# Better than pre-commitment mean-variance portfolio allocation strategies: a semi-self-financing Hamilton-Jacobi-Bellman equation approach *

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**Abstract**

**Expanded Version**

We generalize the idea of semi-self-financing strategies, originally discussed in *Ehrbar*, Journal of Economic Theory (1990), and later formalized in *Cui et al*, Mathematical Finance 22 (2012), for the pre-commitment mean-variance (MV) optimal portfolio allocation problem. The proposed semi-self-financing strategies are built upon a numerical solution framework for Hamilton-Jacobi-Bellman equations, and can be readily employed in a very general setting, namely continuous or discrete re-balancing, jump-diffusions with finite activity, and realistic portfolio constraints. We show that if the portfolio wealth exceeds a threshold, an MV optimal strategy is to withdraw cash. These semi-self-financing strategies are generally non-unique. Numerical results confirming the superiority of the efficient frontiers produced by the strategies with positive cash withdrawals are presented. Tests based on estimation of parameters from historical time series show that the semi-self-financing strategy is robust to estimation ambiguities.

**Keywords:** finance, investment analysis, constrained pre-commitment mean-variance, HJB equation

# 1 Introduction

## 1.1 Motivation

The mean-variance (MV) optimization criteria are popular for portfolio allocation problems, due to their intuitive nature (Basak and Chabakauri, 2010; Bielecki et al., 2005; Leippold et al., 2004; Li and Ng, 2000; Vigna, 2014; Wang and Forsyth, 2010; Zhou and Li, 2000). Under these criteria, risk is quantified by variance, so that investors aim to maximize the expected terminal wealth of their portfolios, given a risk level. Hence, the results can be easily interpreted in terms of the trade-off between the risk and the expected terminal portfolio wealth.

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Mean-variance optimization typically yields *pre-commitment* strategies, which are time inconsistent (Basak and Chabakauri, 2010; Björk and Murgoci, 2010; Cui and Li, 2010; Cui et al., 2012; Wang and Forsyth, 2011, 2012). However, it has been shown in Vigna (2014) that pre-commitment strategies can also be viewed as a target-based optimization which involves minimizing a quadratic loss function. Hence, these strategies are appropriate in the context of pension plan investment and insurance applications (Bauerle, 2005; Delong and Gerrard, 2007; Delong et al., 2008; Jose-Fombellida and Rincon-Zapatero, 2008). In fact, this phenomenon has been also discussed in the literature of MV hedging (see, for example Schweizer (2010)). In addition, it has also been pointed out that, in the context of optimal trade execution, the pre-commitment strategy optimizes trading efficiency as measured in practice (Almgren, 2012).

Previous work on pre-commitment MV optimal portfolio allocation has been dominated by the analytic (closed-form) approach. (See, for example, Bielecki et al. (2005); Li and Ng (2000); Øksendal and Sulem (2009); Zhou and Li (2000), among many others.) However, this approach is not feasible when realistic constraints, such as no trading if insolvent and limited leverage, are imposed. In addition, from a risk management point of view, it is useful to model jumps in asset prices. In this case, it is necessary to impose a liquidation condition if the portfolio wealth jumps into the insolvent region. As a result, in these general situations, a fully numerical approach must be employed. It is important to highlight that realistic portfolio constraints and jumps are found to have pronounced effects on the efficient frontiers (Dang and Forsyth, 2014; Wang and Forsyth, 2010).

It is standard that MV strategies for the optimal portfolio allocation problem are *self-financing*, i.e. no exogenous infusion or withdrawal of cash are allowed under any circumstances. Central to our discussion is the concept of semi-self-financing. The term *semi-self-financing strategy* is usually employed to refer to a strategy that exploits either exogenous infusion or withdrawal of cash, but not both. In our context, we strictly define a semi-self-financing strategy as a strategy that uses only non-negative cash withdrawals.

Ehrbar (1990) is possibly the first published work that touches upon the idea of semi-self-financing in the context of MV optimal portfolio allocation. As illustrated in Ehrbar (1990), even for a single-period model, it is possible to achieve a superior portfolio, i.e. a portfolio having the same standard derivation, but higher expected portfolio wealth, by *not* investing all of the initial wealth. It is further argued in Ehrbar (1990) that the self-financing strategy is unrealistic in the sense that it requires the investors "to invest *all* their money, even if the additional investments do *not* add to their utility". By withdrawing, part of the initial portfolio, the investors can achieve superior results. It is also emphasized in Ehrbar (1990) that semi-self-financing strategies are "not only more straightforward", but also allow "investors to find better uses for the money they cannot invest".

Recently, the idea of semi-self-financing in the context of unconstrained pre-commitment MV optimal portfolio allocation is formalized in Cui and Li (2010); Cui et al. (2012). In these papers, it is shown that, if the portfolio wealth exceeds a threshold at a re-balancing time, by removing a certain amount of cash from the portfolio, one can obtain a portfolio having the same expected wealth and standard deviation as the portfolio obtained by a self-financing MV optimal strategy. In addition, the investor receives a bonus in terms of a free cash flow.

## 1.2 Background and contributions

It is well-known that the MV optimal portfolio allocation problem is a multi-criteria optimization problem. Following a standard scalarization method for multi-criteria optimization, a single criterion can be formed by a positively weighted sum of the criteria (Yu, 1974). The resulting single-objective problem is referred to as the *MV scalarization problem*.

However, for MV optimization in general, and MV optimal portfolio allocation in particular, dynamic programming is not directly applicable to the MV scalarization problem, due to the presence of the variance term. To overcome this difficulty, an embedding technique is proposed in Li and Ng (2000); Zhou and Li (2000) to embed the objective of the MV scalarization problem in a new single-objective optimization problem, namely the *embedded MV optimization problem*. Intuitively, this idea can be viewed as a quadratic target investment strategy (Vigna, 2014). Note that the embedding approach can be applied to general non-convex problems, in contrast to a Lagrange multiplier formulation (Li et al., 2002). Non-convex problems can arise if we consider non-linear effects, such as price impact (Tse et al., 2014).

Optimal solutions with respect to the embedded MV optimization problem can be obtained by solving an associated Hamilton-Jacobi-Bellman (HJB) equation. It has been established in Li and Ng (2000); Zhou and Li (2000) that the MV scalarization optimal set is a *subset* of the embedded MV objective set. However, there may be points in the embedded MV objective set which are not in the MV scalarization optimal set. Methods for eliminating such spurious points are discussed in Dang et al. (2015); Tse et al. (2014). In the rest of the paper, to indicate the optimality of a strategy with respect to the MV scalarization problem and to the embedded MV optimization problem, we respectively use the terms *scalarization MV optimal/optimality* and *embedded MV optimal/optimality*.

The main contributions of the paper can be summarized as follows.

- In this paper, we generalize the idea of semi-self-financing strategies developed in Cui and Li (2010); Cui et al. (2012); Ehrbar (1990) for the pre-commitment MV optimal portfolio allocation problem. Using the results in Dang and Forsyth (2014); Dang et al. (2015); Tse et al. (2014), we formulate the embedded MV optimization problem in terms of the numerical solution of an HJB partial integro-differential equation (PIDE). Utilizing a fully numerical approach, it is straightforward to consider continuous or discrete re-balancing, jump-diffusions with finite activity, and realistic portfolio constraints. We determine an embedded MV optimal strategy over all possible semi-self-financing strategies which satisfy the constraints.

- We find certain cases where it can be proved that an embedded MV optimal semi-self-financing strategy involves withdrawing cash from the portfolio. These cases occur when the portfolio wealth exceeds the discounted optimal terminal wealth of the embedded problem. An embedded MV optimal strategy in this case is to (i) withdraw a specified amount of cash, and (ii) invest the remaining amount in the risk-free asset. However, embedded MV optimal semi-self-financing strategies are generally not unique.

- Using the numerical schemes discussed in Dang and Forsyth (2014) for the solution of the HJB equation, and using the results in Dang et al. (2015); Tse et al. (2014), we can guarantee that scalarization MV optimal points, i.e. those that are on efficient frontiers, can be generated from embedded MV optimal points.

- We include several numerical examples to illustrate the superiority of strategies with positive cash withdrawals in a general setting where continuous and discrete re-balancing, realistic constraints, and jump-diffusions (with finite activity) are allowed.

- We estimate the jump diffusion parameters based on an 89 year time series of market return data. The jump parameter estimates are sensitive to the estimation method. However, the simulated investment results using the semi-self-financing mean-variance strategies are robust to estimated model parameter ambiguities.

# 2 Preliminaries

## 2.1 Underlying processes, allowable portfolios, and admissible sets

Since the portfolio can be either continuously or discretely re-balanced, we denote the set of discrete re-balancing times by

$$\mathcal{T}_M = \{t_0 = 0 < t_1 < \ldots < t_M = T\}.$$

Let

$$\mathcal{T} = \begin{cases} [0, T] & \text{continuous re-balancing,} \\ \mathcal{T}_M & \text{discrete re-balancing.} \end{cases}$$

Define $t^- = t - \epsilon$, where $\epsilon \to 0^+$, i.e. $t^-$ is instant of time just before the (forward) time $t$, $t \in [0, T]$.

For simplicity, we assume that there are only two assets available in the financial market, namely a risky asset and a risk-free asset. We denote by $S(t)$ and $B(t)$ the *amounts* invested in risky and risk-free assets, respectively, at time $t$, $t \in [0, T]$. We denote by $\xi$ the random number representing the jump multiplier. When a jump occurs, we have $S(t) = \xi S(t^-)$. As a specific example, in this paper, we consider $\xi$ following a log-normal distribution $p(\xi)$ given by (Merton, 1976)

$$p(\xi) = \frac{1}{\sqrt{2\pi}\zeta\xi} \exp\big(-\frac{(\log(\xi) - \nu)^2}{2\zeta^2}\big) \ , \tag{2.1}$$

with mean $\nu$ and standard deviation $\zeta$, with $E[\xi] = \exp(\nu + \zeta^2/2)$, where $E[\cdot]$ denotes the expectation operator, and $\kappa = E[\xi] - 1$. In the absence of control, $S$ follows the process

$$\frac{dS(t)}{S(t^-)} = (\mu - \lambda\kappa)dt + \sigma dZ + d\left(\sum_{i=1}^{\pi_t} (\xi_i - 1)\right). \tag{2.2}$$

Here, $dZ$ is the increment of a Wiener process, $\mu$ is the real world drift rate, and $\sigma$ is the volatility, $\pi_t$ is a Poisson process with positive intensity parameter $\lambda$, and $\xi_i$ are i.i.d. positive random variables having distribution (2.1). Also, it is assumed that the dynamics of the risk-free asset $B$, in the absence of control, follows

$$dB(t) = rB(t)dt,$$

where $r$ is the (constant) risk-free rate. We make the standard assumption that the real world drift rate of $S$ is strictly greater than $r$. Since there is only one risky asset, it is never optimal (in a MV setting) to short stock, i.e. $S(t) \geq 0$, $t \in [0, T]$. However, we allow short positions in the risk-free asset, i.e. it is possible that $B(t) < 0$, $t \in [0, T]$.

We denote by $X(t) = (S(t), B(t))$, $t \in [0, T]$, the multi-dimensional (controlled) underlying process. Let $c(\cdot) \equiv (d(\cdot), B(\cdot))$ denote the control as a function of the current state at $t \in \mathcal{T}$, i.e.

$$c(\cdot) : (X(t^-), t^-) \mapsto c = c(X(t^-), t^-) \equiv (d(X(t^-), t^-), B(X(t^-), t^-)) \equiv (d(t), B(t)), \quad t \in \mathcal{T}. \tag{2.3}$$

Here, $d(\cdot)$ denotes the *non-negative* cash amount withdrawn from the portfolio before the re-balancing occurs at time $t$, and $B(\cdot)$ is the amount of portfolio wealth invested in the risk-free asset at the re-balancing time $t \in \mathcal{T}$. Note that $T \in \mathcal{T}$, i.e. cash withdrawals are allowed at $T$.

For $t \in \mathcal{T}$, we denote by $x = (s, b) = (S(t^-), B(t^-))$ the state of the system at time $t^-$, and by $(S(x, c), B(x, c))$ the state of the system after the control $c \equiv (d, B)$ is applied. We then have that

$$S(s, b, c \equiv (d, B)) = (s + b) - d - B \ . \tag{2.4}$$

Let the controlled wealth of the portfolio at time $t \in [0, T]$ be denoted by

$$W_c(t) \equiv W_c(S(t), B(t)) = S(t) + B(t), \quad t \in [0, T].$$

We strictly enforce the solvency condition, i.e. the investor can continue trading at $t \in \mathcal{T}$ only if

$$W_c(s, b) = s + b > 0. \tag{2.5}$$

In the event that insolvency (bankruptcy) occurs, we require that the investor immediately liquidate all investments in the risky asset, and cease trading. That is,

$$S = 0 \quad ; \quad B = W_c(s, b) \quad ; \quad \text{if } W_c(s, b) \le 0 \ .$$

We also constrain the leverage ratio, i.e. the investor must select an allocation satisfying

$$\frac{S}{S + B} \le q_{\max} \ , \tag{2.6}$$

where $q_{\max}$ is a known positive constant with typical value in $[1.5, 2.0]$.

Denote by $\mathcal{Z}_{self}$ the usual self-financing admissible set ($d \equiv 0$)

$$\mathcal{Z}_{self} = \left\{ c \equiv (d, B) \in \{0\} \times (-\infty, +\infty) : S = (s + b) - B, \text{ where } S \ge 0, \text{ and } 0 \le \frac{S}{S + B} \le q_{\max} \right\} .$$

We denote by $\mathcal{Z}_{semi}$ the admissible set under a semi-self-financing strategy.

$$\mathcal{Z}_{semi} = \Big\{ c \equiv (d, B) \in [0, +\infty) \times (-\infty, +\infty) : S = (s + b) - d - B, \text{ where } S \ge 0, \\ \text{and } 0 \le \frac{S}{S + B} \le q_{\max} \Big\}.$$

Clearly

$$\mathcal{Z}_{self} \subseteq \mathcal{Z}_{semi} \ . \tag{2.7}$$

In our subsequent discussions, when describing quantities relevant to the semi-self-financing and self-financing cases, the subscripts *semi* and *self* are used, respectively. However, to avoid repetitions, unless otherwise stated, we occasionally omit these subscripts, with the understanding that the discussion applies to both cases. For example, the admissible sets for both semi-self-financing and self-financing cases are collectively denoted as $\mathcal{Z}$.

## 2.2 Efficient frontiers and embedding methods

We respectively denote by $E^{x,t}_{c(\cdot)}[W_c(T)]$ and $Var^{x,t}_{c(\cdot)}[W_c(T)]$ the expectation and the variance of the terminal portfolio wealth conditional on $(x,t)$ and the control $c(\cdot)$, $t \in \mathcal{T}$. We desire to find controls $c(\cdot)$ which generate Pareto optimal points. We use a standard scalarization method to combine the two conflicting criteria, namely maximizing $E^{x,t}_{c(\cdot)}[W_c(T)]$ and simultaneously minimizing $Var^{x,t}_{c(\cdot)}[W_c(T)]$, into a single objective, by means of a positive weighting parameter $\rho$. Specifically, we desire to find the controls which solve

$$P(x,t;\rho) \quad = \quad \inf_{c(\cdot)\in\mathcal{Z}} \left\{ \rho Var^{x,t}_{c(\cdot)}[W_c(T)] - E^{x,t}_{c(\cdot)}[W_c(T)] \right\} . \tag{2.8}$$

More formally, we have the following definitions.

**Definition 2.1** (Achievable MV objective set)**.** *Let $(x_0,t_0) = (X(t=0), t=0)$ denote the initial state. We denote by*

$$\mathcal{Y} = \{(Var^{x_0,t_0}_{c(\cdot)}[W_c(T)], E^{x_0,t_0}_{c(\cdot)}[W_c(T)]) : c \in \mathcal{Z}\} \tag{2.9}$$

*the* ***achievable MV objective set****, and by $\overline{\mathcal{Y}}$ its closure.*

**Definition 2.2** (Scalarization MV optimal set/efficient frontier)**.** *For $\rho > 0$, let*

$$\mathcal{Y}_{P(\rho)} = \left\{ (\mathcal{V}_*, \mathcal{E}_*) \in \bar{\mathcal{Y}} : \rho\mathcal{V}_* - \mathcal{E}_* = \inf_{(\mathcal{V},\mathcal{E})\in\mathcal{Y}} \rho\mathcal{V} - \mathcal{E} \right\} , \tag{2.10}$$

*where $\overline{\mathcal{Y}}$ denotes the closure of $\mathcal{Y}$. We denote the* **scalarization MV optimal set** *w.r.t. $\mathcal{Y}$ as*

$$\mathcal{Y}_P = \bigcup_{\rho>0} \mathcal{Y}_{P(\rho)}.$$

*The scalarization MV optimal set $\mathcal{Y}_P$ is commonly known as the* ***efficient frontier****.*

**Definition 2.3** (Dominating efficient set)**.** *The set $\mathcal{Y}'$ dominates the set $\mathcal{Y}''$ if*

$$\inf_{(\mathcal{V},\mathcal{E})\in\mathcal{Y}'} \rho\mathcal{V} - \mathcal{E} \;\leq\; \inf_{(\mathcal{V},\mathcal{E})\in\mathcal{Y}''} \rho\mathcal{V} - \mathcal{E} \quad ; \quad \forall\rho > 0 \; , \tag{2.11}$$

*and if $\exists\rho > 0$ s.t. (2.11) is a strict inequality.*

Let $\mathcal{Y}^{self} = \{(Var^{x_0,t_0}_{c(\cdot)}[W_c(T)], E^{x_0,t_0}_{c(\cdot)}[W_c(T)]) : c \in \mathcal{Z}_{self}\}$, with the obvious similar definition of $\mathcal{Y}^{semi}$. Let $\mathcal{Y}^{self}_P$ (resp. $\mathcal{Y}^{semi}_P$) be the scalarization MV optimal set w.r.t. $\mathcal{Y}^{self}$ (resp. $\mathcal{Y}^{semi}$). It follows from (2.7) that $\mathcal{Y}^{self} \subseteq \mathcal{Y}^{semi}$. Hence, we have the following obvious result.

