eta$ appears in the drift term and $r(t)+p$ appears in the discounting term in equation (4.6). In the case that $R=0$, $\eta=1$, which is the total default model with no recovery, the final result is especially simple: we simply solve the full convertible bond problem (2.4), with $r(t)$ replaced by $r(t)+p$. There is no need to solve an additional equation. This has been noted by Takahashi et al. (2001) and Andersen and Buffum (2003).

Defining

$$\mathcal{M}V \equiv -V_t - \left(\frac{\sigma^2}{2} S^2 V_{SS} + (r(t) + p\eta - q) SV_S - (r(t)+p)V\right), \tag{4.7}$$

we can write equation (4.6) for the case where the stock pays a proportional dividend q as

$$\mathcal{M}V - p\max(\kappa S(1-\eta), RX) = 0. \tag{4.8}$$

We are now in a position to consider the complete problem for convertible bonds with risky debt. We can generalize problem (2.4), using equation (4.8):

- $B_c > \kappa S$

$$\begin{aligned} &\begin{pmatrix} \mathcal{M}V - p\max(\kappa S(1-\eta), RX) = 0 \\ (V - \max(B_p, \kappa S)) \geq 0 \\ (V - B_c) \leq 0 \end{pmatrix} \\ \vee &\begin{pmatrix} \mathcal{M}V - p\max(\kappa S(1-\eta), RX) \geq 0 \\ (V - \max(B_p, \kappa S)) = 0 \\ (V - B_c) \leq 0 \end{pmatrix} \\ \vee &\begin{pmatrix} \mathcal{M}V - p\max(\kappa S(1-\eta), RX) \leq 0 \\ (V - \max(B_p, \kappa S) \geq 0 \\ (V - B_c) = 0 \end{pmatrix} \end{aligned} \tag{4.9}$$

- $B_c \leq \kappa S$

$$V = \kappa S. \tag{4.10}$$

Although equations (4.9)-(4.10) appear formidable, the basic concept is easy to understand. The value of the convertible bond is given by

$$\mathcal{M}V - p\max(\kappa S(1-\eta), RX) = 0, \tag{4.11}$$

subject to the constraints

$$\begin{aligned} V &\geq \max(B_p, \kappa S) \\ V &\leq \max(B_c, \kappa S). \end{aligned} \tag{4.12}$$

Again, as with equation (2.4), equation (4.9) simply says that either we are in the continuation region or one of the two constraints (call or put) is binding. In the following, we will refer to the basic model (4.9)-(4.10) as the *hedge model*, since this model is based on hedging the Brownian motion risk, in conjunction with precise assumptions about what occurs on default.

4.1 Recovery Under The Hedge Model

If we recover RX on default, and X is simply the face value of the convertible, or perhaps the discounted cash flows of an equivalent corporate bond (with the same face value), then X can be computed independently of the value of V and so V can be calculated using equations (4.9)-(4.10). Note that in this case there is only a single equation to solve for the value of the convertible V.

However, this decoupling does not occur if we assume that X represents the *bond* component of the convertible. In this case, the bond component value should be affected by put/call provisions, which are applied to the convertible bond as a whole. Under this recovery model, we need to solve another equation for the bond component B, which must be coupled to the total value V.

We emphasize here that this complication only arises for specific assumptions about what happens on default. In particular, if $R = 0$, then equations (4.9)-(4.10) are independent of X.

4.2 Hedge Model: Recover Fraction of Bond Component

Assume that the total convertible bond value is given by equations (4.9)-(4.10). We will make the assumption that upon default, we recover RB, where B is the pre-default bond component of the convertible. We will now devise a splitting of the convertible bond into two components, such that $V = B + C$, where B is the bond component and C is the equity component. The bond component, in the case where there are no put/call provisions, should satisfy an equation similar to equation (3.8).

We emphasize here that this splitting is required only if we assume that upon default the holder recovers RB, with B being the bond component of the convertible, and C, the equity component, is simply $V - B$. There are many possible ways to split the convertible into two components such that $V = B + C$. However, we will determine the splitting such that B can be reasonably (e.g. in a bankruptcy court) taken to be the bond portion of the convertible, to which the holder is entitled to receive a portion RB on default. The actual specification of what is recovered on default is a controversial issue. We include this case in detail since it serves as a representative example to show that our framework can be used to model a wide variety of assumptions. In the case that $B_p = -\infty$ (i.e. there is no put provision), the bond component should satisfy equation (3.8), with initial condition $B = F$, and $X = B$. Under this circumstance, B is simply the value of risky debt with face value F.

Consequently, in the case where the holder recovers RB on default, we propose the following decomposition for the hedge model

$$\begin{aligned}
&\begin{pmatrix} \mathcal{M}C - p\max(\kappa S(1-\eta) - RB, 0) = 0 \\ (C - (\max(B_c, \kappa S) - B)) \leq 0 \\ (C - (\kappa S - B) \geq 0 \end{pmatrix} \\
\vee &\begin{pmatrix} \mathcal{M}C - p\max(\kappa S(1-\eta) - RB, 0) \leq 0 \\ C = \max(B_c, \kappa S) - B \end{pmatrix} \\
\vee &\begin{pmatrix} \mathcal{M}C - p\max(\kappa S(1-\eta) - RB, 0) \geq 0 \\ C = \kappa S - B \end{pmatrix},
\end{aligned} \tag{4.13}$$

$$\begin{pmatrix} \mathcal{M}B - RpB = 0 \\ B - B_c \leq 0 \\ B - (B_p - C) \geq 0 \end{pmatrix} \vee \begin{pmatrix} \mathcal{M}B - RpB \leq 0 \\ B = B_c \end{pmatrix} \vee \begin{pmatrix} \mathcal{M}B - RpB \geq 0 \\ B = B_p - C \end{pmatrix}. \tag{4.14}$$

Adding together equations (4.13)-(4.14), and recalling that $V = B + C$, it is easy to see that equations (4.9)-(4.10) are satisfied. We informally rewrite equations (4.13) as

$$\mathcal{M}C - p\max(\kappa S(1-\eta) - RB, 0) = 0, \tag{4.15}$$

subject to the constraints

$$\begin{aligned} B + C &\leq \max(B_c, \kappa S) \\ B + C &\geq \kappa S. \end{aligned} \tag{4.16}$$

Similarly, we can also rewrite equations (4.14) as

$$\mathcal{M}B - RpB = 0, \tag{4.17}$$

subject to the constraints

$$\begin{aligned} B &\leq B_c \\ B + C &\geq B_p. \end{aligned} \tag{4.18}$$

Note that the constraints (4.16)-(4.18) embody only the fact that $B+C=V$, that V has constraints, and the requirement that $B \leq B_c$. No other assumptions are made regarding the behaviour of the individual B and C components.

We can write the payoff of the convertible as

$$V(S,T) = F + \max(\kappa S - F, 0), \tag{4.19}$$

which suggests terminal conditions of

$$\begin{aligned} C(S,T) &= \max(\kappa S - F, 0) \\ B(S,T) &= F \end{aligned} \tag{4.20}$$

Consider the case of a zero coupon bond where $p = p(t)$, $B < B_c$, $B_p = 0$. In this case, the solution for B is

$$B = F \exp\left[-\int_t^T \left(r(u) + p(u)(1-R)\right) du\right], \tag{4.21}$$

independent of S. We emphasize that we have made specific assumptions about what is recovered on default in this section. However, the framework (4.9)-(4.10) can accommodate many other assumptions.

4.3 The Hedge Model: Some Special Cases

If we assume that $\eta = 0$ (i.e. the partial default case where the stock price does not jump if a default occurs), the recovery rate $R = 0$, and the bond is continuously convertible, then equations (4.13)-(4.14) become

$$\mathcal{M}V + p(V - \kappa S) = 0 \tag{4.22}$$

in the continuation region. This has a simple intuitive interpretation. The convertible is discounted at the risk free rate plus spread when $V \gg \kappa S$ and at the risk free rate when $V \simeq \kappa S$, with smooth interpolation between these values. Equation (4.22) was suggested in Ayache (2001). Note that in this case, we need only solve a single linear complementarity problem for the total convertible value V.