**Proposition 2.1.** *$\mathcal{Y}^{self}_P$ cannot dominate $\mathcal{Y}^{semi}_P$.*

In the context of MV optimal portfolio allocation, our objective is to determine the efficient frontier $\mathcal{Y}_P$. We make use of the result in Li and Ng (2000); Zhou and Li (2000) on the embedding technique.

**Definition 2.4** (Embedded MV objective set). *Let $\mathcal{Y}$ be the achievable MV objective set defined in Definition 2.1. The* ***embedded MV objective set****, denoted by $\mathcal{Y}_Q$, is defined by*

$$\mathcal{Y}_Q = \bigcup_{-\infty<\gamma<+\infty} \mathcal{Y}_{Q(\gamma)},$$

*where*

$$\mathcal{Y}_{Q(\gamma)} \quad = \quad \{(\mathcal{V}_*, \mathcal{E}_*) \in \bar{\mathcal{Y}} : \mathcal{V}_* + \mathcal{E}_*^2 - \gamma\mathcal{E}_* = \inf_{(\mathcal{V},\mathcal{E})\in\mathcal{Y}} (\mathcal{V} + \mathcal{E}^2 - \gamma\mathcal{E})\} \ .$$

The embedding result of Li and Ng (2000); Zhou and Li (2000) is summarized in Theorem 2.1.

**Theorem 2.1** (Embedding result). *Let $\mathcal{Y}$ be the achievable MV objective set defined in Definition 2.1. Assume $(\mathcal{V}_0, \mathcal{E}_0) \in \mathcal{Y}_{P(\rho)}$, $\rho > 0$, i.e.*

$$\rho\mathcal{V}_0 - \mathcal{E}_0 = \inf_{(\mathcal{V},\mathcal{E})\in\mathcal{Y}} \rho\mathcal{V} - \mathcal{E} \ . \tag{2.12}$$

*Then*

$$\mathcal{V}_0 + \mathcal{E}_0^2 - \gamma\mathcal{E}_0 = \inf_{(\mathcal{V},\mathcal{E})\in\mathcal{Y}} \mathcal{V} + \mathcal{E}^2 - \gamma\mathcal{E}, \quad i.e. \ \ (\mathcal{V}_0, \mathcal{E}_0) \in \mathcal{Y}_{Q(\gamma)}, \tag{2.13}$$

*where*

$$\gamma = \frac{1}{\rho} + 2\mathcal{E}_0. \tag{2.14}$$

*That is $\mathcal{Y}_P \subseteq \mathcal{Y}_Q$.*

In Appendix A we give a short proof of this, which shows that the embedding result is essentially a geometric property of $\mathcal{Y}$, and hence is valid for any admissible control set, including that for a semi-self-financing strategy.

The optimization problem arising from Theorem 2.1 is of the form

$$V(x,t) = \inf_{c(\cdot)\in\mathcal{Z}} \left\{ E_{c(\cdot)}^{x,t}[(W_c(T) - \gamma/2)^2] \right\} \ , \tag{2.15}$$

where $V(x,t)$ denotes the value function, and the parameter $\gamma \in (-\infty, +\infty)$. Theorem 2.1 implies that there exists a $\gamma \equiv \gamma(x,t,\rho)$, such that, for a given positive $\rho$, a control $c^* \equiv (d^*, B^*)$ which minimizes the objective function of (2.8) also minimizes that of (2.15).

**Remark 2.1** (Scalarization and embedded MV optimal). *A scalarization (resp. embedded) MV optimal control is a control in $\mathcal{Z}$ which minimizes the objective function of* (2.8) *(resp. of (2.15)). A strategy which generates such a control is a scalarization (resp. embedded) MV optimal strategy.*

**Remark 2.2** (Positive $\gamma$). *Suppose $(\mathcal{V}_0, \mathcal{E}_0) \in \mathcal{Y}_{P(\rho)}$ and $\exists \gamma \leq 0$, s.t. $(\mathcal{V}_0, \mathcal{E}_0) \in \mathcal{Y}_{Q(\gamma)}$. From equation (2.14) this implies that $\mathcal{E}_0 < 0$, and by definition $\mathcal{V}_0 \geq 0$. But the strategy of investing all assets in the risk free bond has positive expectation and zero variance. Hence an embedded MV optimal point $(\mathcal{V}_0, \mathcal{E}_0)$ with $\gamma \leq 0$ cannot be scalarization MV optimal. In view of this, we will assume that $\gamma > 0$ in the remainder of this paper.*

From (2.15), the value $\gamma/2$ can essentially be viewed as the optimal value of the terminal portfolio wealth for the embedded problem. Note that $\gamma/2$ is not the expected terminal wealth (see (2.14)).

**Definition 2.5** (Discounted optimal embedded terminal wealth). *We define the **discounted optimal embedded terminal wealth** at time $t$, $t \in [0,T]$, denoted by $W_{opt}(t)$, as (with $\gamma > 0$ from Remark 2.2)*

$$W_{opt}(t) = \frac{\gamma}{2} \mathrm{e}^{-r(T-t)} > 0. \tag{2.16}$$

# 3 Main result

**Theorem 3.1** (A scalarization MV optimal semi-self-financing strategy). *For any $\gamma > 0$, if $W_c(t^-) > W_{opt}(t^-)$, $t \in \mathcal{T}$, then a scalarization MV optimal semi-self-financing strategy (i.e. $\mathcal{Z} = \mathcal{Z}_{semi}$) is to*

- *Step 1: withdraw $d = W_c(t^-) - W_{opt}(t^-)$ from the portfolio, and*
- *Step 2: invest the remaining amount, i.e. $W_{opt}(t^-)$, in the risk-free asset for the remainder of the investment horizon.*

Since the proof of this result requires a fair amount of machinery, we postpone it until Section 5, where we also discuss the non-uniqueness of this strategy.

# 4 Numerical construction of efficient frontiers

We first describe the computational process associated with the value function (2.15). For each fixed value of $\gamma$, $0 < \gamma < +\infty$, we determine the value function $V(s,b,t)$ by solving the associated HJB equation backward in time (Dang and Forsyth, 2014; Wang and Forsyth, 2010, 2012). The HJB equations are given in Appendix B. The terminal condition at time $T$ is

$$V(s,b,T) = (W_c(T) - \gamma/2)^2 \ , \tag{4.1}$$

where

$$W_c(T) = \begin{cases} s+b & \mathcal{Z} = \mathcal{Z}_{self} \ , \quad (4.2a) \\ \min(\gamma/2, s+b) & \mathcal{Z} = \mathcal{Z}_{semi} \ . \quad (4.2b) \end{cases}$$

(Note that equation (4.2b) assumes that the optimal withdrawal $d(T) = \max(s+b-\gamma/2, 0)$ occurs at $T$.) During this solution process, the optimal control $c^*(\cdot)$ can be determined. We then use this control to find the quantity $U(s,b,t) = E^{x,t}_{c^*(\cdot)}[W_c(T)]$, since this information is needed in order to determine the corresponding embedded MV point $(Var^{x_0,t=0}_{c^*(\cdot)}[W_c(T)], E^{x_0,t=0}_{c^*(\cdot)}[W_c(T)]) \in \mathcal{Y}_{Q(\gamma)}$. This last step primarily involves solving an associated linear PDE/PIDE. For details, see Dang and Forsyth (2014).

The above computation is repeated for different values of $\gamma$, each of which give us an embedded MV point in the corresponding $\mathcal{Y}_{Q(\gamma)}$. However, since our objective is to determine the efficient frontier $\mathcal{Y}_P$, the result that $\mathcal{Y}_P \subseteq \mathcal{Y}_Q$, as given in Theorem 2.1, is insufficient by itself. This is due to the fact that, in a general non-convex setting, there exist spurious points, i.e. points in $\mathcal{Y}_Q$ which are not in $\mathcal{Y}_P$. The identification and elimination of spurious points from the set $\mathcal{Y}_Q$ is primarily based on the concept of scalarization optimal points (SOPs) with respect to a set.

**Definition 4.1.** *Let $\mathcal{X}$ be a non-empty subset of $\overline{\mathcal{Y}}$. We define*

$$\mathcal{A}_\rho(\mathcal{X}) = \big\{(\mathcal{V}_*, \mathcal{E}_*) \in \overline{\mathcal{X}} : \rho\mathcal{V}_* - \mathcal{E}_* = \inf_{(\mathcal{V},\mathcal{E})\in\mathcal{X}} \ \rho\mathcal{V} - \mathcal{E}\big\},$$

*where $\overline{\mathcal{X}}$ is the closure of $\mathcal{X}$. We call a point in $\mathcal{A}_\rho(\mathcal{X})$ a scalarization optimal point (SOP) w.r.t. $(\mathcal{X}, \rho)$. We also define*

$$\mathcal{A}(\mathcal{X}) = \big\{(\mathcal{V}_*, \mathcal{E}_*) : (\mathcal{V}_*, \mathcal{E}_*) \text{ is an SOP w.r.t. } (\mathcal{X}, \rho) \text{ for some } \rho > 0\big\}.$$

*We refer to $(\mathcal{V}_0, \mathcal{E}_0) \in \mathcal{A}(\mathcal{X})$ as* **SOP w.r.t.** $\mathcal{X}$.

We have the following result in Tse et al. (2014), which leads to a computational procedure that guarantees generation of points on the efficient frontier.

**Theorem 4.1** (Theorem 4.7 in Tse et al. (2014))**.** *The SOPs w.r.t. $\mathcal{Y}_Q$ are the same as the SOPs w.r.t. $\mathcal{Y}$, i.e.*

$$\mathcal{A}(\mathcal{Y}_Q) = \mathcal{Y}_P = \mathcal{A}(\mathcal{Y}).$$

We also have the following useful result on uniqueness of points in $\mathcal{Y}_Q$ for fixed $\gamma$ from (Tse et al., 2014).

**Theorem 4.2** (Theorem 4.8 in Tse et al. (2014))**.** *If $(\mathcal{V}, \mathcal{E}) \in \mathcal{A}(\mathcal{Y}_Q)$, then there exits a $\gamma$ such that $(\mathcal{V}, \mathcal{E}) \in \mathcal{Y}_{Q(\gamma)}$, and $\mathcal{Y}_{Q(\gamma)}$ is a singleton.*

An issue associated with numerical construction of the set $\mathcal{Y}_Q$ is that, for each embedding parameter $\gamma$, $0 < \gamma < +\infty$, a numerical algorithm applied to the embedded problem can generate only a single embedded MV point $(\mathcal{V}, \mathcal{E}) \in \mathcal{Y}_Q(\gamma)$, while the set $\mathcal{Y}_{Q(\gamma)}$ may contain multiple embedded MV points. Thus, in the context of computation, what available to us is the *computed MV embedded objective set*, denoted by $\mathcal{Y}_Q^\dagger$. This set is defined as follows.

**Definition 4.2** (Computed MV embedded objective set)**.** *Let $\mathcal{Y}^\dagger_{Q(\gamma)}$ be a singleton subset of $\mathcal{Y}_{Q(\gamma)}$. Specifically, $\mathcal{Y}^\dagger_{Q(\gamma)}$ contains either*

- *the unique single point which is SOP w.r.t. $\mathcal{Y}_Q$ if $\mathcal{Y}_{Q(\gamma)}$ is the singleton set containing a point SOP w.r.t. $\mathcal{Y}_Q$, or*
- *an arbitrarily selected single point of $\mathcal{Y}_{Q(\gamma)}$ otherwise.*

*The computed MV objective set is then defined as* $\mathcal{Y}_Q^\dagger = \bigcup_{0<\gamma<+\infty} \mathcal{Y}^\dagger_{Q(\gamma)}$.

Under some technical conditions, which are satisfied in the present case, it can be shown that $\mathcal{Y}_P$ can be generated from $\mathcal{Y}_Q^\dagger$.

**Theorem 4.3** (Theorem 5.4 in Tse et al. (2014))**.** $\mathcal{A}(\mathcal{Y}_Q^\dagger) = \mathcal{Y}_P = \mathcal{A}(\mathcal{Y})$ .

Another issue is that Theorem 4.3, in principle, requires the entire set $\mathcal{Y}_Q^\dagger$ to be available, but, in practice, we can only solve the embedded problem for a finite number of $\gamma \in (0, +\infty)$ values. More specifically, a sampling discretization for $\gamma$ needs to be implemented. We denote by $\Gamma_k$ the finite discrete set of sampled $\gamma$ values at the sampling discretization level $k$. Examples of how $\Gamma_k$ can be refined are given in Dang et al. (2015). Denote by $(\mathcal{Y}_Q^\dagger)^k = \bigcup_{\gamma \in \Gamma_k} \mathcal{Y}_{Q(\gamma)}^\dagger$ the set of all computed MV embedded points using the sampling set $\Gamma^k$.

For a fixed sampling discretization level $k$, we need to compute the set $(\mathcal{Y}_Q^\dagger)^k$ by repeating the computational process associated with the embedded problem outlined earlier for each $\gamma \in \Gamma_k$. Once the set $(\mathcal{Y}_Q^\dagger)^k$ has been computed, we need to construct its SOPs, i.e. $\mathcal{A}((\mathcal{Y}_Q^\dagger)^k)$. This is easily done by determining the vertices of the upper left convex hull of $(\mathcal{Y}_Q^\dagger)^k$ (Tse et al., 2014). The following theorem captures the asymptotic convergence properties of the set $\mathcal{A}((\mathcal{Y}_Q^\dagger)^k)$ with respect to the discretization of the embedding parameter $\gamma$.

**Theorem 4.4** (Theorem 3.1 of Dang et al. (2015))**.** *Every limit point in $\mathcal{A}((\mathcal{Y}_Q^\dagger)^k)$, $k \to +\infty$, is a point in $\mathcal{A}(\mathcal{Y}_Q^\dagger)$.*

**Remark 4.1.** *Theorem 4.4 implies that a computational procedure which involves (i) discretizing the embedding parameter $\gamma$, (ii) constructing the upper left convex hull of the resulting set $(\mathcal{Y}_Q^\dagger)^k$, and (iii) repeating for finer discretizations of the $\gamma$ set, results in a set of limit points which are points on the efficient frontier.*

# 5 Proof of Theorem 3.1

## 5.1 Preliminaries

Since the value function $V(s,b,t)$ (2.15) is defined as an expectation of a non-negative quantity, it immediately follows that $V(s,b,t) \geq 0$. For use later in the paper, we define $\mathcal{T}_{\geq\alpha} = \{t \in \mathcal{T} : t \geq \alpha\}$.

**Proposition 5.1.** *For both $\mathcal{Z} = \mathcal{Z}_{semi}$ and $\mathcal{Z} = \mathcal{Z}_{self}$, for all $t \in [0,T]$, the state $(s,b) = (0, W_{opt}(t))$, where $W_{opt}(t)$ is defined in* (2.16)*, is a (globally) minimum state of the value function $V(s,b,t)$, i.e. $V(0, W_{opt}(t), t) = 0$, $\forall t \in [0,T]$.*

*Proof.* Consider the state $x = (s,b) = (0, W_{opt}(t))$, $t \in [0,T]$, and the strategy $c^*(\cdot)$ defined by neither withdrawing nor re-balancing in $\mathcal{T}_{\geq t}$. Of course, this strategy is in $\mathcal{Z}$. Under $c^*(\cdot)$, $W_c(T) = \gamma/2$ with certainty, i.e. the optimal embedded terminal wealth $\gamma/2$ is achievable under $c^*(\cdot)$. By definition (2.15), we have, also with certainty, that

$$V(0, W_{opt}(t), t) \equiv \inf_{c(\cdot)\in\mathcal{Z}} \left\{ E_{c(\cdot)}^{x,t}[(W_c(T) - \gamma/2)^2] \right\} = E_{c^*(\cdot)}^{x,t}[(W_c(T) - \gamma/2)^2] = 0.$$

This result holds for both the discrete and continuous re-balancing case. □

We are particularly interested in the case $\mathcal{Z} = \mathcal{Z}_{semi}$, and we summarize the important properties of the state $(s,b) = (0, W_{opt}(t))$ in Property 5.1 for $\mathcal{Z} = \mathcal{Z}_{semi}$.

**Property 5.1.** *The following properties for the state $(s,b) = (0, W_{opt}(t))$, $t \in [0,T]$, where $W_{opt}(t)$ is defined in* (2.16)*, hold for both jump-diffusions and pure diffusions, for both discrete and continuous re-balancing, and for $\mathcal{Z} = \mathcal{Z}_{semi}$*

*(a) The state $(s,b) = (0, W_{opt}(t))$ always satisfies the solvency condition* (2.5) *(see Remark 2.2) and the leverage constraint* (2.6)*.*

*(b) By Proposition 5.1 and (a), $(s,b) = (0, W_{opt}(t))$ is a (globally) minimum state of the value function $V(s,b,t)$, $\forall\, t \in [0,T] \supseteq \mathcal{T}$. Hence, a re-balancing at time $t \in \mathcal{T}$ to this state is embedded MV optimal, and remains embedded MV optimal.*

### 5.2 Proof of Theorem 3.1

We are now in a position to prove our main result, Theorem 3.1.

*Proof.* For any $\gamma > 0$, when $W_c(t^-) > W_{opt}(t^-)$, applying the strategy in Theorem 3.1 results in a re-balancing to the state $(s, b) = (0, W_{opt}(t))$. This state, by Proposition 5.1, is a (globally) minimum state of the value function, and remains a (globally) minimum for times in $[t, T] \supseteq \mathcal{T}_{\geq t}$. As a result, there exist no other strategies in $\mathcal{Z}_{semi}$ which can produce a smaller value function (2.15). Hence, the strategy in Theorem 3.1 is embedded MV optimal.

Suppose $(\mathcal{V}^*, \mathcal{E}^*) \in \mathcal{Y}_P^{semi}$. From Theorem 2.1, $\exists \gamma^*$ s.t. $(\mathcal{V}^*, \mathcal{E}^*) \in \mathcal{Y}_{Q(\gamma^*)}^{semi} \subseteq \mathcal{Y}_Q^{semi}$. But, as shown above, the strategy in Theorem 3.1 is embedded MV optimal for *all* $\gamma > 0$, which includes $\gamma^*$ corresponding to $(\mathcal{V}^*, \mathcal{E}^*) \in \mathcal{Y}_P^{semi}$. Thus, this strategy is also scalarization MV optimal. □

We refer to the strategy described in Theorem 3.1 as the *semi-self-financing MV* strategy. If $W_c(t^-) > W_{opt}(t^-)$, then the allocation $(S(t^-), B(t^-)) \to (S, B) \equiv (0, W_{opt}(t))$ is illustrated in Figure 5.1.

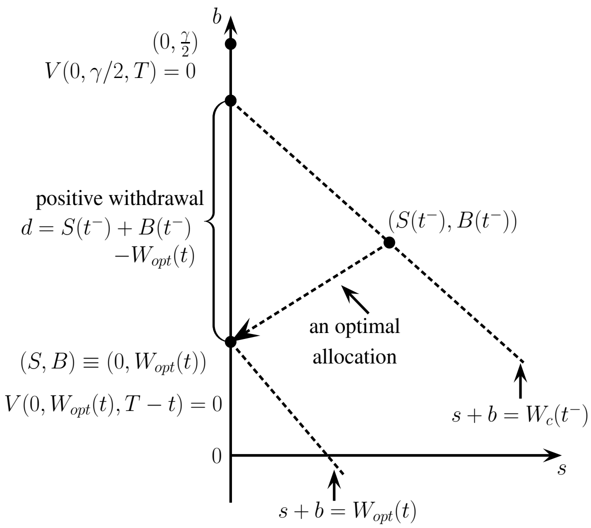



FIGURE 5.1: *Pictorial illustration of the "semi-self-financing MV" strategy*

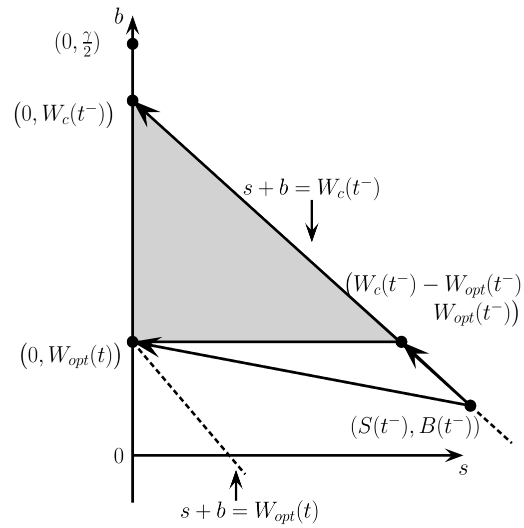



FIGURE 5.2: *Non-uniqueness of embedded MV optimal semi-self-financing strategies.*

### 5.3 Non-uniqueness of embedded MV optimal semi-self-financing strategies

In this subsection, we show that, in general, embedded MV optimal semi-self-financing strategies are non-unique. Recall that $T \in \mathcal{T}$, i.e. withdrawals are permitted at $t = T$.

**Proposition 5.2.** *For both continuous and discrete re-balancing cases, when $\mathcal{Z} = \mathcal{Z}_{semi}$, we have*

$$V(s, b, t) \;\; = \;\; 0 \;\; ; \forall s \geq 0, \; b \geq W_{opt}(t), \forall t \; .$$

*Proof.* Consider the state $x = (s, b)$, $s \geq 0$, and $b \geq W_{opt}(t)$, $t \in [0, T]$. Also consider the control $c^*(\cdot)$ defined by (i) neither withdrawing nor re-balancing in $\mathcal{T}_{\geq t} \setminus \{T\}$, and (ii) applying steps of the strategy in Theorem 3.1 at $T$. Then, under $c^*(\cdot)$, $W_c(T) = \gamma/2$ with certainty, i.e. the optimal embedded terminal wealth $\gamma/2$ is achievable under $c^*(\cdot)$. Thus, we have, also with certainty, that

$$V(s,b,t) \equiv \inf_{c(\cdot)\in\mathcal{Z}} \left\{ E^{x,t}_{c(\cdot)}[(W_c(T)-\gamma/2)^2] \right\} = E^{x,t}_{c^*(\cdot)}[(W_c(T)-\gamma/2)^2] = 0, \quad \forall s \geq 0,\ b \geq W_{opt}(t), \forall t\ .$$

□

If $W_c(t^-) = S(t^-) + B(t^-) > W_{opt}(t^-)$, $t \in \mathcal{T} \setminus \{T\}$, from Proposition 5.2, an embedded MV optimal strategy is

- Step 1: if $B(t^-) < W_{opt}(t^-)$, re-balance the portfolio to a point $(S, B)$ in the triangular region on the $(s, b)$ plane defined by the three points $(0, W_{opt}(t))$, $(W_c(t^-) - W_{opt}(t^-), W_{opt}(t^-))$, and $(0, W_c(t^-))$. This step may involve a positive cash withdrawal.

- Step 2: withdraw from the portfolio, at time $t = T$, the amount exceeding $\gamma/2$.

Possible allocations in Step 1 are illustrated in Figure 5.2 (shaded region, including the boundaries). After Step 2, the optimal terminal wealth for the embedded problem $\gamma/2$ is hit exactly. Note that, if $B(t^-) \geq W_{opt}(t^-)$, then Step 1 can be omitted.

An interesting strategy is the one which corresponds to re-balancing to the point $(W_c(t^-) - W_{opt}(t^-), W_{opt}(t^-))$, i.e. investing $W_{opt}(t^-)$ in the bond, and allocating the remaining wealth to the risky asset. Another strategy is to re-balance to $(0, W_c(t^-))$, i.e. investing all the wealth in bond. No cash withdrawals are needed in these cases at $t$, $t \in \mathcal{T} \setminus \{T\}$. The strategy in Theorem 3.1 corresponds to re-balancing to the point $(0, W_{opt}(t))$. This strategy involves immediately removing $d = W_c(t^-) - W_{opt}(t^-)$ in cash from the portfolio. This corresponds the concept of *free cash flow* as described in Bauerle and Grether (2015); Cui et al. (2012). All of the above strategies involve cash withdrawals of the amount exceeding $\gamma/2$ at $t = T$, and hence, produce the same points in $\mathcal{Y}_Q^{semi}$.

Different choices of strategies simply amount to different ways of handling portfolio wealth in excess of $W_{opt}(t^-)$. Since the strategy in Theorem 3.1 corresponds to the idea of a pure free cash flow of Cui et al. (2012) and Bauerle and Grether (2015), we restrict our attention to this strategy for the remainder of this paper.

**Remark 5.1** (Non-uniqueness of optimal strategies: numerical issues)**.** *We do not compute* $\mathcal{Y}_Q^{semi}$ *but only* $\mathcal{Y}_Q^{\dagger,semi}$ *(see Definition 4.2). However, if* $(\mathcal{V}, \mathcal{E}) \in \mathcal{Y}_{Q(\gamma)}^{\dagger,semi} \subseteq \mathcal{A}(\mathcal{Y}_Q^{\dagger,semi})$*, then from Theorem 4.3,* $(\mathcal{V}, \mathcal{E}) \in \mathcal{Y}_P$*. In other words, even though the embedded MV optimal strategy may be non-unique, the computed point* $(\mathcal{V}, \mathcal{E})$ *is a unique point on the efficient frontier, if* $(\mathcal{V}, \mathcal{E}) \in \mathcal{A}(\mathcal{Y}_Q^{\dagger,semi})$*.*

**Remark 5.2** (Theorem 3.1 strategy: optimality of withdrawing if $W_c(t^-) > W_{opt}(t^-)$)**.** *Consider an alternative strategy: invest all wealth in the risk-free bond if* $W_c(t^-) > W_{opt}(t^-)$*, with no withdrawal at* $t = T$*. In this case, since the total wealth at* $t = T$ *(with certainty) will be larger than* $\gamma/2$*, this will not be embedded MV optimal, hence cannot be scalerization MV optimal. Intuitively, in terms of mean variance, if the terminal wealth exceeds* $\gamma/2$*, then this increases the variance, which is unfavourable from the point of view of mean-variance optimality. Removing the cash from the investable wealth removes this upside penalty.*

### 5.4 Optimality of no withdrawal if $W_c(t^-) \leq W_{opt}(t^-)$, $\forall t \in \mathcal{T}$: $\mathcal{Z} = \mathcal{Z}_{semi}$

We have the following theorem regarding withdrawals when $W_c(t^-) \leq W_{opt}(t^-)$, which holds for cases described in Condition C.1, Appendix C.

**Theorem 5.1.** *When $\mathcal{Z} = \mathcal{Z}_{semi}$ and $W_c(t^-) \leq W_{opt}(t^-)$, $t \in \mathcal{T}$, an optimal embedded MV optimal policy is not to withdraw in the following cases:*

- *continuous re-balancing and jump-diffusions of the form (2.1-2.2), $q_{\max} \geq 1$,*
- *discrete re-balancing and jump-diffusions of the form (2.1-2.2), and no leverage possible (i.e. $q_{\max} = 1$).*
- *continuous or discrete re-balancing and pure diffusions (no jumps), $q_{\max} \geq 1$.*

*Proof.* See Appendix C, specifically Theorem C.2 and Remark C.3. □

Since $\mathcal{Y}_P^{semi} \subseteq \mathcal{Y}_Q^{semi}$, we have the following result.

**Corollary 5.1.** *When $\mathcal{Z} = \mathcal{Z}_{semi}$, for the cases of Theorem 5.1, an optimal scalarization MV policy is to not withdraw when $W_c(t^-) \leq W_{opt}(t^-)$.*

Note that we have not been able to prove that Theorem 5.1 holds for the case of jump-diffusions, discrete rebalancing, and leverage permitted (i.e. $q_{\max} > 1$). In fact, it is not clear that the optimality of no withdrawal for $W_c(t^-) < W_{opt}(t^-)$ holds for this case in general. Nevertheless, in all our numerical tests, we have always observed that, for all cases, it is never embedded MV optimal to withdraw if $W_c(t^-) \leq W_{opt}(t^-)$. However, it remains an open question as to whether this is true in general for any jump processes.

### 5.5 Non-attainability of $W_c(t) > W_{opt}(t)$, $\forall t$

It is interesting to note that in some cases, the portfolio wealth $W_c(t)$ never exceeds $W_{opt}(t)$ $\forall t$. in Vigna (2014), it is proven that, for the case of pure diffusions, continuous re-balancing, no leverage permitted and no solvency constraints, then $W_c(t) < W_{opt}(t)$, i.e. the optimal terminal wealth for the embedded problem is always approached from below. The continuous re-balancing case was also discussed in Cui et al. (2012).

From Bauerle and Grether (2015) we learn that in a complete market, it is never optimal to withdraw cash from the investment portfolio. However, this is a sufficient but not necessary condition. Consider an incomplete market with downward jumps and continuous rebalancing. In this case, an optimal control will always produce $W_c(t) \leq W_{opt}(t)$, hence it is never optimal to withdraw cash in this case.

### 5.6 Free cash flow

The positive cash withdrawal $d$ certainly falls outside the scope of MV framework. However, this positive cash amount is an extra bonus for the investor that should be taken into account. To handle this free cash amount, we follow the concept of free cash flow in Bauerle and Grether (2015); Cui and Li (2010); Cui et al. (2012). Let

$$E_{c^*(\cdot)}^{x_0,0}[d_{tot}] = E\left[\int_0^T d(s)e^{r(T-s)}\, ds\right] \tag{5.1}$$

be the expected value of the cash withdrawals, under the control $c^*(\cdot)$, including interest. This expectation can added to the expected wealth of the portfolio. We refer to this strategy as the *semi-self-financing MV plus free cash* strategy. Note that if we allow only a finite number of rebalancing times $t_\alpha$, then

$$d(s) = \sum_\alpha \delta(s - t_\alpha) d(t_\alpha) \tag{5.2}$$

where $\delta(s - t_\alpha)$ is a Dirac function. In the continuous withdrawal case, we expect that it will be optimal to withdraw cash only once (see Section 5.4). In this case, we denote by $\hat{t}$ the random variable representing the *first* time the cash withdrawal $d(\cdot)$ is positive. This random variable is defined as

$$\hat{t} = \inf\{t \in \mathcal{T} : d(t) > 0\}. \tag{5.3}$$

In this case, equation (5.2) becomes

$$d(s) = \delta(s - \hat{t}) d(\hat{t}) \ . \tag{5.4}$$

To be clear in subsequent discussions, in Table 5.1, we summarize how the expectations and the variances of the terminal wealth are computed for different strategies. These expectations and variances are used to plot efficient frontiers in numerical examples presented in Section 6.

| Strategy | Exp. Val. | Var. |
|---|---|---|
| self-financing MV | $E^{x_0,0}_{c^*(\cdot)}[W_c(T)]$ | $Var^{x_0,0}_{c^*(\cdot)}[W_c(T)]$ |
| semi-self-financing MV | $E^{x_0,0}_{c^*(\cdot)}[W_c(T)]$ | $Var^{x_0,0}_{c^*(\cdot)}[W_c(T)]$ |
| semi-self-financing MV plus free cash | $E^{x_0,0}_{c^*(\cdot)}[W_c(T)] + E^{x_0,0}_{c^*(\cdot)}[d_{tot}]$ | $Var^{x_0,0}_{c^*(\cdot)}[W_c(T)]$ |

TABLE 5.1: *Details as to how the expectation (Exp. Val) and the variance (Var.) of the portfolio terminal wealth are computed for different strategies.*

# 6 Numerical results: representative parameters

In this section, we present selected numerical results of our proposed strategies applied to the MV portfolio allocation problem. In the experiments, we assume process (2.2) in the jump diffusion case, and $\dfrac{dS_t}{S} = \mu dt + \sigma dZ$, for the Geometric Brownian Motion (GBM) case.

We solve the HJB equations in Appendix B using the finite difference method described in Dang and Forsyth (2014). For computational purposes, we localize the original domain to $[0, s_{\max}) \times [-b_{\max}, b_{\max}] \times [0, T]$, where $s_{\max}$ and $b_{\max}$ are positive and sufficiently large. Unless otherwise noted, the details of grid and timestep refinement levels used are given in Table 6.1. For the

| Refinement | Timesteps | $S$ Nodes | $B$ Nodes |
|---|---|---|---|
| 0 | 60 | 70 | 137 |
| 1 | 120 | 139 | 273 |
| 2 | 240 | 277 | 545 |

TABLE 6.1: *Grid and timestep refinement levels used during numerical tests. On each refinement, a new grid point is placed halfway between all old grid points, and the number of timesteps is doubled. Non-uniform grids are used for $s$ and $b$, and a constant timestep size is employed. For the localized domain, we use $s_{\max} = 7 \times 10^6$, $b_{\max} = 3.5 \times 10^6$.*

construction of the efficient frontier, we also need to discretize $\gamma$. In our numerical experiments, when constructing efficient frontiers using refinement levels 1 and 2, we respectively use a total of 30 and 60 values of $\gamma$. Theorem 4.4 guarantees that successive refinements of the $\gamma$ discretization, and constructing the upper left convex hull of the these points, generates limit points which are on the efficient frontier. Details of the numerical scheme are given in Dang and Forsyth (2014).