Making the assumptions that $\eta = 1$ (i.e. the total default case where the stock price jumps to zero upon default) and that the recovery rate $R = 0$, equations (4.13)-(4.14) reduce to

$$\mathcal{M}V = 0 \tag{4.23}$$

in the continuation region, which agrees with Takahashi et al. (2001). In this case, there is no need to split the convertible bond into equity and bond components. If the recovery rate is non-zero, our model is slightly different from that in Takahashi et al.. There it is assumed that upon default the holder recovers RV, compared to model (4.13)-(4.14) where the holder recovers RB. Consequently, for nonzero R, approach (4.13)-(4.14) requires the solution of the coupled set of linear complementarity problems, while the assumption in Takahashi et al. requires only the solution of a single linear complementarity problem. Since the total convertible bond value V includes a fixed income component and an option component, it seems more reasonable to us that in the event of total default (the assumption made in Takahashi et al. (2001)), the option component is by definition worthless and only a fraction of the bond component can be recovered. The total default case also appears to be similar to the model suggested in Davis and Lischka (1999). A similar total default model is also suggested in Andersen and Buffum (2003), for the case $R = 0, \eta = 1$.

As an aside, it is worth observing that if we assume that the stock price of a firm jumps to zero on default, then we can use the above arguments to deduce the PDE satisfied by vanilla puts and calls on the issuer's equity. If the price of an option is denoted by $U(S,t)$, then U is given by the solution to

$$U_t + (r+p)SU_S + \frac{\sigma^2 S^2}{2} U_{SS} - (r+p)U + pU(0,t) = 0. \tag{4.24}$$

This suggests that information about the hazard rate is contained in the market prices of vanilla options.

5 Comparison With Previous Work

There have been various attempts to value convertibles by splitting the total value of a convertible into *bond* and *equity* components, and then valuing each component separately. An early effort along these lines is described in a research note published in 1994 by Goldman Sachs. In this article, the probability of conversion is estimated, and the discount rate is a weighted average of the risk free rate and the risk free rate plus spread, where the weighting factor is the probability of conversion.

More recently, the model described in Tsiveriotis and Fernandes (1998) has become popular. In the following, we will refer to it as the TF model. This model is outlined in the latest edition of Hull's standard text, and has been adopted by several software vendors. We will discuss this model in some detail.

5.1 The TF Model

The basic idea of the TF model is that the *equity* component of the convertible should be discounted at the risk-free rate (as in any other contingent claim), and the *bond* component should be discounted at a risky rate. This leads to the following equation for the convertible value V

$$V_t + \frac{\sigma^2}{2} S^2 V_{SS} + (r_g - q) S V_S - r(V - B) - (r + s)B = 0 \tag{5.1}$$

subject to the constraints

$$\begin{aligned} V &\geq \max(B_p, \kappa S) \\ V &\leq \max(B_c, \kappa S). \end{aligned} \tag{5.2}$$

In equation (5.1), r_g is the growth rate of the stock, s is the spread, and B is the bond component of the convertible. Following the description of this model in Hull (2003), we will assume here that the "growth rate of the stock" is the risk free rate, i.e. $r_g = r$. The bond component satisfies

$$B_t + rSB_S + \frac{\sigma^2 S^2}{2} B_{SS} - (r + s)B = 0. \tag{5.3}$$

Comparing equations (3.5) and (5.3), setting $X = B$, and assuming that s and p are constant, we can see that the spread can be interpreted as $s = p(1 - R)$.

Although not stated in Tsiveriotis and Fernandes (1998), we deduce that the model described therein is a partial default model (stock price does not jump upon default) since the equity part of the convertible is discounted at the risk free rate. Of course, we can extend their model to handle other assumptions about the behaviour of the stock price upon default, while keeping the same decomposition into bond and equity components.

We can write the equation satisfied by the total convertible value V in the TF model as the following linear complementarity problem

- $B_c > \kappa S$

$$\begin{pmatrix} \mathcal{L}V + p(1-R)B = 0 \\ (V - \max(B_p, \kappa S)) \geq 0 \\ (V - B_c) \leq 0 \end{pmatrix} \vee \begin{pmatrix} \mathcal{L}V + p(1-R)B \geq 0 \\ (V - \max(B_p, \kappa S)) = 0 \\ (V - B_c) \leq 0 \end{pmatrix} \vee \begin{pmatrix} \mathcal{L}V + p(1-R)B \leq 0 \\ (V - \max(B_p, \kappa S)) \geq 0 \\ (V - B_c) = 0 \end{pmatrix} \tag{5.4}$$

- $B_c \leq \kappa S$

$$V = \kappa S. \tag{5.5}$$

It is convenient to describe the decomposition of the total convertible price as $V = B + C$, where B is the bond component, and C is the equity component. In general, we can express the solution for $\{V, B, C\}$ in terms of a coupled set of equations. Assuming that equations (5.4)-(5.5) are also being solved for V, then we can specify $\{B, C\}$. In the TF model, the following decomposition is suggested:

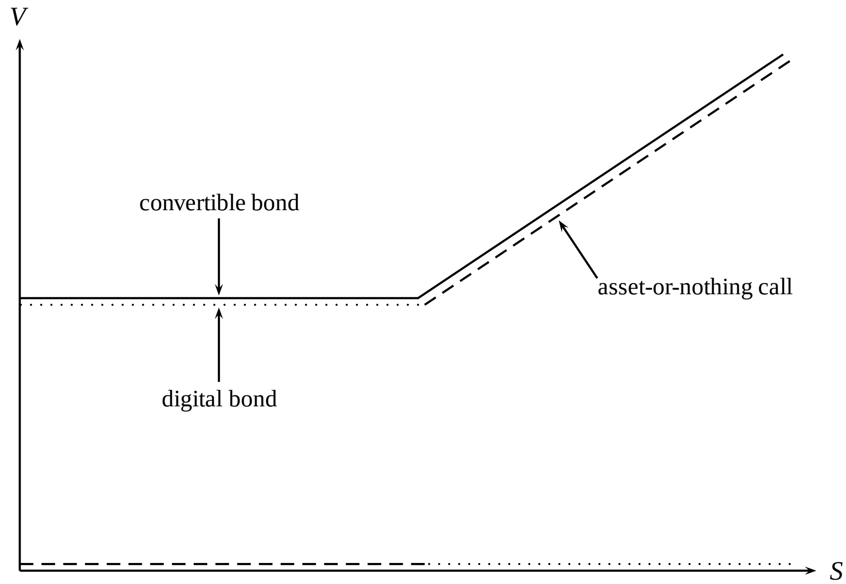

FIGURE 1: *Illustration of the TF method for decomposing a convertible bond into a digital bond plus an asset-or-nothing call.*

- $B_p > \kappa S$

$$\begin{aligned} \mathcal{L}C = 0; \quad \mathcal{L}B + p(1-R)B = 0 \qquad & \text{if } V \neq B_p \text{ and } V \neq B_c \\ B = B_p; \quad C = 0 \qquad & \text{if } V = B_p \\ B = 0; \quad C = B_c \qquad & \text{if } V = B_c. \end{aligned} \tag{5.6}$$

- $B_p \leq \kappa S$

$$\begin{aligned} \mathcal{L}C = 0; \quad \mathcal{L}B + p(1-R)B = 0 \qquad & \text{if } V \neq \max(\kappa S, B_c) \\ C = \max(\kappa S, B_c); \quad B = 0 \qquad & \text{if } V = \max(\kappa S, B_c). \end{aligned} \tag{5.7}$$

It is easy to verify that the sum of equations (5.6)-(5.7) gives equations (5.4)-(5.5), noting that $V = B + C$.