To illustrate the effect of our proposed strategies on the efficient frontiers, we carry out experiments where, in the case of jump diffusions, the mean jump size is upward (i.e. $\nu > 0$, see equation (2.1)) and mean jump size downward (i.e. $\nu < 0$ ), as well as with pure diffusions. Input parameters and data for these test cases are given in Table 6.2. (The parameters for the mean downward jump-diffusion and pure diffusion cases are used in Dang and Forsyth (2014).) Note

| Parameters | jump-diffusion | | pure |
|---|---|---|---|
| | mean downward | mean upward | diffusion (GBM) |
| $\lambda$ (jump intensity) | 0.05851 | 0.05851 | N/A |
| $\nu$ (jump multiplier mean) | -0.78832 | 0.10000 | N/A |
| $\zeta$ (jump multiplier std) | 0.45050 | 0.45050 | N/A |
| $\mu$ (drift) | 0.07955 | 0.12168 | 0.07955 |
| $\sigma$ (volatility) | 0.17650 | 0.17650 | 0.28175 |
| $\kappa$ (exp. rel. jump amplitude) | -0.49684 | 0.22321 | N/A |
| $\mu - \lambda\kappa$ (effective drift) | 0.10862 | 0.10862 | N/A |
| initial wealth | 100 | 100 | 100 |
| $q_{\max}$ (leverage constraint) | 1.5 | 1.5 | 1.5 |
| $r$ (risk-free interest rate) | 0.04450 | 0.04450 | 0.04450 |
| $T$ (investment horizon) | 20. (years) | 20. (years) | 20. (years) |
| $t_{i+1} - t_i$ (discrete re-balancing time period) | 1.0 (years) | 1.0 (years) | 1.0 (years) |

TABLE 6.2: *Input parameters for mean downward/upward jump-diffusion and pure diffusion test cases. The parameters for mean downward jump-diffusion and pure diffusion are used in Dang and Forsyth (2014). See definitions of jump diffusion parameters in equation (2.1).*

that, compared to the mean downward jump case, for the mean upward jump case, we increase the mean ($\nu$) of the jump multiplier ($\xi$) from -0.78832 to 0.1, and the drift $\mu$ from 0.07955 to 0.12168, while keeping other parameters the same. Hence, both jump cases have the same compensated drift. As a result, the changes in the efficient frontiers observed in these two cases entirely come from the effect of the jump term, and not from the drift term. In this section, we will show the superiority of the efficient frontiers produced by the strategies with positive cash withdrawals.

## 6.1 Effect of truncated boundaries and a discretization error check

As we mentioned earlier, errors are introduced in truncating an infinite domain for the localized problem. However, we can make these errors small by choosing $b_{\max}$ and $s_{\max}$ sufficiently large. As an illustrative example of this point, we carry out experiments with the semi-self-financing control with discrete re-balancing and upward jumps. Table 6.3 shows the expectation values (Exp. Val.) and the standard deviations (Std. Dev.) of the portfolio wealth obtained with various large boundary values for $s_{\max}$ and $b_{\max}$. It is observed that, as long as $s_{\max}$ and $b_{\max}$ are sufficiently large, the values of the expectation and the variance are insensitive to the location of the truncated boundaries.

| Strategy | $s_{\max}$ | $b_{\max}$ | Exp. Val. | Std. Dev. |
|---|---|---|---|---|
| semi-self-financing MV | $5 \times 10^6$ | $2.5 \times 10^6$ | 443.975 | 97.001 |
| | $7 \times 10^6$ | $3.5 \times 10^6$ | 443.967 | 97.014 |
| | $9 \times 10^6$ | $4.5 \times 10^6$ | 443.967 | 97.014 |

TABLE 6.3: *Effect of the finite boundaries, "semi-self-financing MV" strategy with discrete re-balancing and upward jumps. For this test, $\gamma = 1000$ and refinement level 2 are used.*

Next, we numerically show that the differences between the efficient frontiers obtained by a semi-self-financing strategy and those obtained by its self-financing counterpart are much larger than the discretization errors of the numerical methods. As an illustrative example, we consider a self-financing MV strategy and the *semi-self-financing MV* strategy for the case of discrete re-balancing and upward jumps.

In Figure 6.1 we present the computed MV embedded objective sets $(\mathcal{Y}_Q^{\dagger})^k$, $k = 1, 2$, with spurious points removed, i.e. $\mathcal{A}((\mathcal{Y}_Q^{\dagger})^k)$, $k = 1, 2$. Note that the expected value is plotted versus standard deviation, which is a more practically meaningful display of the results. For each strategy, the set $\mathcal{A}((\mathcal{Y}_Q^{\dagger})^k)$ for $k = 2$ visually coincides with that for $k = 1$. Further refinement steps show negligible changes. This suggests convergence of the numerical solution, as well as convergence of $\mathcal{A}((\mathcal{Y}_Q^{\dagger})^k)$ to the efficient frontiers. Results from Figure 6.1 indicate that the discretization errors of the numerical methods are negligibly small compared to the differences between the efficient frontiers obtained by different MV strategies. In the following, unless otherwise stated, refinement level 2 is used, and $\mathcal{A}((\mathcal{Y}_Q^{\dagger})^2)$ is considered to be the efficient frontier.

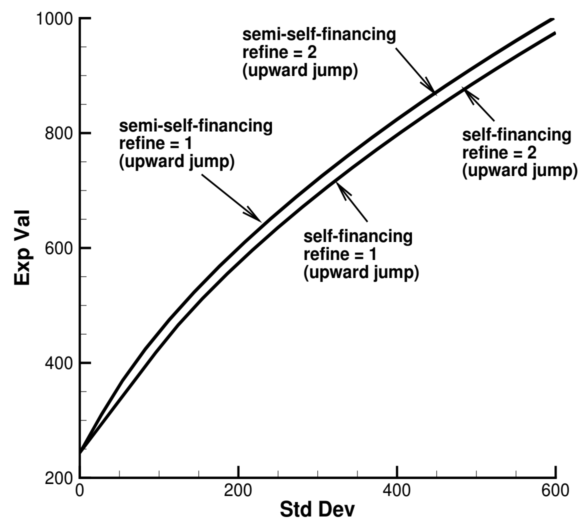


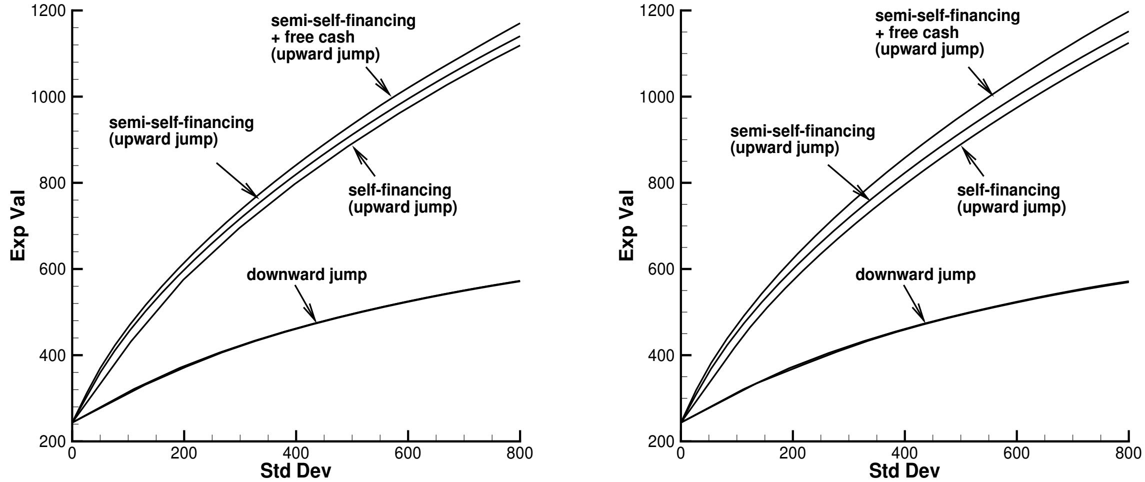

FIGURE 6.1: *Efficient frontiers for different refinement levels, self-financing MV' and "semi-self-financing MV" strategies, discrete re-balancing, upward jumps.*

## 6.2 Comparison of efficient frontiers

In this subsection, we compare the efficient frontiers obtained using semi-self-financing MV strategies with those obtained using a self-financing MV strategy. We only discuss the continuous and discrete re-balancing with jump-diffusions. Findings in the discrete case under pure diffusions are similar, and hence, omitted.

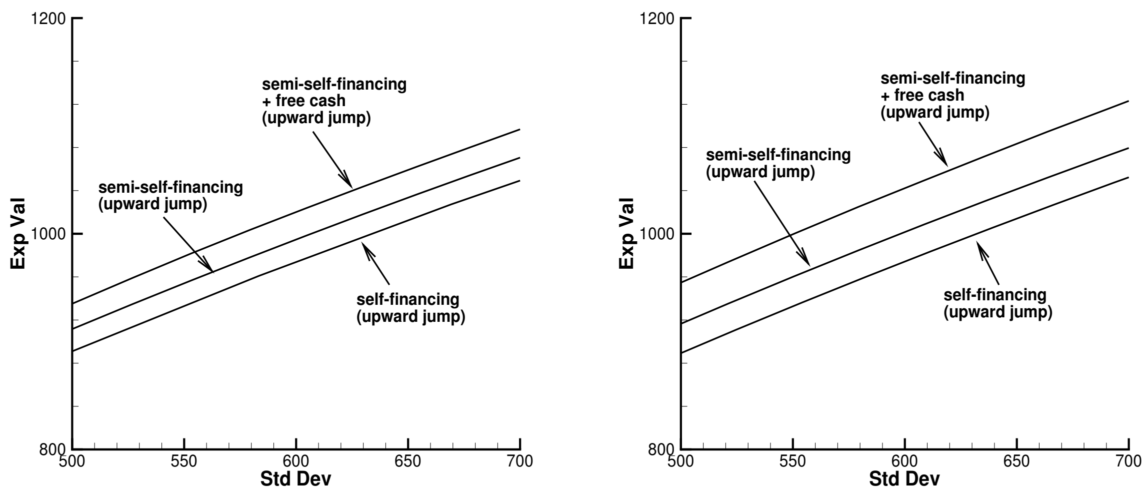
In Figure 6.2, we present plots of efficient frontiers for continuous and discrete re-balancing cases. Both mean downward and mean upward jumps are considered. In Figures 6.3-6.4, we present the close-up versions of these efficient frontiers for the mean upward jump case and mean downward jump case, respectively. We make the following observations:

- Overall, for both upward and downward jump cases, the efficient frontiers produced by the




(a) *continuous re-balancing, jumps* (b) *discrete re-balancing, jumps*

Figure 6.2: *Efficient frontiers for different strategies -* ***continuous and discrete re-balancing, jumps****. Level 2 refinement used.*

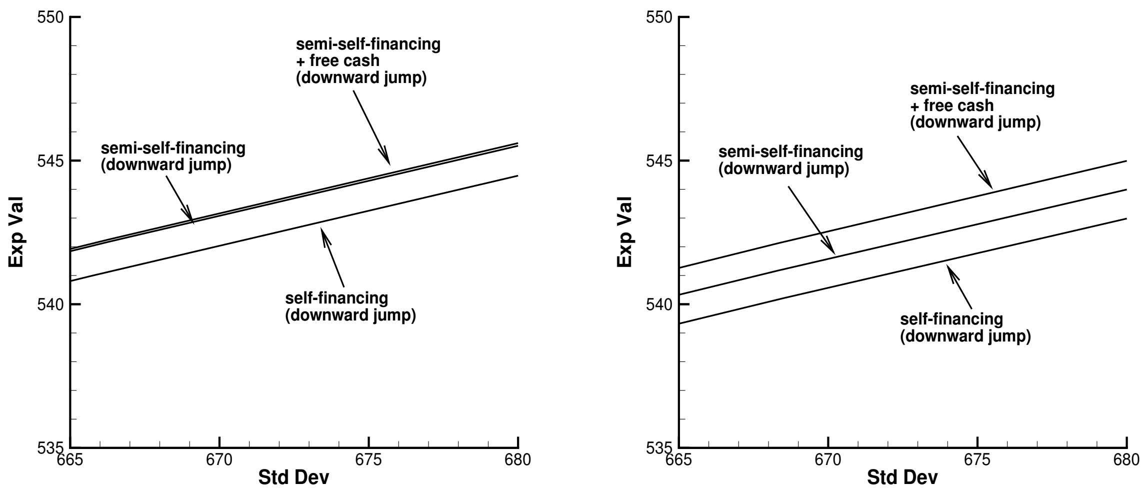



(a) *continuous re-balancing, upward jumps* (b) *discrete re-balancing, upward jumps*

Figure 6.3: *Zoomed-in efficient frontiers for different strategies -* ***continuous and discrete re-balancing, mean upward jumps****. Level 2 refinement used.*

semi-self-financing MV strategies dominate those produced by a self-financing MV strategy.

- The effect on the MV efficient frontiers of the semi-self-financing MV strategies are more pronounced for mean upward jumps than with mean downward jumps. This is an expected result, since for mean downward jumps, the probability that the $W_c(t)$ exceeds $W_{opt}(t)$ is much smaller than that in the mean upward jump case.




(a) *continuous re-balancing, downward jump*  (b) *discrete re-balancing, downward jump*

FIGURE 6.4: *Zoomed-in efficient frontiers for different strategies -* ***continuous and discrete re-balancing, mean downward jumps****. Level 2 refinement used.*

For mean downward jumps, the effect of our proposed MV strategies on the efficient frontiers is very small (see Figure 6.4).

- The effect on the MV efficient frontiers of our proposed MV strategies appear to be more pronounced with discrete re-balancing than with continuous re-balancing, assuming the same dynamics for the risky asset. This behavior is also expected.

We conclude that the semi-self-financing strategies are clearly more advantageous than self-financing strategies. From Proposition 2.1, we are ensured that the semi-self-financing strategies are never inferior to a self-financing strategy. But, in some cases, i.e. $W_c(t) > W_{opt}(t)$, for some $t \in \mathcal{T}$, which are likely to occur in a general setting, the semi-self-financing strategies are superior to those obtained by a self-financing strategy. This is because semi-self-financing efficient frontiers can be no worse than those obtained using a self-financing strategy, and the semi-self-financing strategies have the ability to generate a positive free cash flow during the investment.

# 7 Empirical data analysis

We assume that the SDE followed by a stock market index is given by equations (2.1) and (2.2). Recall that the log-normal distribution for the jump size density $p(\xi)$ (from equation (2.1)) has mean $\nu$ and standard deviation $\zeta$, with $E[\xi] = \exp(\nu + \zeta^2/2)$, where $E[\cdot]$ denotes the expectation operator, and $\kappa = E[\xi] - 1$.

In order to determine appropriate parameters for the jump diffusion model, we use the daily total return data from the Center for Research in Security Prices (CRSP)[1] . The CRSP VWD index is a value (capitalization) weighted index of all securities traded on major US exchanges, dating from 1925. The returns include all dividends and distributions.

[1]See `http://www.crsp.com/`

We use the daily total return (including dividends) series from December 31, 1925 to December 31, 2014, a span of 89 years. We also extract monthly returns from the same series. We convert the daily simple returns into index prices.

Consider a discrete series of index prices $S(t_i) = S_i, i = 1, \ldots, N+1$, observed at equally spaced time intervals $\Delta t = t_{i+1} - t_i, \forall i$, with $T = N\Delta t$ [2]. Let

$$\Delta X_i = \log\left(\frac{S_{i+1}}{S_i}\right), \tag{7.1}$$

be the log return. We also define the detrended log returns $\Delta \hat{X}_i$ as

$$\begin{aligned} \Delta \hat{X}_i &= \Delta X_i - \hat{m}\Delta t \\ \hat{m} &= \frac{\log(S_{N+1}) - \log(S_1)}{T} . \end{aligned} \tag{7.2}$$

## 7.1 GBM Estimates

As a first example, we assume that there are no jumps (i.e. $\lambda = 0$ in equation (2.2)), so that the index is assumed to follow pure Geometric Brownian Motion (GBM). We determine the two parameters $\mu, \sigma$ by maximum likelihood estimation (MLE), which in this case are given by the simple expressions

$$\begin{aligned} \mu - \frac{\sigma^2}{2} &= \hat{m} \quad (\text{ from equation (7.2) }) \\ \sigma^2 &= \frac{1}{\Delta t}\, var\Big(\{\Delta X_i\}\Big), \end{aligned} \tag{7.3}$$

where $var$ is the variance.

The results for both daily and monthly log returns are shown in Table 7.1. The estimates for $\mu$ and $\sigma$ are insensitive to the choice of daily or monthly observations. Table 7.2 also shows the mean treasury rates for the entire period as well.

| Series | $\mu$ | $\sigma$ |
|---|---|---|
| Daily | .1119 | .1862 |
| Monthly | .1121 | .1874 |

TABLE 7.1: *Data: CRSP VWD value weighted total return series, December 31, 1925 to December 31, 2014. GBM assumed, parameters estimated using maximum likelihood (MLE).*

## 7.2 Jump Diffusion Estimates

In order to determine the set of parameters for the full jump-diffusion model, use of maximum likelihood is well known to be problematic, due to multiple local maxima and the ill-posedness of attempting to distinguish high frequency small jumps from diffusion (Honore, 1998).

From a long term investor perspective, the most important feature of a jump diffusion model is that it allows modelling of infrequent large jumps in asset prices. Small, frequent jumps look like

[2] We assume equally spacing for ease of exposition

| Mean Treasury Rate | |
|---|---|
| 3-month | 1-year |
| .0369 | .0499 |

Table 7.2: *Data: CRSP, December 31, 1925 to December 31, 2014. Mean treasury rates.*

enhanced volatility, when examined on a large scale, hence these effects are probably not important in constructing a long term investment strategy.

Ait-Sahalia and Jacod (2012) discuss many econometric techniques that have been developed for detecting the presence of jumps in high frequency data (i.e. with the time scale of seconds). However, these high frequency jumps are not of particular interest to the long-term investor, hence we will use the thresholding technique described in Mancini (2009) and Cont and Mancini (2011). This technique is considered to be more efficient for low frequency data.

Suppose we have an estimate for the diffusive volatility component $\hat{\sigma}$, then we detect a jump in period $i$ if (Shimizu, 2013)

$$|\Delta \hat{X}_i| \quad > \quad \alpha \ \hat{\sigma} \ \frac{\sqrt{\Delta t}}{(\Delta t)^{\beta}} \tag{7.4}$$

where $\beta, \alpha > 0$ are tuning parameters. The intuition behind equation (7.4) can be explained simply. If we choose $\alpha = 4$, say, and $\beta \ll 1$, then equation (7.4) labels a return as a jump if the observed return is larger than a 4 standard deviation Brownian motion change, which would be extremely improbable. Hence we would consider this return to be due to a jump. If we choose a smaller time interval, i.e. reduce $\Delta t$, keeping the total time $T$ fixed, then equation (7.4) indicates that we should increase the threshold which filters out the Brownian motion increments. Intuitively, this is because we have increased the number of samples, and we expect to observe some large deviation events purely by chance. Typically, $\beta$ in equation (7.4) is quite small, $\beta \simeq .01 - .02$.

In Figure 7.1 we show a histogram of the monthly and daily log returns from the CRSP index, scaled to unit standard deviation and zero mean. We also plot a standard normal density as well.

Based on the monthly log returns in Figure 7.1 we set the jump detection indicator $\mathbf{1}_i$ as follows

$$\mathbf{1}_i \quad = \quad \begin{cases} 1 & \text{if } \Delta \hat{X}_i > \alpha^{up} \ \hat{\sigma} \sqrt{\Delta t} \quad \text{or } \Delta \hat{X}_i < \alpha^{low} \ \hat{\sigma} \sqrt{\Delta t} \\ 0 & \text{otherwise} \end{cases} \tag{7.5}$$

Criteria (7.5) allows us to separate out the downward jumps and the upward jumps. From an investment risk management perspective, we may be more concerned with downward as opposed to upward jumps.

Define

$$\sum_{i=1}^{N} \mathbf{1}_i = N^{jps} \quad ; \quad \sum_{i=1}^{N} (1 - \mathbf{1}_i) = N^{gbm} \tag{7.6}$$

where $N^{jps}$ is the number of jumps detected, and $N^{gbm}$ is the number of GBM increments. Our estimate of the volatility is then

$$\hat{\sigma}^2 \quad = \quad \frac{1}{\Delta t} \, var\Big( \{ \Delta \hat{X}_i \mid \mathbf{1}_i = 0 \} \Big) \ . \tag{7.7}$$

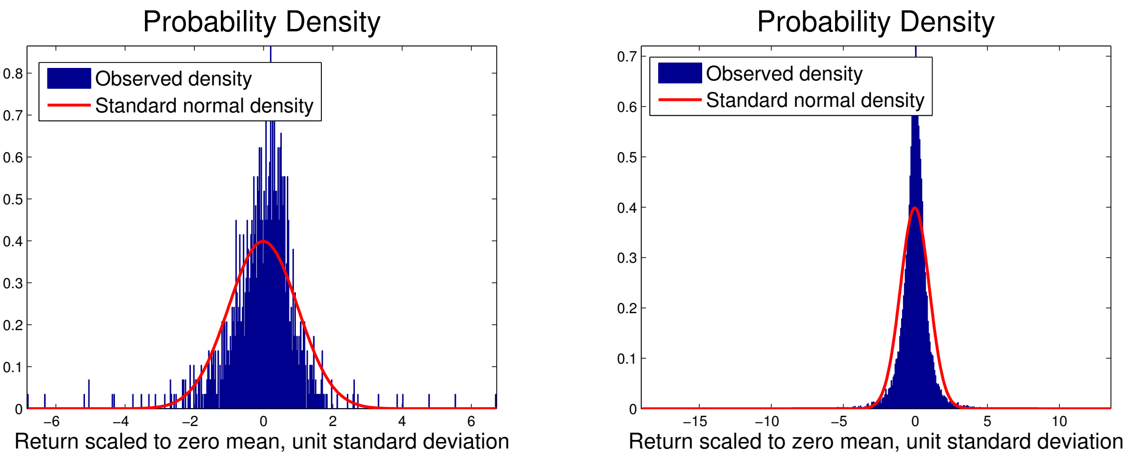



FIGURE 7.1: *Probability density of* log *returns, CRSP VWD value weighted total return series, December 31, 1925 to December 31, 2014. Scaled to unit standard deviation and zero mean. Standard normal density also shown. Left: monthly sampling, Right: daily sampling.*

Note that equations (7.5-7.7) constitute an implicit equation for $\hat{\sigma}$, which must be solved by an iterative method (Clewlow and Strickland, 2000). Once we have an estimate for the Brownian Motion volatility $\hat{\sigma}$, we can estimate the jump parameters, the jump intensity $\lambda$, the mean jump size $\nu$ and the jump size standard deviation $\zeta$ using the method suggested by Tauchen and Zhou (2011), assuming only one jump occurs in $[t_i, t_{i+1}]$

$$\begin{aligned} \lambda &= \frac{N^{jps}}{T} \\ \nu &= mean\Big(\{\Delta \hat{X}_i \mid \mathbf{1}_i = 1\}\Big) \\ \zeta^2 &= var\Big(\{\Delta \hat{X}_i \mid \mathbf{1}_i = 1\}\Big) \end{aligned} \tag{7.8}$$

Once we fix the estimates for $\sigma, \lambda, \nu, \zeta$, we estimate the drift term $\mu$ in two ways. The simplest method is to note that, then from equation (2.2) we have ($X = \log S$)

$$dX = \left(\mu - \lambda\kappa - \frac{\sigma^2}{2}\right) dt + \sigma\ dZ + d\left(\sum_{i=1}^{\pi_t} \log \xi_i\right) . \tag{7.9}$$

Taking expectations of both sides of equation (7.9), and assuming only one jump takes place in $[t, t + dt]$ gives

$$E[dX] = \left(\mu - \lambda\kappa - \frac{\sigma^2}{2}\right) dt + \lambda E[\log \xi]\ dt . \tag{7.10}$$

Writing equation (7.10) in discrete time, gives

$$\begin{aligned} \frac{mean\Big(\{\Delta X_i\}\Big)}{\Delta t} &= \left(\mu - \lambda\kappa - \frac{\sigma^2}{2}\right) + \lambda\nu \\ & \Delta X_i = \log(S_{i+1}) - \log(S_i) . \end{aligned} \tag{7.11}$$

Alternatively, we can determine an estimate for the drift rate $\mu$ using maximum likelihood (MLE) (Honore, 1998). Let $f_{\mathcal{N}}(x; \mathcal{M}, \mathcal{V}^2)$ be the normal density function with mean $\mathcal{M}$ and variance $\mathcal{V}^2$, evaluated at point $x$. The density function for the log return $\Delta X$ is then (assuming only one jump occurs in $[t_i, t_{i+1}]$)

$$\begin{aligned} \mathcal{P}(\Delta X) &= (1 - \lambda \Delta t) f_{\mathcal{N}}(\Delta X; (\mu - \lambda\kappa - \sigma^2/2)\Delta t, \sigma^2 \Delta t) \\ &\quad + \lambda \Delta t f_{\mathcal{N}}(\Delta X; (\mu - \lambda\kappa - \sigma^2/2)\Delta t + \nu, \sigma^2 \Delta t + \zeta^2) \; . \end{aligned} \tag{7.12}$$

Assuming $\sigma, \lambda, \nu, \zeta$ are known, the MLE estimate for $\mu$ is determined from

$$\max_{\mu} \left( \sum_i \log \mathcal{P}(\Delta X_i) \right) \; . \tag{7.13}$$

We will use both methods in the following.

Table 7.3 shows the estimates for the jump diffusion parameters using various values of the cutoff thresholds $\alpha^{low}, \alpha^{high}$. We can see from this table that as we increase $|\alpha|$, we find smaller jump intensities with an increasing estimate for the Brownian volatility. In other words, the smaller more frequent jumps are now considered to modelled by a diffusion process. We can also see that use of a one sided downward jump detection threshold has larger, more infrequent jumps, as expected. From an investment perspective, we are mainly concerned with the downward jumps, since upward jumps are a pleasant surprise. Both maximum likelihood (MLE, as in equation (7.13)) and the expected value (EVal, as in equation (7.11) give consistent estimates of the drift rate $\mu$.

| Cutoff Parameters | | Estimated Parameters | | | | | |
|---|---|---|---|---|---|---|---|
| $\alpha^{low}$ | $\alpha^{up}$ | $\mu$ (MLE) | $\mu$ (EVal) | $\sigma$ | $\lambda$ | $\nu$ | $\zeta$ |
| -3 | 3 | .1164 | .1124 | .1442 | .3488 | -.0755 | .1988 |
| -4 | 4 | .1216 | .1127 | .1575 | .1573 | -.0664 | .2674 |
| -5 | 5 | .1210 | .1129 | .1666 | .0899 | -.0646 | .3131 |
| -4 | $\infty$ | .1212 | .1122 | .1715 | .0899 | -.2631 | .0476 |
| -5 | $\infty$ | .1263 | .1122 | .1779 | .0494 | -.3000 | .0351 |

TABLE 7.3: *Jump diffusion parameter estimates, monthly* log *returns, CRSP VWD value weighted total return series, December 31, 1925 to December 31, 2014. MLE (maximum likelihood, equation(7.13)), EVal (expected value, equation (7.11)).*

Figure 7.2 shows a zoom of the right hand plot in Figure 7.1, the daily return CRSP index (1925-2015). As before, we have scaled the plot to have unit standard deviation and zero mean. The standard normal density is also shown. Now that we have a higher sampling frequency, the *large* jumps are much smaller in (relative) numbers, with a much wider range of jump sizes, in terms of standard deviations, compared to the monthly series. As we expect, our criteria for a *jump* has to become stricter compared to the monthly return series, otherwise we detect a very large number of (relatively) small jumps. It becomes more difficult now to separate the jumps from the diffusion increments. In fact, it is perhaps more desirable, for the long term investor, to use a coarser sampling (i.e. monthly) since in this case it is easy to differentiate the significant jumps. We can see from these plots, that in some sense, our idea of a jump for a long term investor, depends on the time scale of interest.

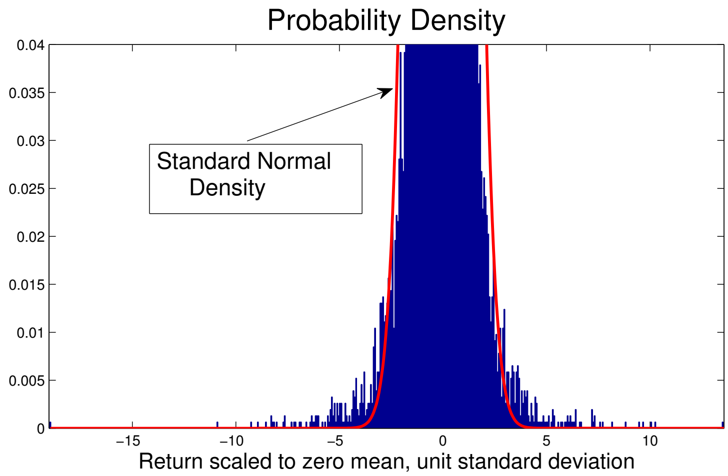



FIGURE 7.2: *Daily* log *returns, CRSP VWD value weighted total return series, December 31, 1925 to December 31, 2014. Scaled to unit standard deviation and zero mean. Standard normal density also shown. Zoom of right hand plot in Figure 7.1.*

In Table 7.4, we show the results for the CRSP index, daily log returns. In this case, the MLE estimate for the drift rate $\mu$ and the expected value estimates differ considerably for strict jump threshold detection parameters. At all values of jump cutoff parameters, the jumps appear to occur with a high frequency compared to the monthly observations.

| Cutoff Parameters | | Estimated Parameters | | | | | |
|---|---|---|---|---|---|---|---|
| $\alpha^{low}$ | $\alpha^{up}$ | $\mu$ (MLE) | $\mu$ (EVal) | $\sigma$ | $\lambda$ | $\nu$ | $\zeta$ |
| -4 | 4 | .1240 | .1121 | .1505 | 3.787 | -.0034 | .0571 |
| -5 | 5 | .1169 | .1120 | .1631 | 1.528 | -.00759 | .0733 |
| -6 | 6 | .1242 | .1120 | .1702 | .7640 | -.00288 | .0877 |
| -5 | $\infty$ | .1400 | .1119 | .1746 | .7416 | -.0722 | .0235 |
| -6 | $\infty$ | .1478 | .1119 | .1784 | .3483 | -.0870 | .0277 |

TABLE 7.4: *Jump diffusion parameter estimates, daily* log *returns, CRSP VWD value weighted total return series, December 31, 1925 to December 31, 2014. MLE (maximum likelihood, equation(7.13)), EVal (expected value, equation (7.11)).*

# 8 Numerical results: empirical parameters

We can see from Section 7 that the estimated parameters which model the SDE of a market index depend on several tuning parameters. We choose three possible sets of parameters, and we will compute the mean-variance results for each set.

We select the following cases

- Pure-diffusion model (GBM), i.e. $\lambda = 0$ in equation (2.2). Parameters obtained from *daily* log returns, Table 7.1.

- Jump-diffusion with parameters obtained from *daily* log returns. This is the case $\alpha^{low} = -5$, and $\alpha^{up} = +5$, from Table 7.4. $\mu$ estimated using equation (7.11).

- Jump-diffusion with parameters obtained from *monthly* log returns. This is the case $\alpha^{low} = -4$, $\alpha^{up} = +\infty$, in Table 7.3. $\mu$ estimated using equation (7.11).

In all cases, we use the one year T-bill rate from Table 7.2. Note that the drift rates estimated from equation (7.11) are always lower than the drift rates obtained using MLE (maximum likelihood), hence we use these more conservative estimates. The parameters for the representative cases are summarized in Table 8.1. Note that the drift rates and volatilities are not too different for all cases, however the jump parameters are quite different for the jump diffusion cases. Since the jump parameters are difficult to estimate, we can examine the effect of differing estimates on the investment results.

| Parameters | jump-diffusion | | pure diffusion |
|---|---|---|---|
| | daily | monthly | daily (GBM) |
| $\lambda$ (jump intensity) | 1.528 | 0.0899 | N/A |
| $\nu$ (jump multiplier mean) | -0.00759 | -0.2631 | N/A |
| $\zeta$ (jump multiplier std) | 0.0733 | 0.0476 | N/A |
| $\mu$ (drift) | 0.1120 | 0.1122 | 0.1119 |
| $\sigma$ (volatility) | 0.1631 | 0.1715 | 0.1862 |
| initial wealth | 100 | 100 | 100 |
| $q_{\max}$ (leverage constraint) | 1.5 | 1.5 | 1.5 |
| $r$ (risk-free interest rate) | 0.0499 | 0.0499 | 0.0499 |
| $T$ (investment horizon) | 30. (years) | 30. (years) | 30. (years) |
| $t_{i+1} - t_i$ (discrete re-balancing time period) | 1.0 (years) | 1.0 (years) | 1.0 (years) |

TABLE 8.1: *Parameters for the empirical data tests for three cases: pure-diffusion (GBM) with daily data, jump-diffusions with parameter estimates from daily and monthly* log *returns.*

## 8.1 Sensitivity of efficient frontiers

In this test, we use $T = 30$ (years), 360 timesteps, and the same numbers of $S$ and $B$ nodes as for refinement 2 in Table 6.1. In Figure 8.1, we present efficient frontiers obtained from three sets of parameters in Table 8.1. In all cases we use the semi-self-financing strategy. We do not include the free cash. We observe from Figure 8.1 that the efficient frontiers obtained from the three sets of parameters are essentially the same.