The terminal conditions for the TF decomposition are

$$\begin{aligned} C(S, t = T) &= H(\kappa S - F)(\max(\kappa S - F, 0) + F) \\ B(S, t = T) &= H(F - \kappa S)F, \end{aligned} \tag{5.8}$$

where

$$H(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0. \end{cases} \tag{5.9}$$

However, the splitting in equations (5.6)-(5.7) does not seem to be based on theoretical arguments which require specifying precisely what happens in the case of default. Tsiveriotis and Fernandes (1998) provide no discussion of the actual events in the case of default, and how this would affect the hedging portfolio. There is no clear statement in their paper as to what happens to the stock price in the event of default.

Figure 1 illustrates the decomposition of the convertible bond using equation (5.8). Note that the convertible bond payoff is split into two discontinuous components, a digital bond and an asset-or-nothing call. The splitting occurs at the conversion boundary. This can be expected to cause some difficulties for a numerical scheme, as we have to solve for a problem with a discontinuity which moves over time (as the conversion boundary moves).

5.2 TF Splitting: Call Just Before Expiry

We now turn to discussing some inconsistencies in the TF model. As a first example, consider a case where there are no put provisions, there are no coupons, $\kappa = 1$, conversion is allowed only at the terminal time (or at the call time), and the bond can only be called the instant before maturity, at $t = T^-$. The call price $B_c = F - \varepsilon$, $\varepsilon > 0$, $\varepsilon \ll 1$.

Suppose that the bond is called at $t = T^-$. From equations (5.7) and (5.8), we conclude that we end up effectively solving the original problem with the altered payoff at $t = T^-$

$$\begin{aligned} \mathcal{L}V + pB &= 0 \\ V(S, T^-) &= \max(S, F - \varepsilon) \\ \mathcal{L}B + pB &= 0 \\ B(S, T^-) &= 0. \end{aligned} \tag{5.10}$$

Note that the condition on B at $t = T^-$ is due to the boundary condition (5.7). Now, since the solution of equation (5.10) for B (with $B = 0$ initially) is $B \equiv 0$ for all $t < T^-$, the equation for the convertible bond is simply

$$\begin{aligned} \mathcal{L}V &= 0 \\ V(S, T^-) &= \max(S, F - \varepsilon). \end{aligned} \tag{5.11}$$

In other words, there is no effect of the hazard rate in this case. This peculiar situation comes about because the TF model requires that the bond value be zero if $V = B_c$, even if the effect of the call on the total convertible bond value at the instant of the call is infinitesimally small. This result indicates that calling the bond the instant before expiry with $B_c = F - \varepsilon$ makes the convertible bond value independent of the credit risk of the issuer, which is clearly inappropriate.

5.3 Hedging

As a second example of an inconsistency in the TF framework, we consider what happens if we attempt to dynamically hedge the convertible bond. Since there are two sources of risk (Brownian risk and default risk), we expect that we will need to hedge with the underlying stock and another contingent claim, which we denote by I. This second claim could be, for instance, another bond issued by the same firm. Given the presence of this second hedging instrument, in this context we will drop the assumption that default risk is diversifiable. Thus, in the following λdt is the actual probability of default during $[t, t + dt]$, whereas $p dt$ is its risk-adjusted value.

Consider the hedging portfolio

$$\Pi = V - \beta S - \beta' I + A \tag{5.12}$$

where A is the cash component, which has value $A = -(V - \beta S - \beta' I)$. Assume a real world process of the form

$$dS = (\mu + \lambda\eta) S dt + \sigma S dz - \eta S dq \tag{5.13}$$

where μ is the drift rate and the Poisson default process

$$dq = \begin{cases} 1 & \text{with probability } \lambda\, dt \\ 0 & \text{with probability } 1 - \lambda\, dt \end{cases}.$$

Suppose we choose

$$V_S - \beta' I_S - \beta = 0. \tag{5.14}$$

Using Itô's Lemma, we obtain (from equations (5.12) and (5.14))

$$\begin{aligned} d\Pi = &\left[\frac{\sigma^2 S^2}{2} V_{SS} + V_t - \beta' \left(\frac{\sigma^2 S^2}{2} I_{SS} + I_t \right) \right] dt \\ &+ (\beta S + \beta' I - V)\, r\, dt \\ &+ [\text{ change in } \Pi \text{ on default }]\, dq. \end{aligned} \tag{5.15}$$

We have implicitly assumed in equation (5.15) that the second contingent claim I defaults at precisely the same time as the convertible V.

To avoid tedious algebra, we will assume that the recovery rate of the bond component $R = 0$. If the contingent claims are not called, put, converted, or defaulted in $[t, t + dt]$, then

- hedge model (from equation (4.6))

$$
\begin{aligned}
V_t + \frac{\sigma^2 S^2}{2} V_{SS} &= -\left[(r+p\eta)SV_S - (r+p)V + p\kappa S(1-\eta)\right] \\
I_t + \frac{\sigma^2 S^2}{2} V_{SS} &= -\left[(r+p\eta)SI_S - (r+p)I + p\kappa' S(1-\eta)\right]
\end{aligned}
\tag{5.16}
$$

- TF model (from equation (5.1))

$$
\begin{aligned}
V_t + \frac{\sigma^2 S^2}{2} V_{SS} &= -\left[rSV_S - rV - pB\right] \\
I_t + \frac{\sigma^2 S^2}{2} V_{SS} &= -\left[rSI_S - rI - pB'\right].
\end{aligned}
\tag{5.17}
$$

Note that κ' is the number of shares that a holder of the second claim I would receive in the event of a default, and B' is the bond component of I. We assume that in all cases (noting that $\beta = V_S - \beta' I_S$)

$$
\begin{aligned}
[\text{ change in } \Pi \text{ on default }] &= \kappa S(1-\eta) - \beta S(1-\eta) - \beta'\kappa' S(1-\eta) - (V - \beta' I - \beta S) \\
&= \kappa S(1-\eta) + (V_S - \beta' I_S)S\eta - \beta'\kappa' S(1-\eta) - V + \beta' I.
\end{aligned}
\tag{5.18}
$$

Consequently, for both the hedge model and the TF model, we obtain (from equations (5.15) and (5.18))

$$
\begin{aligned}
d\Pi = &\left[\frac{\sigma^2 S^2}{2} V_{SS} + V_t - \beta'\left(\frac{\sigma^2 S^2}{2} I_{SS} + I_t\right)\right] dt \\
&+ \left[(V_S - \beta' I_S)S + \beta' I - V\right] r\, dt \\
&+ \left[\kappa S(1-\eta) + (V_S - \beta' I_S)S\eta - \beta'\kappa' S(1-\eta) - V + \beta' I\right] dq.
\end{aligned}
\tag{5.19}
$$

For the hedge model, using equation (5.16) in equation (5.19) gives

$$
\begin{aligned}
d\Pi &= -p\, dt\left[SV_S\eta - V + \kappa S(1-\eta) - \beta'\left(\eta SI_S - I + \kappa' S(1-\eta)\right)\right] \\
&\quad + dq\left[\kappa S(1-\eta) + (V_S - \beta' I_S)S\eta - \beta'\kappa' S(1-\eta) - V + \beta' I\right] \\
&= -p\, dt\left[SV_S\eta - V + \kappa S(1-\eta) - \beta'\eta SI_S - I + \kappa' S(1-\eta)\right] \\
&\quad + dq\left[\kappa S(1-\eta) + V_S S\eta - V + \beta' I - I_S S\eta - \kappa'(1-\eta)S\right].
\end{aligned}
\tag{5.20}
$$

Choosing

$$
\beta' = \frac{SV_S\eta - V + \kappa S(1-\eta)}{\eta SI_S - I + \kappa'(1-\eta)S}
\tag{5.21}
$$

and substituting equation (5.21) into equation (5.20) gives

$$
d\Pi = 0,
\tag{5.22}
$$

so that the hedging portfolio is risk free and self-financing under the real world measure.