As a further check on our results, we carry out the following tests. We assume that, for each set of parameters in Table 8.1, the real world dynamics for $S$ follows the corresponding stochastic differential equations. We then we use the PDE method, described earlier, to find the optimal semi-self-financing strategy strategies for a given value of $\gamma$. These controls are stored for each discrete state value and timestep. We then carry out Monte-Carlo (MC) simulations from $t = 0$ to $t = T$ following these stored PDE-computed optimal strategies. Finally, we compare the MC-computed means and variances with the PDE-computed counterparts. In Table 8.2, as an illustrative example, we present MC-computed means and standard deviations for the three cases when when $\gamma = 1510$. We observe that the MC-computed means and standard deviations of the three cases agree with

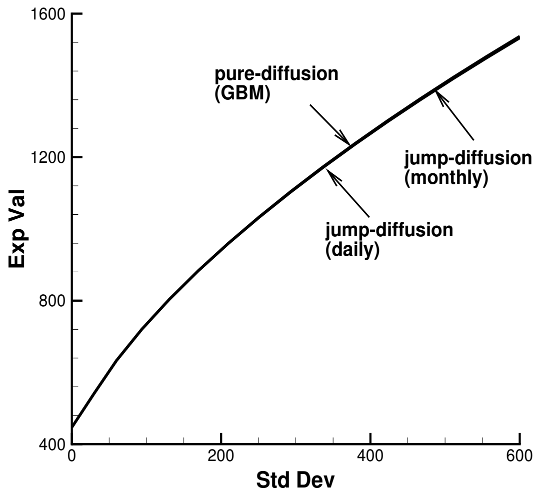



FIGURE 8.1: *Sensitivity of the the semi-self-financing strategy with respect to various estimates of market parameters. Parameters are from Table 8.1, with* 360 *timesteps, and the same numbers of* $S$ *and* $B$ *nodes, refinement 2, Table 6.1.*

| samples size | timesteps | pure-diffusion (GBM) | | jump-diffusion (daily) | | jump-diffusion (monthly) | |
|---|---|---|---|---|---|---|---|
| | | mean | std. | mean | std. | mean | std. |
| $1 \times 10^7$ | 720 | 720.43 | 93.25 | 719.29 | 93.49 | 720.77 | 92.75 |
| PDE-computed | | 720.24 | 93.76 | 720.38 | 93.87 | 720.35 | 93.68 |

TABLE 8.2: *Monte-Carlo-computed mean and standard deviations.* $\gamma = 1510$*. The PDE results are obtained with* 720 *timesteps, and the same numbers of* $S$ *and* $B$ *as for Level 2. Data in Table 8.1.*

each other, and with the PDE-computed results, and that we obtain very similar mean and variance for each case.

## 8.2 Robustness to misspecified parameters

To study the method's robustness with respect to parameter estimation ambiguity, we proceed as follows. First, using the PDE method, we compute and store the optimal strategies under a specific model assumption, for example, when the risky asset follows a pure-diffusion (GBM) model. We then carry out MC simulations for the portfolio from $t = 0$ to $t = T$ following these stored PDE-computed optimal strategies, but assuming the real world's dynamics of the risky asset follow a different model, for example jump-diffusion (daily). We then compare the MC-computed mean and variance for each pair of real world model and strategy computing model. In other words, we assume that the real world follows a jump diffusion process, but the investor assumes that the process is a GBM, and computes the optimal strategy based on GBM parameter estimates. This, then, is a test of strategy robustness in the face of model parameter mis-specification.

Table 8.3 shows the results for all combinations of representative test cases.

Table 8.3 demonstrates that the strategy appears to be insensitive to model mis-specification,

| real world model | strategy computing models | | | |
|---|---|---|---|---|
| | mean | std. | mean | std. |
| pure-diffusion GBM | jump-diffusion (daily) | | jump-diffusion (monthly) | |
| | 719.38 | 92.93 | 720.06 | 92.84 |
| jump-diffusion (daily) | pure-diffusion (GBM) | | jump-diffusion (monthly) | |
| | 720.46 | 94.05 | 720.08 | 93.94 |
| jump-diffusion (monthly) | pure-diffusion (GBM) | | jump-diffusion (daily) | |
| | 721.02 | 93.08 | 719.97 | 92.30 |

TABLE 8.3: *MC-computed mean and variance for each pair of different real world and computing models.* $\gamma = 1510$. *Same level of refinement as in Table 8.2 is used. Data in Table 8.1.*

which is, of course, a very desirable result. We should also mention that a test of model robustness for the case of a stochastic volatility model compared to GBM, also shows that the GBM strategy produces excellent results compared with the true stochastic volatility strategy (Ma and Forsyth, 2015).

# 9 Conclusions

In this paper, we generalize the idea of semi-self-financing strategies developed in Ehrbar (1990), Cui et al. (2012), and Cui and Li (2010) for the MV optimal portfolio allocation problem, which can be re-formulated as an embedded MV optimization problem (Li and Ng, 2000; Zhou and Li, 2000) in terms of the numerical solution of an HJB equation. Under this fully numerical approach, it is straightforward to determine an MV embedded optimal strategy over all possible semi-self-financing strategies, in a very general setting, namely continuous or discrete re-balancing, jump-diffusions with finite activity, and realistic portfolio constraints. If the portfolio wealth is above a critical threshold, then we prove that a scalarization MV optimal strategy is (i) to withdraw wealth exceeding this threshold, and (ii) to invest the remaining wealth in the risk-free asset for the remainder of the investment horizon.

In certain cases, we can prove that an optimal scalarization MV strategy is to not withdraw cash if the portfolio wealth is below the critical threshold. However, it remains an open question as to whether or not this is true in general for the case of jump-diffusions and discrete re-balancing. Nonetheless, we always observe that it is non-optimal to withdraw below the threshold in all of our numerical experiments.

We show that, in general, embedded MV optimal semi-self-financing strategies are not unique. However, in case of non-uniqueness, all embedded MV optimal semi-self-financing strategies produce the same embedded MV points. Using the results of Tse et al. (2014) and Dang et al. (2015), we show that all of these strategies generate the same set of points on the MV efficient frontier. Moreover, semi-self-financing strategies have the ability to produce a free cash flow during the investment, and can never be inferior, in terms of MV efficiency, to a self-financing strategy.

We have carried out an empirical data analysis using historical long term market returns. We obtain estimates for a GBM model, as well as estimates for jump diffusion models. Jump diffusion models are probably the simplest models which account for the observed *fat tails* of market returns. Based on several representative test cases, we find that the semi-self-financing pre-commitment mean-variance strategies are robust to model parameter estimation errors.

# Appendix

## A Proof of embedding result in Theorem 2.1

We present a characterization of the main property of the embedding technique given in Li and Ng (2000); Zhou and Li (2000) in terms of the achievable objective set. This main property is summarized in Theorem 2.1. We follow along the lines of Li and Ng (2000); Zhou and Li (2000) to prove this result, although we use slightly different steps. We include this proof to illustrate the generality of the embedding result, i.e. it is essentially independent of the specification of the admissible set for the control $c(\cdot)$.

*Proof.* Assume to the contrary that (2.13) does not hold. Then,

$$\inf_{(\mathcal{V},\mathcal{E})\in\mathcal{Y}} \mathcal{V} + \mathcal{E}^2 - \gamma\mathcal{E} < \mathcal{V}_0 + \mathcal{E}_0^2 - \gamma\mathcal{E}_0. \tag{A.1}$$

Then there exists $(\mathcal{V}_*, \mathcal{E}_*) \in \mathcal{Y}$ such that

$$\mathcal{V}_* + \mathcal{E}_*^2 - \gamma\mathcal{E}_* < \mathcal{V}_0 + \mathcal{E}_0^2 - \gamma\mathcal{E}_0. \tag{A.2}$$

Rearranging equation (A.2) and multiplying by $\rho > 0$ gives

$$\rho(\mathcal{V}_* + \mathcal{E}_*^2) - \rho(\mathcal{V}_0 + \mathcal{E}_0^2) - \gamma\rho(\mathcal{E}_* - \mathcal{E}_0) < 0 \tag{A.3}$$

Substitute equation (2.14) into equation (A.3) to obtain

$$\rho(\mathcal{V}_* + \mathcal{E}_*^2) - \rho(\mathcal{V}_0 + \mathcal{E}_0^2) \ - (1 + 2\rho\mathcal{E}_0)(\mathcal{E}_* - \mathcal{E}_0) < 0 \ . \tag{A.4}$$

Further manipulation gives

$$(\rho\mathcal{V}_* - \mathcal{E}_*) - (\rho\mathcal{V}_0 - \mathcal{E}_0) + \rho\big(\mathcal{E}_*^2 - \mathcal{E}_0^2 - 2\mathcal{E}_0\mathcal{E}_* + 2\mathcal{E}_0^2\big) < 0 \ , \tag{A.5}$$

and

$$\rho\mathcal{V}_* - \mathcal{E}_* \ \ < \ \ \rho\mathcal{V}_0 - \mathcal{E}_0 - \rho(\mathcal{E}_* - \mathcal{E}_0)^2 \ , \tag{A.6}$$

and hence

$$\rho\mathcal{V}_* - \mathcal{E}_* < \rho\mathcal{V}_0 - \mathcal{E}_0,$$

which contradicts equation (2.12). Hence (2.13) holds. □

Finally, we note that the embedded objective function can be written as

$$\begin{aligned} \mathcal{V} + \mathcal{E}^2 - \gamma\mathcal{E} &= E_{c(\cdot)}^{x_0,t_0}[W_c(T)^2] - \big(E_{c(\cdot)}^{x_0,t_0}[W_c(T)]\big)^2 + \big(E_{c(\cdot)}^{x_0,t_0}[W_c(T)]\big)^2 - \gamma E_{c(\cdot)}^{x_0,t_0}[W_c(T)] \\ &= E_{c(\cdot)}^{x_0,t_0}[W_c(T)^2 - \gamma W_c(T)] \\ &= E_{c(\cdot)}^{x_0,t_0}[(W_c(T) - \gamma/2)^2] - (\gamma/2)^2 \end{aligned} \tag{A.7}$$

So minimizing the above embedded objective function is equivalent to (2.15).

# B HJB PIDEs

In this section, we briefly describe the HJB PIDEs for the case of continuous re-balancing and jump-diffusions. For brevity, we omit the PIDEs for the case of discrete re-balancing which can be found in Dang et al. (2015). For the case of continuous re-balancing with jump-diffusions, the MV optimal portfolio allocation problem can be formulated as the solution to a 2-dimensional (2-D) impulse control problem, in the form of a non-linear HJB PIDE. This approach is suggested in Dang and Forsyth (2014). We refer the reader to Dang and Forsyth (2014) for a complete discussion of the formulation. We define the solution domain as

$$\Omega = \{(s, b, t) \in [0, \infty) \times (-\infty, +\infty) \times [0, T]\}. \tag{B.1}$$

We define the solvency region, denoted by $\mathcal{S}$, as

$$\mathcal{S} \ = \ \{(s, b) \in [0, \infty) \times (-\infty, +\infty) : W_c(s, b) > 0\} \ , \tag{B.2}$$

and the bankruptcy (insolvency) region $\mathcal{B} = \Omega \backslash \mathcal{S}$. Let $\mathcal{Q} = \{(s, b) \in \mathcal{S} : s/(s + b) > q_{\max}\}$ be the region where the leverage constraint is violated. We respectively denote by $\mathcal{L}V$ and $\mathcal{J}V$ the diffusion and jump operators, where

$$\mathcal{L}V \ \equiv \ \frac{\sigma^2 s^2}{2} V_{ss} + (\mu - \lambda\kappa) s V_s + rbV_b - \lambda V \ , \tag{B.3}$$

$$\mathcal{J}V \ \equiv \ \int_0^\infty p(\xi) V(\xi s, b, \tau) \ d\xi \ . \tag{B.4}$$

Following standard arguments (Øksendal and Sulem, 2009; Pham, 2009), the value function $V(s, b, t)$ is the viscosity solution of the HJB PIDE

$$\max\left[V_t + \mathcal{L}V + \mathcal{J}V, V - \inf_{c \in \mathcal{Z}} (\mathcal{M}(c) \ V)\right] = 0 \quad ; \quad \text{if } (s, b) \in \mathcal{S} \backslash \mathcal{Q} \ , \tag{B.5}$$

$$\max\left[V_t + rbV_b, V - \inf_{c \in \mathcal{Z}} (\mathcal{M}(c) \ V)\right] = 0 \quad ; \quad \text{if } s = 0 \ , \tag{B.6}$$

$$V - \inf_{c \in \mathcal{Z}} (\mathcal{M}(c) \ V) = 0 \quad ; \quad \text{if } (s, b) \in \mathcal{Q} \ , \tag{B.7}$$

$$V(s, b, t) = V(0, W_c(s, b), t) \quad ; \quad \text{if } (s, b) \in \mathcal{B} \ , \tag{B.8}$$

$$V(s, b, T) = \begin{cases} (s + b - \gamma/2)^2 & \mathcal{Z} = \mathcal{Z}_{self} \\ (\max(\gamma/2 - (s + b), 0))^2 & \mathcal{Z} = \mathcal{Z}_{semi} \end{cases} \ , \tag{B.9}$$

defined on the domain $\Omega$. The intervention operator $\mathcal{M}(c) \ V(s, b, t)$ is defined as

$$\mathcal{M}(c) \ V(s, b, t) \ = \ V(S(s, b, c), B, t) + \varrho. \tag{B.10}$$

Here, $\varrho > 0$ is an arbitrarily small switching cost required to ensure that the impulse control HJB PIDE problem is well-posed, the control $c \equiv (d, B)$ is defined in (2.3), and $S(s, b, c)$ is defined in (2.4).

For computational purposes, we localize the original domain to

$$\Omega_{loc} = \{(s, b, t) \in [0, s_{\max}) \times [-b_{\max}, b_{\max}] \times [0, T]\}, \tag{B.11}$$

where $s_{\max}$ and $b_{\max}$ are positive and sufficiently large. We assume that the boundary conditions at $s_{\max}$, $-b_{\max}$, and $b_{\max}$ used in this paper are the asymptotic forms of the HJB PDE/PIDE as $s, |b| \to \infty$ (Dang and Forsyth, 2014).

# C Embedded MV optimal controls

In general, $\mathcal{Z}_{semi}$ allows cash withdrawals at any time and any value of the state $(s, b)$. In this Appendix, we show that only a subset of all possible withdrawal strategies is optimal.

Our plan is the following. Using a specific discretization, we will prove certain properties of the optimal control set. This can be regarded as a reduction of the size of $\mathcal{Z}_{semi}$ to be used in an implementation using this particular discretization. However, this discretization method can be easily shown to satisfy all the requirements required for convergence to the optimal control problem (B.5-B.9) (see Dang and Forsyth (2014)). As a result, we then take the limit as $h \to 0$, and these properties of $\mathcal{Z}_{semi}$ must also be properties of the viscosity solution of problem (B.5-B.9). Hence, we can apply these results (concerning $\mathcal{Z}_{semi}$) to any convergent discretization of (B.5-B.9).

For ease of exposition, we consider only the continuously observed case in this appendix. We will indicate under what circumstances we can extend these results to the discretely observed case. We give a brief description of the discretization method used to solve equations (B.5-B.9) here. We refer the reader to Dang and Forsyth (2014) for details. We consider the localized problem defined on $\Omega_{loc} = [0, s_{\max}] \times [-b_{\max}, +b_{\max}] \times [0, T]$, as described in Dang and Forsyth (2014). Artificial boundary conditions are applied at $s = s_{\max}$ and $b = \pm b_{\max}$. In the following, it will also be convenient to write

$$\mathcal{L}V = \mathcal{P}V + rbV_b \ \ ; \quad \text{where } \mathcal{P}V \equiv \frac{\sigma^2 s^2}{2} V_{ss} + (\mu - \lambda\kappa) s V_s - \lambda V \ . \tag{C.1}$$

When we analyze the PIDE solve, to avoid tedious algebraic manipulation, we will make extensive use of the transformation $z = \log s$. Let $\hat{V}(z, b, \tau) = V(e^z, b, \tau)$, where, with some abuse of notation $\tau = T - t$. Then

$$\mathcal{L}^z \hat{V} \ = \ \mathcal{P}^z V + rb\hat{V}_b \ \ ; \quad \text{where } \mathcal{P}^z \hat{V} \equiv \frac{\sigma^2}{2} \hat{V}_{zz} + (\mu - \lambda\kappa) \hat{V}_z \ - \lambda V.$$

In the transformed $z = \log s$ coordinates the jump term becomes

$$\mathcal{J}^z \hat{V} \ = \ \int_{-\infty}^{+\infty} \bar{p}(y) \hat{V}(z + y) \ dy, \qquad \text{where } \bar{p}(y) = e^y p(y) \ ; \ y = \log J \ ,$$

where $p(J)$ is the density of the jump size defined in equation (2.1).

## C.1 Discretization

Let $\mathcal{S}_{loc}$ denote the localized solvency region $\mathcal{S}_{loc} = \{(s, b) \in \Omega_{loc} \mid (s + b) > 0\}$. Define a set of nodes in the $z$-direction by $\{z_0, z_1, \ldots, z_{i_{\max}}\}$ ($z = \log s$), and in the $b$-direction $\{b_0, \ldots, b_{j_{\max}}\}$. Let

$$\begin{aligned} z_i &= z_{\min} + i\Delta z \ ; \ i = 0, \ldots, i_{\max}, \qquad \text{where} \quad e^{z_{\min}} \simeq 0 \ , \\ b_j &= -b_{\max} + j\Delta b \ . \end{aligned}$$

Denote the $n^{th}$ discrete timestep by $\tau^n$. For ease of notation, we assume constant timestep sizes, i.e. $\Delta\tau = \tau^{n+1} - \tau^n$ is constant, and that

$$\Delta z = C_1 h \ ; \ \Delta b = C_2 h \ ; \ \Delta\tau = C_3 \Delta\tau \ , \tag{C.2}$$

where $h$ is a positive discretization parameter, and $C_i$ are positive constants. In order to simplify some of the technical analysis, we make the following assumptions:

$$(e^{r\Delta\tau} - 1) < \Delta b \quad ; \quad b_{\max} > \gamma/2 \quad . \tag{C.3}$$

These assumptions can be removed, but at the cost of considerable algebraic complication.

We denote by $V(s, b, \tau^n)$ the exact solution to the non-linear value equations (B.5-B.9). Let $V_h(s, b, \tau)$ be the approximate solution at the point $(s, b, \tau)$ obtained using the discretization parameter $h$. Similarly, let $\hat{V}_h(z_i, b_j, \tau^n)$ be the approximate solution at the reference node $(z_i, b_j, \tau^n)$. In the event that we need to evaluate $V_h$ or $\hat{V}_h$ at a point other than nodal values, linear interpolation is used. It will also be understood that the arguments of $V_h$ are truncated if necessary to remain in $\Omega_{loc}$, i.e.

$$V_h(s, b, \tau) \quad \equiv \quad V_h(\min(\max(s, 0), s_{\max}), \min(\max(b, -b_{\max}), b_{\max}), \tau) \ .$$

With some abuse of notation, will understand that the following definitions are overloaded

$$\log s \equiv \log \max(s, e^{z_{\min}}) \quad ; \quad e^z \equiv e^z - e^{z_{\min}} \ . \tag{C.4}$$

Using equations (C.4) we also define $s_i = e^{z_i}$ and $z_i = \log s_i$. We trust that this use of $\hat{V}_h$ and $V_h$ as well as the unconventional notation (C.4) will make the analysis less tedious for the reader.

Let $\tau^n_+ = T - t^-$, i.e the instant before rebalancing in forward time. As described in Dang and Forsyth (2014), the semi-Lagrangian timestepping method proceeds in the following two steps.

1. The first step solves a local optimization problem

$$V_h(s_i, b_j, \tau^n_+) = \min \left[ V_h(s_i, b_j e^{r\Delta\tau}, \tau^n), \min_{(d,B)\in\mathcal{Z}_{semi}} V_h(S(s_i, b_j e^{r\Delta\tau}, B, d), B, \tau^n) \right] ,$$
$$(s_i, b_j) \in \mathcal{S}_{loc} \ , \tag{C.5}$$

   where $S(\cdot)$ is given below, and $\mathcal{Z}_{semi}$ refers to the controls which satisfy

$$S = (s + b) - B - d, \quad \text{where } d \geq 0 \quad ; \quad S \geq 0 \quad ; \quad \frac{S}{S + B} \leq q_{\max}, \quad (S, B) \in \mathcal{S}_{loc} \quad .$$

   In equation (C.5, we adopt the notational convention that

$$V_h(s_i, b_j e^{r\Delta\tau}, \tau^n) \quad = \quad +\infty \quad ; \quad \text{if } \{(s_i, b_j e^{r\Delta\tau}) \in \mathcal{Q}. \tag{C.6}$$

2. The second step consists of a time advance from $\tau^n_+$ to $\tau^{n+1}$ with the initial condition obtained from the previous step. For this step, we will use the $z = \log s$ coordinate system. More specifically, we use $\hat{V}_h(z_i, b_j, \tau^n_+)$ as the initial condition. Denote the discrete forms of $\mathcal{P}^z, \mathcal{J}^z$ as $\mathcal{P}^z_h, \mathcal{J}^z_h$. Using implicit-explicit timestepping, we have

$$\hat{V}_h(z_i, b_j, \tau^{n+1}) - \Delta\tau(\mathcal{P})^z_h \hat{V}_h(z_i, b_j, \tau^{n+1}) - \Delta\tau(\mathcal{J})^z_h \hat{V}_h(z_i, b_j, \tau^n) = \hat{V}_h(z_i, b_j, \tau^n_+) \ . \tag{C.7}$$

   Note that in contrast to Dang and Forsyth (2014), we evaluate the jump term explicitly, which is unconditionally stable (d'Halluin et al., 2005). This scheme can also be shown be convergent to the viscosity solution of equations (B.5-B.9) using the techniques in(Dang and Forsyth, 2014).

**Condition C.1.** *In this Appendix, we assume the following conditions*

*(a) The volatility $\sigma$, jump intensity $\lambda$, and jump size probability denisty $p(J)$ are independent of $s$.*

*(b) The leverage ratio $q_{\max} \geq 1$.*

*(c) The optimization step (C.5) is carried out every timestep. Hence, this converges to the continuously re-balanced solution as $h \to 0$*

## C.2 Preliminary results

**Proposition C.1** (Properties of value function at $\tau = 0$)**.** *After applying the initial control, the value function is, from equation (B.9),*

$$V_h(s, b, 0) \;\; = \;\; \Big( \max(\gamma/2 - (s+b), 0) \Big)^2 \; , \tag{C.8}$$

*which is non-increasing in $s$ for fixed $b$, i.e.*

$$V_h(s, b, 0) \;\; \geq \;\; V_h(s', b, 0) \; ; \; s' > s \; , \quad \forall (s, b), (s', b) \in \Omega_{loc} \; . \tag{C.9}$$

**Proposition C.2** (Insolvent region properties)**.** *In the insolvent region $(s+b) \leq 0$, the solution is*

$$V_h(s, b, \tau) \;\; = \;\; (\gamma/2 - (s+b)e^{r\tau})^2 \; . \tag{C.10}$$

*Hence,*

$$V_h(s, b, \tau) \;\; \geq \;\; V_h(s', b, \tau) \;\; ; \;\; s' > s \;\; ; \;\; \forall (s, b), (s', b) \in \Omega_{loc} \backslash \mathcal{S}_{loc} \; , \forall \tau, \tag{C.11}$$

*and*

$$\begin{aligned} V_h(s, b, \tau) \;\; &= \;\; (\gamma/2)^2 \;\; ; \;\; (s+b) = 0 \\ V_h(s, b, \tau) \;\; &\geq \;\; (\gamma/2)^2 \;\; ; \;\; \forall (s, b) \in \Omega_{loc} \backslash \mathcal{S}_{loc} \; . \end{aligned} \tag{C.12}$$

**Proposition C.3.** *If $(s, b) \in \mathcal{S}_{loc}$ and $(s, b)$ satisfies the leverage constraint*

$$\frac{s}{s+b} \;\; \leq \;\; q_{\max} \;\; ; \;\; q_{\max} \geq 1 \;\; , \tag{C.13}$$

*then the point $(s + \eta, b) \in \mathcal{S}_{loc}$, $\eta > 0$, also satisfies the leverage constraint.*

**Lemma C.1** (Embedded MV non-optimality of withdrawing)**.** *Let $\mathcal{G}_h(s, b, \tau)$ be an arbitrary grid function (i.e. defined by linear interpolation of nodal values specified at $(s_i, b_j, \tau^n)$) defined on $\Omega_{loc}$. If $q_{\max} \geq 1$ and*

$$\mathcal{G}_h(s, b, \tau^n) \;\; \geq \;\; \mathcal{G}_h(s', b, \tau^n) \; ; \; s' > s \; , \quad \forall (s, b), (s', b) \in \Omega_{loc} \; , \tag{C.14}$$

*then*

$$\min_{(d,B) \in \mathcal{Z}_{semi}} \mathcal{G}_h(s + be^{r\Delta\tau} - B - d, B, \tau^n) \;\; = \;\; \min_{B \in \mathcal{Z}_{semi}} \mathcal{G}_h(s + be^{r\Delta\tau} - B, B, \tau^n) \tag{C.15}$$

*Proof.* Assume to the contrary that $\exists (d^*, B^*) \in \mathcal{Z}_{semi}$ with $d^* > 0$, such that

$$\begin{aligned} \min_{B \in \mathcal{Z}_{semi}} \mathcal{G}_h(s + be^{r\Delta\tau} - B - d^*, B, \tau^n) &= \mathcal{G}_h(s + be^{r\Delta\tau} - B^* - d^*, B^*, \tau^n) \\ &< \min_{B \in \mathcal{Z}_{semi}} \mathcal{G}_h(s + be^{r\Delta\tau} - B, B, \tau^n) \ . \end{aligned} \tag{C.16}$$

But, from Proposition C.3, we note that if $(s + be^{r\Delta\tau} - B^* - d^*, B^*)$ satisfies the leverage constraint, then the point $(s + be^{r\Delta\tau} - B^*, B^*)$ is also admissible. That is,

$$\min_{B \in \mathcal{Z}_{semi}} \mathcal{G}_h(s + be^{r\Delta\tau} - B, B, \tau^n) \leq \mathcal{G}_h(s + be^{r\Delta\tau} - B^*, B^*, \tau^n) \ . \tag{C.17}$$

Consequently, from equations (C.16) and (C.17) we have that

$$\mathcal{G}_h(s + be^{r\Delta\tau} - B^* - d^*, B^*, \tau^n) < \mathcal{G}_h(s + be^{r\Delta\tau} - B^*, B^*, \tau^n) \tag{C.18}$$

which contradicts equation (C.14). □

**Remark C.1** (Non-uniqueness of embedded MV optimal strategy)**.** *Lemma C.1 does not imply that $d = 0$ is a unique embedded MV optimal strategy if equation (C.14) is satisfied.*

## C.3 Properties of time advancement

We refer the reader to d'Halluin et al. (2004, 2005); Huang et al. (2012) for details of the discretization of equation (C.7). Let

$$\begin{aligned} \hat{p}(y_k) &= \hat{p}_k = \int_{y_k - \Delta J/2}^{y_k + \Delta J/2} \overline{p}(y) \ dy \\ & \quad \Delta J = \Delta z \ ; \ z_i = z_{\min} + i\Delta z \ ; \ y_k = k\Delta J \ , \\ & \quad 0 \leq \hat{p}_k \leq 1 \ \ ; \ \sum_{k=-k_{\max}}^{k=k_{\max}} \hat{p}_k \leq 1 \ . \end{aligned} \tag{C.19}$$

Let $\hat{V}_{i,j}(\tau^{n+1}) = \hat{V}_h(z_i, b_j, \tau^{n+1})$, then the discrete form of equation (C.7) is then

$$\begin{aligned} \hat{V}_{i,j}(\tau^{n+1}) &= \hat{V}_{i,j}(\tau^n_+) + \Delta\tau\alpha(\hat{V}_{i-1,j}(\tau^{n+1}) - \hat{V}_{i,j}(\tau^{n+1})) + \Delta\tau\beta(\hat{V}_{i+1,j}(\tau^{n+1}) - \hat{V}_{i,j}(\tau^{n+1})) \\ & \quad + \lambda\Delta\tau \sum_{k=-k_{\max}}^{k=k_{\max}} \hat{p}_k \hat{V}_h(z_i + y_k, b_j, \tau^n_+) - \lambda\Delta\tau\hat{V}_{i,j}(\tau^{n+1}) \qquad \text{(C.20)} \\ i &= i_{\min}(j), \ldots, i^* \\ i_{\min}(j) &= \begin{cases} 1 & \text{if } b_j \geq 0 \\ \min\{i \ |(e^{z_i} + b_j) > 0\} & \text{if } b_j < 0 \end{cases} \\ \hat{V}_{i,j}(\tau^{n+1}) &= \hat{V}_{i,j}(\tau^n_+) = 0 \ ; \ i = i^* + 1, \ldots, i_{\max} \\ \hat{V}_{0,j}(\tau^{n+1}) &= \hat{V}_{0,j}(\tau^n_+) \ ; \ b_j > 0 \\ \hat{V}_{i,j}(\tau^n) &= \hat{V}_{i,j}(\tau^n_+) = \begin{cases} (\gamma/2 - (e^{z_i} + b_j)e^{r\tau^n})^2 & ; \ b_j < 0 \ ; \ i = 0, \ldots, i_{\min}(j) - 2 \ , \\ (\gamma/2)^2 & ; \ b_j \leq 0 \ ; \ i = i_{\min}(j) - 1 \end{cases} \end{aligned}$$

where we assume that the grid size is sufficiently small so that the positive coefficient condition is satisfied

$$\alpha \geq 0 \ ; \ \beta \geq 0 \ ; \ \lambda \geq 0 \ , \tag{C.21}$$

and that the grid is constructed so that

$$s_{i_{\min}(j)-1} + b_j = 0 \ \ . \tag{C.22}$$

The approximate Dirichlet condition at the points $i = i^* + 1, \ldots, i_{\max}$ follows from equation (C.8). Since the jump term is computed using an FFT, $i^*$ and $i_{\max}$ are selected so as to minimize wrap-around error (d'Halluin et al., 2005). Also, $k_{\max}$ is selected sufficiently large so that errors in the approximation of the integral are minimized (d'Halluin et al., 2005).

**Lemma C.2** (Bounds on the solution)**.** *If the discrete time advance equations are given by equation (C.20), and assuming that the positive coefficient conditions (C.19) and (C.21) are satisfied, then*

$$\begin{aligned} \max_{i \in [i_{\min}(j), i^*]} \hat{V}_{i,j}(\tau^{n+1}) \ &\leq \ \max_{i \in [0, i_{\max}]} \hat{V}_{i,j}(\tau^n_+) \ , \\ \min_{i \in [i_{\min}(j), i^*]} \hat{V}_{i,j}(\tau^{n+1}) \ &\geq \ \min_{i \in [0, i_{\max}]} \hat{V}_{i,j}(\tau^n_+) \ . \end{aligned} \tag{C.23}$$

*Proof.* This follows from a maximum analysis of equation (C.20). □

**Lemma C.3** (Upper bound for embedded MV optimal solution)**.** *If grid restrictions (C.3) and (C.22) hold, the initial condition is given as in Proposition C.1, and the conditions for Lemma C.2 are satisfied, then*

$$V_h(s, b, \tau^{n+1}_+) \ \ \leq \ \ (\gamma/2)^2 \ \ ; \ \ \{(s,b) \in \Omega_{loc} \mid (s+b) \geq 0\} \ \ ; \ \ \forall n \geq 0 \ \ . \tag{C.24}$$

*Proof.* Suppose

$$V_h(s, b, \tau^n_+) \ \ \leq \ \ (\gamma/2)^2 \ \ \{(s,b) \in \Omega_{loc} \mid b \geq 0\} \ . \tag{C.25}$$

Since linear interpolation is used to move from $V_h \rightarrow \hat{V}_h$

$$\hat{V}_h(z_i, b_j, \tau^n_+) \ \ \leq \ \ (\gamma/2)^2 \ \ ; \ \ i = 0, \ldots, i_{\max} \ \ ; \ \ b_j \geq 0 \ . \tag{C.26}$$

From Lemma C.2 and noting the boundary condition $\hat{V}_{0,j}(\tau^{n+1}) = \hat{V}_{0,j}(\tau^n_+)$, $b_j > 0$, and $\hat{V}_{0,0}(\tau^{n+1}) = (\gamma/2)^2$, we have

$$\hat{V}_{i,j}(\tau^{n+1}) \leq \ \ (\gamma/2)^2 \ \ ; \ \ i = 0, \ldots, i_{\max} \ ; \ b_j \geq 0 \ , \tag{C.27}$$

and in addition from equations (C.12), we have the boundary condition

$$\hat{V}_{i,j}(\tau^{n+1}) = \hat{V}_{i,j}(\tau^{n+1}_+) = (\gamma/2)^2 \ \ \ ; \ \ (e^{z_{i_{\min}(j)-1}} + b_j) = (s_{i_{\min}(j)-1} + b_j) = 0 \ \ ; \ \ b_j \leq 0 \tag{C.28}$$

Now, using linear interpolation to move from $\hat{V}_h \rightarrow V_h$

$$V_h(s, b, \tau^{n+1}) \leq (\gamma/2)^2 \ \ ; \ \ b \geq 0 \ . \tag{C.29}$$

Grid condition (C.3) ensures that if $(s_i, b_j) \in \mathcal{S}_{loc}$ then $(s_i, \min[\max(b_j e^{r\Delta\tau}, -b_{\max}), b_{\max}]) \in \mathcal{S}_{loc}$. In addition if $(s_i, b_j) \in \mathcal{S}_{loc}$, then $\exists B \geq 0 \in \mathcal{Z}_{semi}$ such that $(s_i + b_j e^{r\Delta\tau} - B, B) \in \mathcal{S}_{loc}$. Consequently we can bound the solution by only examining the values of $V_h(s, b, \tau^{n+1})$ for $b \geq 0$. Noting these simplifications in (C.5) we have

$$\begin{aligned} V_h(s_i, b_j, \tau_+^{n+1}) &\leq \min_{(d,B)\in\mathcal{Z}_{semi}} V_h(s_i + b_j e^{r\Delta\tau} - B - d, B, \tau^{n+1}) \ , \ (s_i, b_j) \in \mathcal{S}_{loc} \ . \\ &\leq \max_{\substack{(d,B)\in\mathcal{Z}_{semi} \\ B\geq 0}} V_h(s_i + b_j e^{r\Delta\tau} - B - d, B, \tau^{n+1}) \\ &\leq (\gamma/2)^2 \end{aligned} \tag{C.30}$$

where the last step follows from equation (C.29). Combining equation (C.30) with boundary condition (C.28), we obtain

$$V_h(s, b, \tau_+^{n+1}) \ \leq \ (\gamma/2)^2 \ ; \ \{(s,b) \in \Omega_{loc} \mid (s+b) \geq 0\} \ . \tag{C.31}$$