On the other hand, in the case of the TF model, substituting equation (5.17) into equation (5.19) gives

$$
d\Pi = p\, dt(B - \beta' B') + dq\left[\kappa S(1-\eta) + V_S S\eta - V + \beta'\left(I - I_S S\eta - \kappa'(1-\eta)S\right)\right].
\tag{5.23}
$$

If we choose β' as in equation (5.21), and substitute in equation (5.23), we obtain

$$
d\Pi = \left[B - \beta' B'\right] p\, dt.
\tag{5.24}
$$

This means that the hedging portfolio is no longer self-financing. Another possibility is to require

$$
E\left[d\Pi\right] = 0.
\tag{5.25}
$$

Using equations (5.14), (5.23), and (5.25) gives

$$\beta' = \frac{-\lambda\left(\kappa S(1-\eta) + V_S S\eta - V\right) - pB}{\lambda\left(I - I_S S\eta - \kappa'(1-\eta)S\right) - pB'}. \tag{5.26}$$

Note that in this case β' depends in general on λ. With this choice of β', the variance in the hedging portfolio in $[t, t+dt]$ is

$$\text{Var}\ [d\Pi] = E\left[(d\Pi)^2\right] \tag{5.27}$$

which in general is nonzero, so that the hedging portfolio is not risk free.

Consequently, the hedge model can be used to generate a self-financing hedging zero risk portfolio under the real probability measure. In contrast, the TF model will not generate a hedging portfolio which is both risk free and self-financing. This is simply because in the hedge model we have specified what happens on default, so that the PDE is consistent with the default model.

6 Numerical Examples

A detailed description of the numerical algorithms is provided in Appendix A. In this section, we provide some convergence tests of the numerical methods for some simple and easily reproducible cases, as well as some more realistic examples.

In order to be precise about the way put and call provisions are handled, we will describe the method used to calculate the effects of accrued interest and the coupon payments in some detail. The payoff condition for the convertible bond is (at $t = T$)

$$V(S,T) = \max(\kappa S, F + K_{last}), \tag{6.1}$$

where K_{last} is the last coupon payment. Let t be the current time in the forward direction, t_p the time of the previous coupon payment, and t_n be the time of the next pending coupon payment, i.e. $t_p \leq t \leq t_n$. Then, define the accrued interest on the pending coupon payment as

$$\text{AccI}(t) = K_n \frac{t - t_p}{t_n - t_p} \tag{6.2}$$

where K_n is the coupon payment at $t = t_n$.

The dirty call price B_c and the dirty put price B_p, which are used in equations (4.13)-(4.14) and equations (5.6)-(5.7), are given by

$$\begin{aligned} B_c(t) &= B_c^{cl}(t) + \text{AccI}(t) \\ B_p(t) &= B_p^{cl}(t) + \text{AccI}(t), \end{aligned} \tag{6.3}$$

where B_c^{cl} and B_p^{cl} are the clean prices.

Let t_i^+ be the forward time the instant after a coupon payment, and t_i^- be the forward time the instant before a coupon payment. If K_i is the coupon payment at $t = t_i$, then the discrete coupon payments are handled by setting

$$\begin{aligned} V(S,t_i^-) &= V(S,t_i^+) + K_i \\ B(S,t_i^-) &= B(S,t_i^+) + K_i \\ C(S,t_i^-) &= C(S,t_i^+), \end{aligned} \tag{6.4}$$

where V is the total convertible value and B is the bond component. The coupon payments are modelled in the same way for both the TF and the hedge models.

The data used for the numerical examples is given in Table 1, which is similar to the data used in Tsiveriotis and Fernandes (1998) (except that some data, such as the volatility of the stock price, was not provided in that paper). We will confine these numerical examples to the two limiting assumptions of total default ($\eta = 1.0$) or partial default ($\eta = 0.0$) (see equation (4.1)).

Table 2 demonstrates the convergence of the numerical methods for both models. It is interesting to note that the hedge model partial and total default models appear to give solutions correct to \$.01 with coarse grids/timesteps, while considerably finer grids/timesteps are required to achieve this level of accuracy for the TF model. This reflects

T	5 years
Clean call price	110 in years $2-5$
	0 in years $0-2$
Clean put price	105 at 3 years
r	.05
p	.02
σ	.20
Conversion ratio	1.0
Recovery factor R	0.0
Face value of bond	100
Coupon dates	$.5, 1.0, 1.5, \ldots, 5.0$
Coupon payments	4.0
Total default	$\eta = 1.0$
Partial default	$\eta = 0.0$

TABLE 1: *Data for numerical example. Partial and total default cases defined by equation* (4.1).

Nodes	Timesteps	Hedge Model (Partial Default)	Hedge Model (Total Default)	TF
200	200	124.9158	122.7341	124.0025
400	400	124.9175	122.7333	123.9916
800	800	124.9178	122.7325	123.9821
1600	1600	124.9178	122.7319	123.9754
3200	3200	124.9178	122.7316	123.9714

TABLE 2: *Comparison of hedge (partial and total default) and TF models. Value at* $t = 0, S = 100$. *Data given in Table 1. For the TF model, partially implicit application of constraints. Total default* ($\eta = 1.0$) *and partial default* ($\eta = 0.0$) *defined in equation* (4.1).

our earlier comment (at the end of Section 5.1) that the bond component of the TF model effectively involves a time dependent knock-out barrier, which is difficult to solve accurately. Note that the partial default hedge model gives a price which is about \$1.00 higher than the TF price. In contrast, the total default hedge model is about \$1.00 less than the TF price.

The results in Table 2 should be contrasted with the results in Table 3, where the hazard rate p is set to zero. In this case, the value of the convertible bond is about \$2.00 more than for the TF model, and about \$1.00 more than for the partial default hedge model.

In Appendix A.1, a technique is suggested which decouples the coupled PDEs for B and C for the TF model. In contrast to the methods in Tsiveriotis and Fernandes (1998), we include an extra implicit step at each timestep. Table 4 shows the convergence of the TF model, where the last fully implicit solution of the total bond value in equations (A.12)-(A.14) is included/omitted. In this case, we can compare the results in Table 4 to those in Table 2. We observe

Nodes	Timesteps	Value ($p = 0$)
200	200	125.9500
400	400	125.9523
800	800	125.9528
1600	1600	125.9529
3200	3200	125.9529

TABLE 3: *Value of convertible bond at* $t = 0, S = 100$. *Data given in Table 1, except that the hazard rate* $p = 0$. *In this case, both the TF and the hedge models give the same result.*

Nodes	Timesteps	TF (Table 2) (partially implicit constraints)	TF (explicit constraints)
200	200	124.00249	124.09519
400	400	123.99160	124.05384
800	800	123.98210	124.02508
1600	1600	123.97538	124.00798
3200	3200	123.97141	123.99433
6400	6400	123.97050	123.98531

TABLE 4: *TF model value at $t = 0, S = 100$. Data given in Table 1. Comparison of partially implicit constraints (use equations* (A.12)-(A.14)*) and explicit application of constraints (omit equations* (A.12)-(A.14)*).*

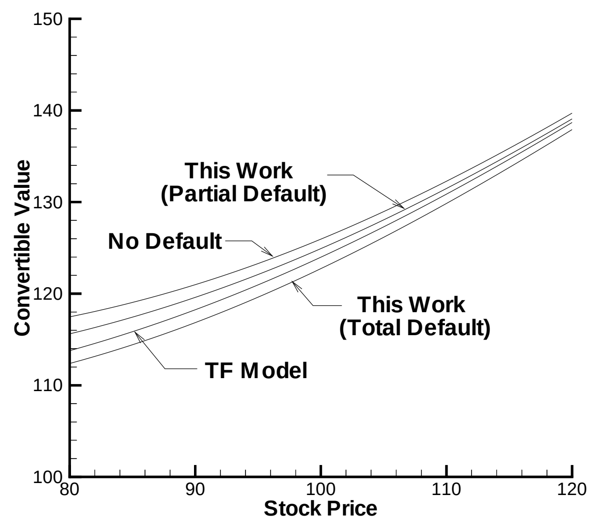

FIGURE 2: *Convertible bond values at $t = 0$, showing the results for no default, the TF model, and the hedge (partial default ($\eta = 0.0$) and total default ($\eta = 1.0$) models. (see equation* (4.1)*). Data as in Table 1.*

that the extra implicit solve (equations (A.12)-(A.14)) does indeed speed up convergence as the grid is refined and the timestep size is reduced.

Figure 2 provides a plot for the cases of no default, the TF model, and the two hedge models (partial and total default). For high enough levels of the underlying stock price, the bond will be converted and all of the models converge to the same value. Similarly, although it is not shown in the figure, as $S \to 0$ all of the models (except for the no default case) converge to the same value as the valuation equation becomes an ordinary differential equation which is independent of η (though not of p). Between these two extremes, the graph reflects the behavior shown in Table 2, with the hedge partial default value above the TF model which is in turn above the hedge total default value. The figure also shows the additional intuitive feature not documented in the table that the case of no default yields higher values than any of the models with default.

It is interesting to see the behavior of the TF bond component and the TF total convertible value an instant before $t = 3$ years. Recall from Table 1 the bond is puttable at $t = 3$, and there is a pending coupon payment as well. Figure 3 shows the discontinuous behaviour of the bond component near the put price for the TF model. Since $V = B + C$, the call component also has a discontinuity.

Figure 4 shows results for the total default hedge model with different recovery factors R (equation (4.9)). We also show the case with no default risk ($p = 0$) for comparison. Note the rather curious fact that for the admittedly unrealistic case of $R = 100\%$, the value of the convertible bond is above the value with no default risk. This can be explained with reference to the hedging portfolio (4.2). Note that the portfolio is long the bond and short the stock. If

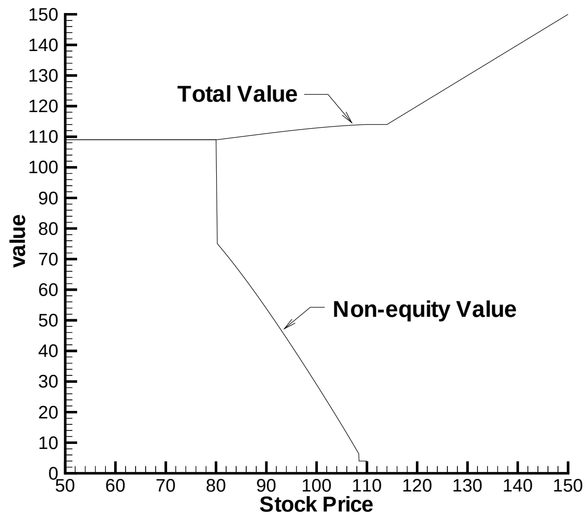

FIGURE 3: *TF model, total and non-equity (bond) component at* $t = 3$ *years, just before coupon payment and put provision. Data as in Table 1.*

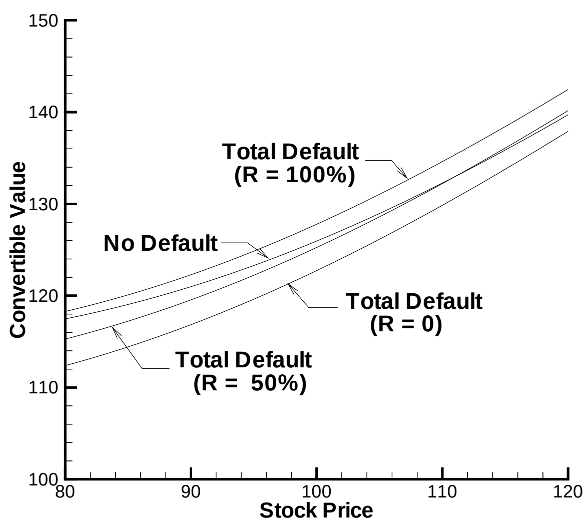

FIGURE 4: *Total default hedge model with different recovery rates. Data as in Table 1.*

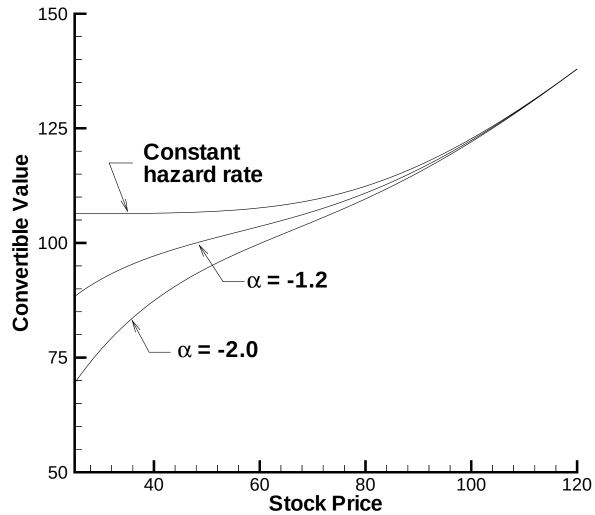

FIGURE 5: *Total default hedge model with constant hazard rate and non-constant hazard rate with different exponents in equation* (6.5)*.* $S_0 = 100$*,* $p_0 = .02$*; other data as in Table 1.*

there is a default, and the recovery factor is high, the hedger obtains a windfall profit, since there is a gain on the short position, and a very small loss on the bond position.

The previous examples used a constant hazard rate (as specified in Table 1). However, it is more realistic to model the hazard rate as increasing as the stock price decreases. A parsimonious model of the hazard rate is given by

$$p(S) = p_0 \left(\frac{S}{S_0} \right)^{\alpha} \tag{6.5}$$

where p_0 is the estimated hazard rate at $S = S_0$. In Muromachi (1999), a function of the form of equation (6.5) was observed to be a reasonable fit to bonds rated BB+ and below in the Japanese market. Typical values for α are in the range from -1.2 to -2.0 (Muromachi, 1999).

In Figure 5 we compare the value of the total default hedge model for constant $p(S)$ as well as for $p(S)$ given by equation (6.5). The data are as in Table 1, except that for the non-constant $p(S)$ cases, we use equation (6.5) with $p_0 = .02$, $S_0 = 100$. Figures 6 and 7 show the corresponding delta and gamma values.

7 Risk Neutral Hedging Simulations

We can gain further insight into the difference between the TF model and the hedge model by considering the hedging performance of these models, but in a risk neutral setting (in contrast to the real world measure considered above in Section 5.3).

Consider the hedging portfolio

$$\Pi_{tot} = V - \beta S + A, \tag{7.1}$$

where the total portfolio Π_{tot} also includes the amount in the risk free bank account which is required to finance the portfolio. Note that $A = \beta S - V$ in cash. Let dG be the gain in the portfolio if no default occurs, and dL be the losses due to default, in the interval $[t, t + dt]$. By definition

$$d\Pi_{tot} - dG - dL = 0.$$

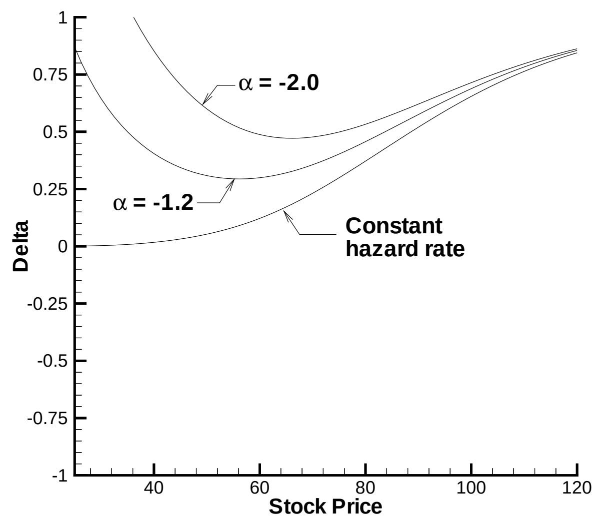

FIGURE 6: *Delta for the total default hedge models with constant hazard rate and non-constant hazard rate with different exponents in equation* (6.5)*.* $S_0 = 100$*,* $p_0 = .02$*; other data as in Table 1.*

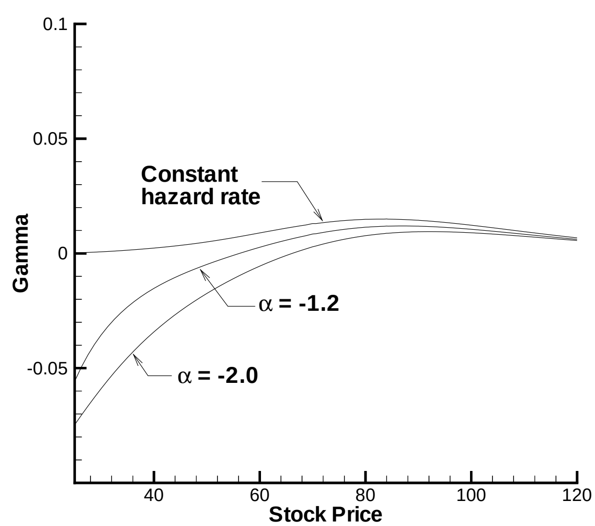

FIGURE 7: *Gamma for the total default hedge models with constant hazard rate and non-constant hazard rate with different exponents in equation* (6.5)*.* $S_0 = 100$*,* $p_0 = .02$*; other data as in Table 1.*

Assume that no default has occurred in $[0,t]$, and that no default occurs in $[t,t+dt]$, then for $\beta = V_S$ we obtain

$$\begin{aligned} d\Pi_{tot} - dG &= 0 \\ &= \left[V_t + \frac{\sigma^2}{2} V_{SS} \right] dt + [\beta S - V]\, r\, dt - dG + o(dt). \end{aligned} \tag{7.2}$$

Equation (7.2) holds for both the TF and the hedge models.

For simplicity in the following, we will assume that the recovery rate $R = 0$. With the further assumption that the convertible bond is not called, put, converted or defaulted in $[t,t+dt]$, it follows that

- hedge model (from equation (4.6))

$$V_t + \frac{\sigma^2 S^2}{2} V_{SS} = -\left[(r + p\eta) S V_S - (r+p) V + p\kappa S(1-\eta) \right]. \tag{7.3}$$

- TF model (from equation (5.1))

$$V_t + \frac{\sigma^2 S^2}{2} V_{SS} = -\left[rSV_S - rV - pB \right]. \tag{7.4}$$

If $p = 0$, then equations (7.2), (7.3), and (7.4) give $dG = 0$. This is to be expected, since setting $\beta = V_S$ eliminates the Brownian risk. However, if $p \neq 0$, then the expected gain in value of the portfolio assuming no default in $[t,t+dt]$ is

- hedge model (from equations (7.2) and (7.3))

$$dG = \left[-p\eta S V_S + pV - p\kappa S(1-\eta) \right] dt. \tag{7.5}$$

- TF model (from equations (7.2) and (7.4))

$$dG = p\, B\, dt. \tag{7.6}$$

Let $E[dG(t)]$ be the expected value of the excess amount in the portfolio if no default occurs in $[t,t+dt]$. Then, given that the probability of no default occurring in $[t,t+dt]$ is $1 - p\,dt$, it follows that

- hedge model (from equation (7.5))

$$E[dG] = \left[-p\eta S V_S + pV - p\kappa S(1-\eta) \right] dt + o(dt). \tag{7.7}$$

- TF model (from equation (7.6))

$$E[dG] = p\, B\, dt + o(dt). \tag{7.8}$$

In the risk neutral measure, expected gains in value of the hedging portfolio must compensate for expected losses due to default. Let S_t^i be the value of S at time t on the i-th realized path of the underlying stock price process. Let $\chi(S_t^i, t)$ be the probability of no default in $[0,t]$, along path S_t^i. Then, the discounted value of the expected no-default gain is

- hedge model (from equation (7.7))

$$E[G]_d = E_{<S_t^i>} \int_0^T \chi(S_t^i, t) e^{-rt} \left[-p\eta S V_S + pV - p\kappa S(1-\eta) \right]_{S_t^i} dt \tag{7.9}$$

- TF model (from equation (7.8))

$$E[G]_d = E_{<S_t^i>} \int_0^T \chi(S_t^i, t) e^{-rt} \left[pB \right]_{S_t^i} dt \tag{7.10}$$

Now consider the losses due to default. Given

$$\Pi_{tot} = V - \beta S + A \tag{7.11}$$

where $A = (\beta S - V)$ in cash, assume that no default has occurred in $[0,t]$, but that default occurs in $[t, t+dt]$. Consequently, on default we have (assuming $R = 0$, and that conversion is possible)

$$\begin{aligned} V &\rightarrow \kappa S(1-\eta) \\ S &\rightarrow S(1-\eta) \\ A &\rightarrow A. \end{aligned} \tag{7.12}$$

Thus

$$\begin{aligned} d\Pi_{tot} &= \Pi_{after} - \Pi_{before} \\ &= \Pi_{after} \\ &= \kappa S(1-\eta) + V_S S \eta - V, \end{aligned} \tag{7.13}$$

which gives

$$dL = \kappa S(1-\eta) + V_S S \eta - V. \tag{7.14}$$

Now, default occurs in $[t, t+dt]$ with probability $p\,dt$, so that the expected discounted losses due to default are

$$E[L]_d = E_{<S_t^i>} \int_0^T \chi(S_t^i, t) e^{-rt} p \left[\kappa S(1-\eta) + V_S S \eta - V\right]_{S_t^i} \, dt. \tag{7.15}$$

Equation (7.15) is valid for both the hedge and the TF models. Moreover, from equations (7.9) and (7.15) we have

$$E[G]_d + E[L]_d = 0 \tag{7.16}$$

for the hedge model. In other words, the expected no-default gains exactly offset the expected default losses for a delta hedged portfolio under the hedge model. Of course, the Brownian motion risk is identically zero along all paths for this model as well. However, from equations (7.10) and (7.15), we see that in general equation (7.16) may not hold for the TF model.

We can verify these results using Monte Carlo simulations. First, we compute and store the discrete PDE linear complementarity solutions for both the TF and the hedge models. The discrete values of V and V_S are stored at each grid point and timestep. We also store flags to indicate whether the convertible bond has been called, put or converted at every grid node and discrete time t_j. We then compute a realized path S_t^i, assuming a process of the form (5.13), but in a risk neutral setting (i.e. with μ replaced by r and λ by its risk neutral counterpart p). At each discrete time $t_j = j\Delta t$, $S = S(t_j)$, we carry out the following steps:

- If the convertible has been called, converted or put, then the simulation along this path ends.
- A random draw is made to determine if default occurs in $[t, t+dt]$. If default occurs, increment the losses using equation (7.14). The simulation ends.
- If the convertible bond is not called, put, converted or defaulted, we can compute the gain from equation (7.9) for the hedge model, or from equation (7.10) for the TF model.
- Repeat for $S(t_{j+1})$ until $t_j = T$.

We then repeat the above for many realized paths to obtain an estimate of equations (7.9), (7.10), and (7.15).

The Monte Carlo hedging simulations were carried out using the data in Table 1 except that we use the variable hazard rate (6.5), with $p_0 = .02, S_0 = 100, \alpha = -1.2$. Various values of η will be used. Figure 8 shows a convergence study of the hedging simulation for $\eta = 1.0$. The expected discounted net value is shown

$$E[\text{Net}] = E[G]_d + E[L]_d. \tag{7.17}$$

Each timestep of the PDE solution was divided into five substeps for the Monte Carlo simulation. Based on the results in Figure 8, it appears that using a PDE solution with 400 nodes and timesteps, and 2×10^6 Monte Carlo trials is accurate to within a cent.

Table 5 shows that to within the accuracy of the Monte Carlo simulations, the hedge model has expected gains (no-default) which exactly compensate for expected losses due to default. In general, this is not true for the TF model, except for a particular choice for the stock jump parameter η.

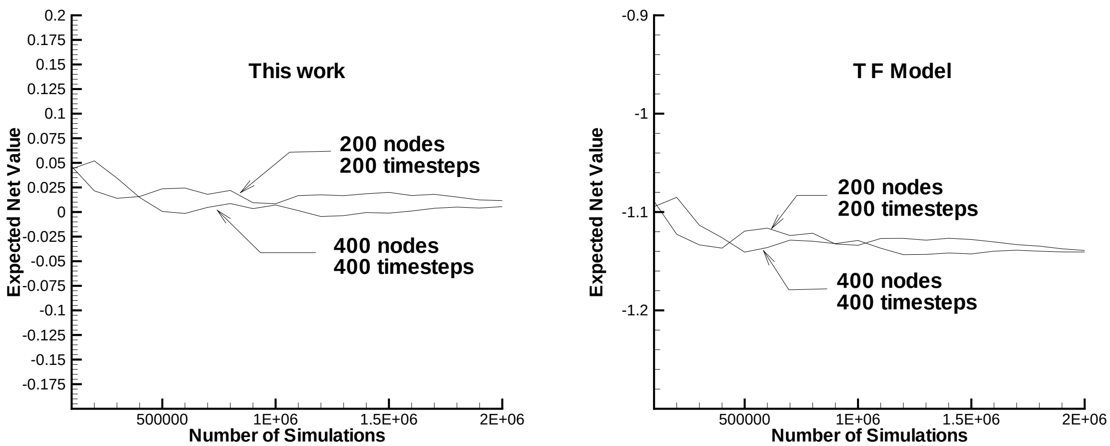

FIGURE 8: *Convergence test for hedging in the risk neutral measure. Expected net value from equation* (7.17)*. Data given in Table 1, but with variable hazard rate* (6.5)*, with* $p_0 = .02, S_0 = 100, \alpha = -1.2$*,* $\eta = 1$.

η	Expected Gain	Expected Loss	Expected Net
Hedge Model			
0.0	1.19521	-1.19575	-0.00054
0.5	2.42083	-2.42162	-0.00079
1.0	3.38514	-3.37966	0.00548
TF Model			
0.0	2.32902	-1.11340	1.21562
0.5	2.32902	-2.29152	0.03750
1.0	2.32902	-3.46964	-1.14062

TABLE 5: *Hedging simulations. Data given in Table 1, with variable hazard rate* (6.5)*, but with* $p_0 = .02, S_0 = 100, \alpha = -1.2$*,* $\eta = 1$*.* 400 *nodes and timesteps in the PDE solve,* 2×10^6 *Monte Carlo trials.*

8 Conclusions

Even in the simple case where the single risk factor is the stock price (interest rates being deterministic), there have been several models proposed for default risk involving convertible bonds. In order to value convertible bonds with credit risk, it is necessary to specify precisely what happens to the components of the hedging portfolio in the event of a default.

In this work, we consider a continuum of possibilities for the value of the stock price after default. Various assumptions can also be made about what is recovered on default. Two special cases which we have examined in detail are:

- Partial default: the stock price is unchanged upon default. The holder of the convertible bond can elect to

  (a) receive a recovery factor times the bond component value, or

  (b) convert the bond to shares.

- Total default: the stock price jumps to zero upon default. The equity component of the convertible bond is, by definition, zero. A fraction of the bond value of the convertible is recovered.

In the case of total default with a recovery factor of zero, this model agrees with that in Takahashi et al. (2001). In this situation, there is no need to split the convertible bond into equity and bond components. In the case of non-zero recovery, our model is slightly different from that in Takahashi et al. (2001). This would appear to be due to a different definition of the term *recovery factor*.

In the partial default case, the model developed in this work uses a different splitting (i.e. bond and equity components) than that used in Tsiveriotis and Fernandes (1998). We have presented several arguments as to why we think their model is somewhat inconsistent. Both the TF model and the model developed here hedge the Brownian risk. However, in the risk neutral measure, the model developed in this paper ensures that the expected value of the net gains and losses due to default is zero. This is not the case for the TF model. Monte Carlo simulations (in a risk neutral setting) demonstrate that the net gain/loss of the TF model due to defaults is significant. It is also possible (using an additional contingent claim) to construct a hedging portfolio which is self-financing and eliminates risk for the hedge model, under a real world default process. This is not possible for the TF model. The impact of model assumptions on real world hedging is also presented.

It is possible to make other assumptions about the behavior of the stock price on default. As well, there may be limits on conversion rights on default, and other assumptions can be made about recovery on default.

The convertible pricing equation is developed by following the following steps

- The usual hedging portfolio is constructed.
- A Poisson default process is specified.
- Specific assumptions are made about the behaviour of the stock price on default, and recovery after default.

It is then straightforward to derive a risk-neutral pricing equation. There are no *ad-hoc* decisions required about which part of the convertible is discounted at the risky rate, and which part is discounted at the risky rate. We emphasize that the framework developed here can accommodate many different assumptions.

Convertible bond pricing generally results in a complex coupled system of linear complementarity problems. We have used a partially implicit method to decouple the system of linear complementarity problems at each timestep. The final value of the convertible bond is computed by solving a full linear complementarity problem (but with explicitly computed source terms), which gives good convergence as the mesh and timestep are reduced, and also results in smooth delta and gamma values.

It is clear that the value of a convertible bond depends on the precise behavior assumed when the issuer goes into default. Given any particular assumption, it is straightforward to model these effects in the framework presented in this paper. A decision concerning which assumptions are appropriate requires an extensive empirical study for different classes of corporate debt.

A Numerical Method

Define $\tau = T - t$, so that the operator $\mathcal{L}V$ becomes

$$\mathcal{L}V = V_\tau - \left(\frac{\sigma^2}{2} S^2 V_{SS} + (r(t) - q) S V_S - r(t) V \right), \tag{A.1}$$

and

$$\mathcal{M}V = V_\tau - \left(\frac{\sigma^2}{2} S^2 V_{SS} + (r(t) + p\eta - q) S V_S - (r(t) + p) V \right). \tag{A.2}$$

It is also convenient to define

$$\mathcal{H}V \equiv \left(\frac{\sigma^2}{2} S^2 V_{SS} + (r(t) - q) S V_S \right), \tag{A.3}$$

and

$$\mathcal{P}V \equiv \left(\frac{\sigma^2}{2} S^2 V_{SS} + (r(t) + p\eta - q) S V_S \right), \tag{A.4}$$

so that equation (A.1) can be written

$$\mathcal{L}V = V_\tau - (\mathcal{H}V - r(t)V), \tag{A.5}$$

and equation (A.2) becomes

$$\mathcal{M}V = V_\tau - (\mathcal{P}V - (r(t) + p)V). \tag{A.6}$$

The terms $\mathcal{H}V$ and $\mathcal{P}V$ are discretized using standard methods (see Zvan et al., 2001; Forsyth and Vetzal, 2001, 2002). Let $V_i^n \equiv V(S_i, \tau^n)$, and denote the discrete form of $\mathcal{H}V$ at (S_i, τ^n) by $(\mathcal{H}V)_i^n$, and the discrete form of $\mathcal{M}V$ by $(\mathcal{M}V)_i^n$. In the following, for ease of exposition, we will describe the timestepping method for a fully implicit discretization of equation (A.1). In actual practice, we use Crank-Nicolson timestepping with the modification suggested in Rannacher (1984) to handle non-smooth initial conditions (which generally occur at each coupon payment). The reader should have no difficulty generalizing the equations to the Crank-Nicolson or BDF (Becker, 1998) case. We also suppress the dependence of r on time for notational convenience.

A.1 The TF Model: Numerical Method

In this section, we describe a method which can be used to solve equations (5.6)-(5.7). We denote the total value of the convertible bond computed using explicit constraints by V^E. A corrected total convertible value, obtained by applying estimates for the constraints in implicit fashion, is denoted by V^I. Given initial values of $(V^E)_i^n, (V^I)_i^n$ and B_i^n, the timestepping proceeds as follows. First, the value of B_i^{n+1} is estimated, ignoring any constraints. We denote this estimate by $(B^*)_i^{n+1}$:

$$\frac{(B^*)_i^{n+1} - B_i^n}{\Delta\tau} = (\mathcal{H}B^*)_i^{n+1} - (r + p)_i^{n+1} (B^*)_i^{n+1}. \tag{A.7}$$

This value of $(B^*)_i^{n+1}$ is then used to compute $(V^E)_i^{n+1}$ from

$$\frac{(V^E)_i^{n+1} - (V^E)_i^n}{\Delta\tau} = (\mathcal{H}V^E)_i^{n+1} - (rV^E)_i^{n+1} - (pB^*)_i^{n+1}. \tag{A.8}$$

Then, we check the minimum value constraints:

For $i = 1, \ldots,$
  $B_i^{n+1} = (B^*)_i^{n+1}$
    If $(B_p > \kappa S)$ then
      If $((V^E)_i^{n+1} < B_p)$ then
        $(B)_i^{n+1} = B_p; \quad (V^E)_i^{n+1} = B_p$
      Endif
    Else
      If $((V^E)_i^{n+1} < \kappa S)$ then
        $(B)_i^{n+1} = 0.0; \quad (V^E)_i^{n+1} = \kappa S$

Endif

Endif

Endfor

Next, the maximum value constraints are applied:

For $i = 1, \ldots,$

If $((V^E)_i^{n+1} > \max(B_c, \kappa S))$ then

$(B)_i^{n+1} = 0; \quad (V^E)_i^{n+1} = \max(B_c, \kappa S)$

Endif

Endfor

In principle, we could simply go on to the next timestep at this point using B_i^{n+1} and $(V^E)_i^{n+1}$. However, we have found that convergence (as the timestep size is reduced) is enhanced and the delta and gamma values are smoother if we add the following steps. Let

$$(QV)_i^{n+1} \equiv \left(\frac{(V^I)_i^{n+1} - (V^I)_i^n}{\Delta\tau} \right) - \left((\mathcal{H}V^I)_i^{n+1} - (rV^I)_i^{n+1} - (pB)_i^{n+1} \right). \tag{A.9}$$

Then if $B_c > \kappa S$, $(V^I))_i^{n+1}$ is determined by solving the discrete linear complementary problem

$$\begin{pmatrix} (QV^I)^{n+1} = 0 \\ ((V^I)^{n+1} - \max(B_p, \kappa S)) \geq 0 \\ ((V^I)^{n+1} - B_c)) \leq 0 \end{pmatrix} \tag{A.10}$$

$$\vee \begin{pmatrix} (QV^I)^{n+1} \geq 0 \\ ((V^I)^{n+1} - \max(B_p, \kappa S)) = 0 \\ ((V^I)^{n+1} - B_c) < 0 \end{pmatrix} \tag{A.11}$$

$$\vee \begin{pmatrix} (QV^I)^{n+1} \leq 0 \\ ((V^I)^{n+1} - \max(B_p, \kappa S)) \geq 0 \\ ((V^I)^{n+1} - B_c) = 0 \end{pmatrix}, \tag{A.12}$$

while if $B_c \leq \kappa S$, we apply the Dirichlet conditions

$$(V^I)_i^{n+1} = \kappa S_i. \tag{A.13}$$

A penalty method (Forsyth and Vetzal, 2002) is used to solve the discrete complementarity problem (A.12). Finally, we set

$$\begin{aligned} (V^E)^{n+1} &= (V^I)^{n+1} \\ B_i^{n+1} &= \min(B_i^{n+1}, (V^I)^{n+1}). \end{aligned} \tag{A.14}$$

The above algorithm essentially decouples the system of linear complementarity problems for B and V by applying the constraints in a partially explicit fashion. However, we apply the constraints as implicitly as possible, without having to solve the fully coupled linear complementarity problem. Consequently, we can only expect first order convergence (in the timestep size $\Delta\tau$), even if Crank-Nicolson timestepping is used. However, this approach makes it comparatively straightforward to experiment with different convertible bond models. As well, it is unlikely that the overhead of the fully coupled approach will result in lower computational cost compared to the decoupled method above (at least for practical convergence tolerances).

A.2 The Hedge Model: Numerical Method

In this section, we describe the numerical method used to solve discrete forms of (4.9)-(4.10) and (4.13)-(4.14). Given initial values of C_i^n and B_i^n, and the total value V_i^{n+1}, the timestepping proceeds as follows. First, the value of B_i^{n+1} is estimated, ignoring any constraints. We denote this estimate by $(B^*)_i^{n+1}$:

$$\frac{(B^*)_i^{n+1} - B_i^n}{\Delta\tau} = (\mathcal{P}B^*)_i^{n+1} - (r+p)_i^{n+1}(B^*)_i^{n+1} + (pRB^*)_i^{n+1}. \tag{A.15}$$

Then, C^{n+1} is estimated, also ignoring constraints. We denote this estimate by $(C^*)_i^{n+1}$:

$$\frac{(C^*)_i^{n+1} - C_i^n}{\Delta\tau} = (\mathcal{P}C)_i^{n+1} - ((r+p)C^*)_i^{n+1} + p\max(\kappa S(1-\eta) - R(B^*)_i^{n+1}, 0). \tag{A.16}$$

Then, we check the minimum value constraints

For $i = 1, \ldots,$
    $B_i^{n+1} = \min(B_c, (B^*)_i^{n+1})$ // overriding maximum constraint
    $C_i^{n+1} = (C^*)_i^{n+1}$
        If $(B_p > \kappa S_i)$ then
            $B_i^{n+1} = \max(B_i^{n+1}, B_p - C_i^{n+1})$
        Else
            $C_i^{n+1} = \max(\kappa S_i - B_i^{n+1}, C_i^{n+1})$
        Endif
Endfor

Then, the maximum value constraints are applied:

For $i = 1, \ldots,$
    $C_i^{n+1} = \min(C_i^{n+1}, \max(\kappa S_i, B_c) - B_i^{n+1})$
Endfor

In principle, we could simply continue on to the next timestep, setting $V_i^{n+1} = B_i^{n+1} + C_i^{n+1}$. However, as for the case of TF splitting, convergence is enhanced if we carry out the following additional steps. Let

$$(\mathcal{T}V)_i^{n+1} \equiv \left(\frac{V_i^{n+1} - V_i^n}{\Delta t}\right) - \left((\mathcal{P}V)_i^{n+1} - ((r+p)V)_i^{n+1} + p\max(\kappa S_i(1-\eta), (RB)_i^{n+1})\right). \tag{A.17}$$

Then V_i^{n+1} (the total convertible value) is determined by solving the discrete linear complementary problem

$$\begin{pmatrix} (\mathcal{T}V)^{n+1} = 0 \\ ((V)^{n+1} - \max(B_p, \kappa S)) \geq 0 \\ ((V)^{n+1} - B_c) \leq 0 \end{pmatrix} \vee \begin{pmatrix} (\mathcal{T}V^I)^{n+1} \geq 0 \\ (V^{n+1} - \max(B_p, \kappa S)) = 0 \\ (V^{n+1} - B_c) < 0 \end{pmatrix} \vee \begin{pmatrix} (\mathcal{T}V^I)^{n+1} \leq 0 \\ (V^{n+1} - \max(B_p, \kappa S)) \geq 0 \\ (V^{n+1} - B_c) = 0 \end{pmatrix} \tag{A.18}$$

if $B_c > \kappa S$, while if $B_c \leq \kappa S$ we apply the Dirichlet conditions

$$V_i^{n+1} = \kappa S_i. \tag{A.19}$$

As with the TF case, a penalty method is used to solve the discrete complementarity problem (A.12). Finally, we set

$$\begin{aligned} B_i^{n+1} &= \min(B_i^{n+1}, V_i^{n+1}) \\ C_i^{n+1} &= V_i^{n+1} - B_i^{n+1}, \end{aligned} \tag{A.20}$$

which ensures that

$$V_i^{n+1} = C_i^{n+1} + B_i^{n+1}. \tag{A.21}$$

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