The result follows $\forall n$ since from equation (C.8)

$$V_h(s, b, 0) \ \leq \ (\gamma/2)^2 \ ; \ \{(s,b) \in \Omega_{loc} \mid (s+b) \geq 0\} \ . \tag{C.32}$$

□

We now proceed to verify the conditions required for Lemma C.1. Before beginning, we note the following, which we obtain by writing equation (C.20) for node $i+1$ and subtracting from equation (C.20) for node $i$.

$$\begin{aligned} &\Big(\hat{V}_{i+1,j}(\tau^{n+1}) - \hat{V}_{i,j}(\tau^{n+1})\Big)(1 + \lambda\Delta\tau + \Delta\tau\alpha + \Delta\tau\beta) \\ &\quad - \Delta\tau\beta\Big(\hat{V}_{i+2,j}(\tau^{n+1}) - \hat{V}_{i+1,j}(\tau^{n+1})\Big) - \Delta\tau\alpha\Big(\hat{V}_{i,j}(\tau^{n+1}) - \hat{V}_{i-1,j}(\tau^{n+1})\Big) \\ = &\Big(\hat{V}_{i+1,j}(\tau_+^n) - \hat{V}_{i,j}(\tau_+^n)\Big) + \lambda\Delta\tau \sum_{k=-k_{\max}}^{k=k_{\max}} \hat{p}_k \Big(\hat{V}_h(z_{i+1} + y_k, b_j, \tau_+^n) - \hat{V}_h(z_i + y_k, b_j, \tau_+^n)\Big) \\ &\quad ; \ i = i_{\min}(j), \ldots, i^* - 1 \ . \end{aligned} \tag{C.33}$$

**Lemma C.4.** *[Non-increasing value function in s: $b \geq 0$] If Condition C.1 holds, the conditions required for Lemma C.3 are satisfied, with the discrete equations and boundary conditions as given in equation (C.20), and if $b_j \geq 0$ and*

$$\hat{V}_{i+1,j}(\tau_+^n) - \hat{V}_{i,j}(\tau_+^n) \ \leq \ 0 \ ; \ i = 0, \ldots, i_{\max} - 1 \ ; \ b_j \geq 0 \tag{C.34}$$

*then*

$$\hat{V}_{i+1,j}(\tau^{n+1}) - \hat{V}_{i,j}(\tau^{n+1}) \ \leq \ 0 \ ; \ i = 0, \ldots, i_{\max} - 1 \ ; \ b_j \geq 0 \tag{C.35}$$

*Proof.* From Lemma C.2, equation (C.23) and equation (C.34), along with $\hat{V}_{0,0}(\tau) = (\gamma/2)^2$, $\hat{V}_{i^*+1,j}(\tau^{n+1}) = 0$ (see equation (C.20)), we obtain

$$\begin{aligned} \max_i \hat{V}_{i,j}(\tau^{n+1}) &= \hat{V}_{0,j}(\tau^{n+1}) = \hat{V}_{0,j}(\tau_+^n) \ ; \ j \geq 0 \\ \min_i \hat{V}_{i,j}(\tau^{n+1}) &= \hat{V}_{i^*+1,j}(\tau_+^n) = 0 \ ; \ \forall j, \end{aligned} \tag{C.36}$$

hence

$$\left(\hat{V}_{1,j}(\tau^{n+1}) - \hat{V}_{0,j}(\tau^{n+1})\right) \leq 0 \ ; \ j \geq 0 \tag{C.37}$$

$$\left(\hat{V}_{i^*+1,j}(\tau^{n+1}) - \hat{V}_{i^*,j}(\tau^{n+1})\right) \leq 0 \ ; \ j \geq 0 \ . \tag{C.38}$$

Hence, we can write equation (C.33) as

$$\mathcal{M}\mathcal{X} = \mathcal{B}, \quad \text{where } \mathcal{X}_i = \left(\hat{V}_{i+1,j}(\tau^{n+1}) - \hat{V}_{i,j}(\tau^{n+1})\right) ,$$

and where $\mathcal{M}$ is a diagonally dominant $M$ matrix, and from condition (C.34), boundary conditions (C.37 - C.38) and equation (C.33) we have that $\mathcal{B} \leq 0$, hence

$$\mathcal{X}_i = \left(\hat{V}_{i+1,j}(\tau^{n+1}) - \hat{V}_{i,j}(\tau^{n+1})\right) \leq 0 \ , \quad = i_{\min}(j), \ldots, i^* - 1 \ ; \ j \geq 0 \ .$$

In view of the fact that $\hat{V}_{i,j}(\tau^{n+1}) \equiv 0$, $i = i^* + 1, \ldots, i_{\max}$ and that boundary condition (C.37) holds, we then have equation (C.35). □

For the case that $b_j < 0$ we have to proceed in a different fashion. If we assume equation (C.34) holds for $b_j < 0$, the problem is that the boundary condition at $s_{i_{\min}(j)-1}$ is, noting equations (C.12) and (C.22),

$$\hat{V}_{i_{\min}(j)-1,j}(\tau^{n+1}) = (\gamma/2 - (s_{i_{\min}(j)-1} + b_j)e^{r\tau^n})^2 = (\gamma/2)^2 \ , \tag{C.39}$$

while bound (C.23) gives us (using equation (C.10) and recalling that $b_j < 0$)

$$\hat{V}_{i_{\min}(j),j}(\tau^{n+1}) \leq (\gamma/2 - b_j e^{r\tau^n})^2 \tag{C.40}$$

so that we cannot conclude that

$$\left(\hat{V}_{i_{\min}(j),j}(\tau^{n+1}) - \hat{V}_{i_{\min}(j)-1,j}(\tau^{n+1})\right) \leq 0 \ . \tag{C.41}$$

The problem can be traced to the nonlocal jump term. An intuitive explanation of this is that we can imagine a case where the investor has only a small positive amount of wealth and has a large amount of leverage. In this case, the probability of a jump into insolvency is large, and hence this is a worse situation than actually having zero wealth. One might suppose that this would be a situation where a withdrawal of wealth would be embedded MV optimal. But there is clearly a better strategy here. For example, the investor would be better off de-leveraging and simply investing the small positive wealth in the market. We will now formalize this argument in the following.

**Lemma C.5.** *Assume the discrete equations and boundary conditions are as given in equation (C.20). If $b_j < 0$, then there exists a node $\hat{i}$, $i_{min}(j) - 1 \leq \hat{i} \leq i^* + 1$ such that*

$$\left(\hat{V}_{\hat{i}+1,j}(\tau^{n+1}) - \hat{V}_{\hat{i},j}(\tau^{n+1})\right) \leq 0 \ , \tag{C.42}$$

*and there is no other node $i < \hat{i}$ having this property.*

*Proof.* From equation (C.12) the boundary conditions are

$$\begin{aligned} \hat{V}_{i_{\min}(j)-1,j}(\tau^{n+1}) &= (\gamma/2)^2 > 0 \ ; \ (s_{i_{\min}(j)-1,j} + b) = 0 \ , \\ \hat{V}_{i^*+1,j}(\tau^{n+1}) &= 0 \ , \end{aligned} \tag{C.43}$$

hence at least one node $\hat{i}$ satisfying condition (C.42) exists. If there is more than one such node, let $\hat{i}$ be the node with the smallest index. □

**Lemma C.6.** *[Non-increasing value function in s: $b < 0$] Assume the conditions required for Lemma C.3 are satisfied, with the discrete equations and boundary conditions as given in equation (C.20). If $b_j < 0$, and*

$$\hat{V}_{i+1,j}(\tau_+^n) - \hat{V}_{i,j}(\tau_+^n) \ \leq \ 0 \ ; \ i = 0, \ldots, i_{\max} - 1 \ ; \ b_j < 0 \ , \tag{C.44}$$

*then there exists a node $\hat{i}$ such that*

*(a)*

$$\left( \hat{V}_{i+1,j}(\tau^{n+1}) - \hat{V}_{i,j}(\tau^{n+1}) \right) \leq 0 \ ; \ i = \hat{i}, \ldots, i_{\max} - 1 \tag{C.45}$$

*(b) Either $\hat{i} = i_{\min}(j) - 1$ or $\hat{V}_{i,j}(\tau^{n+1}) > \hat{V}_{i_{\min}(j)-1,j}(\tau^{n+1}) = (\gamma/2)^2 \ ; \ i = i_{\min}(j), \ldots, \hat{i}$*

*Proof.* From Lemma C.5, the node $\hat{i}$ exists and satisfies equation (C.42). Noting equation (C.44), (C.42) and following the same steps as used to prove Lemma C.4, we obtain (a). For (b), note that from Lemma C.5, $\hat{i}$ is smallest index node satisfying property (C.42). Assume $\hat{i} > i_{\min} - 1$. Hence

$$\left( \hat{V}_{i+1,j}(\tau^{n+1}) - \hat{V}_{i,j}(\tau^{n+1}) \right) \ > \ 0 \ ; \ i = i_{\min} - 1, \ldots, \hat{i} - 1 \tag{C.46}$$

and (b) follows.

□

**Lemma C.7** (Non-increasing value function)**.** *If the conditions required for Lemma C.3 are satisfied, with the discrete equations and boundary conditions as given in equation (C.20), and*

$$\hat{V}_{i+1,j}(\tau_+^n) - \hat{V}_{i,j}(\tau_+^n) \ \leq \ 0 \ ; \ i = 0, \ldots, i_{\max} - 1 \ ; \forall j \tag{C.47}$$

*then the function*

$$\begin{aligned} \mathcal{G}_h(s, b, \tau^{n+1}) &= \min((\gamma/2)^2, V_h(s, b, \tau^{n+1})) \ ; \ (s, b) \in \mathcal{S}_{loc} \\ &= V_h(s, b, \tau^{n+1}) \ ; \ (s, b) \in \Omega_{loc} \backslash \mathcal{S}_{loc} \end{aligned} \tag{C.48}$$

*has the property that*

$$\mathcal{G}_h(s, b, \tau^{n+1}) \geq \mathcal{G}_h(s', b, \tau^{n+1}); \ \ s' > s \ ; \ \forall (s, b), (s', b) \in \Omega_{loc} \ . \tag{C.49}$$

*Proof.* For $b_j \geq 0$, then from Lemma C.3, equation (C.27) we have

$$\min(\hat{V}_{i,j}(\tau^{n+1}), (\gamma/2)^2) \;\; = \;\; \hat{V}_{i,j}(\tau^{n+1}) \;\; ; \; b_j \geq 0 \; , \tag{C.50}$$

and from Lemma C.4 and the fact that $V_h(s_i, b_j, \tau^{n+1}) = \hat{V}_h(\log s_i, b_j, \tau^{n+1})$, and the properties of linear interpolation, we then conclude that equation (C.49) holds for $b_j \geq 0$.

For the case $b_j < 0$, note that, from Proposition C.2 that for $(s+b) \leq 0$, $V_h(s, b, \tau^{n+1})$ is nonincreasing in $s$ for fixed $b$, and $V_h(\cdot) = (\gamma/2)^2$ at $(s+b) = 0$, hence $\mathcal{G}_h(\cdot)$ is continuous at $s + b = 0$ and nonincreasing for $s > -b$ from Lemma C.6, equation (C.48), and the properties of linear interpolation.

□

**Lemma C.8** (Local embedded MV non-optimality of withdrawing). *If conditions C.1 hold, and the conditions required for Lemma C.3 are satisfied, with the discrete equations and boundary conditions as given in equation (C.20), and if*

$$V_h(s, b, \tau_+^n) \geq V_h(s', b, \tau_+^n) \; ; \; s' > s \; , \quad \forall (s, b), (s', b) \in \Omega_{loc} \tag{C.51}$$

*then there exist embedded MV optimal strategies which move $V_h(\cdot, \tau^{n+1}) \rightarrow V_h(\cdot, \tau_+^{n+1})$ such that $d \equiv 0$, and*

$$V_h(s, b, \tau_+^{n+1}) \geq V_h(s', b, \tau_+^{n+1}) \; ; \; s' > s \; . \;\; ; \;\; \forall (s, b), (s', b) \in \Omega_{loc} \tag{C.52}$$

*Proof.* Write equation (C.5) for $\tau_+^{n+1}$

$$\begin{aligned} V_h(s_i, b_j, \tau_+^{n+1}) \;\; &= \;\; \min \left[ V_h(s_i, b_j e^{r\Delta\tau}, \tau^{n+1}), \min_{(d,B) \in \mathcal{Z}_{semi}} V_h(S(s_i, b_j e^{r\Delta\tau}, B, d), B, \tau^{n+1}) \right] \\ & \qquad (s_i, b_j) \in \mathcal{S}_{loc} \; . \end{aligned} \tag{C.53}$$

From Lemma C.3, we can write, for $(s_i, b_j) \in \mathcal{S}_{loc}$,

$$\begin{aligned} V_h(s_i, b_j, \tau_+^{n+1}) \;\; &= \;\; \min\big(V_h(s_i, b_j, \tau_+^{n+1}), (\gamma/2)^2\big) \\ &= \;\; \min \Big[ \min\big(V_h(s_i, b_j e^{r\Delta\tau}, \tau^{n+1}), (\gamma/2)^2\big), \\ & \qquad \min_{(d,B) \in \mathcal{Z}_{semi}} \Big( \min\big(V_h(S(s_i, b_j e^{r\Delta\tau}, B, d), B, \tau^{n+1}), (\gamma/2)^2\big) \Big) \Big] \; , \\ &= \;\; \min \left[ \mathcal{G}_h(s_i, b_j e^{r\Delta\tau}, \tau^{n+1}), \min_{(d,B) \in \mathcal{Z}_{semi}} \mathcal{G}_h(S(s_i, b_j e^{r\Delta\tau}, B, d), B, \tau^{n+1}) \right] \; , \end{aligned} \tag{C.54}$$

where $\mathcal{G}_h(\cdot)$ is defined in Lemma C.7. We have also used the fact that grid condition (C.3) ensures that if $(s_i, b_j) \in \mathcal{S}_{loc}$ then $(s_i, \min[\max(b_j e^{r\Delta\tau}, -b_{\max}), b_{\max}] \in \mathcal{S}_{loc}$, and that if $(s_i, b_j) \in \mathcal{S}_{loc}$, any $B \in \mathcal{Z}_{semi}$ is such that $(s_i + b_j e^{r\Delta\tau} - B, B) \in \mathcal{S}_{loc}$,

From Lemma C.7 and equation (C.51), $\mathcal{G}_h(\cdot)$ is a non-increasing function of $s$, it follows from Lemma C.1 that $d = 0$ is an embedded MV optimal strategy. We can then write equation (C.54) as

$$V_h(s_i, b_j, \tau_+^{n+1}) \;\; = \;\; \min \left[ \mathcal{G}_h(s_i, b_j e^{r\Delta\tau}, \tau^{n+1}), \min_{B \in \mathcal{Z}_{semi}} \mathcal{G}_h(s + b e^{r\Delta\tau} - B, B, \tau^{n+1}) \right] \; , \tag{C.55}$$

As a result $\forall (s,b),(s',b) \in \mathcal{S}_{loc}$, $s' > s$

$$\begin{aligned} V_h(s,b,\tau_+^{n+1}) - V_h(s',b,\tau_+^{n+1}) &= \min\Big[\mathcal{G}_h(s, be^{r\Delta\tau}, \tau^{n+1}), \min_{B\in\mathcal{Z}_{semi}} \mathcal{G}_h(s+be^{r\Delta\tau}-B, B, \tau^{n+1})\Big] \\ &\quad - \min\Big[\mathcal{G}_h(s', be^{r\Delta\tau}, \tau^{n+1}), \min_{B\in\mathcal{Z}_{semi}} \mathcal{G}_h(s'+be^{r\Delta\tau}-B, B, \tau^{n+1})\Big] \\ &\geq \min\Big[\mathcal{G}_h(s, be^{r\Delta\tau}, \tau^{n+1}) - \mathcal{G}_h(s', be^{r\Delta\tau}, \tau^{n+1}), \\ &\quad \min_{B\in\mathcal{Z}_{semi}} \mathcal{G}_h(s+be^{r\Delta\tau}-B, B, \tau^{n+1}) \\ &\quad\quad - \min_{B\in\mathcal{Z}_{semi}} \mathcal{G}_h(s'+be^{r\Delta\tau}-B, B, \tau^{n+1})\Big] \ . \end{aligned} \tag{C.56}$$

From grid conditions (C.3) and (C.22) and Proposition C.3, $\exists B^*$ s.t. $(s+be^{r\Delta\tau}-B^*, B^*)$ and $(s'+be^{r\Delta\tau}-B^*, B^*)$ are admissible, and

$$\begin{aligned} \min_{B\in\mathcal{Z}_{semi}} \mathcal{G}_h(s+be^{r\Delta\tau}-B, B, \tau^{n+1}) &= \mathcal{G}_h(s+be^{r\Delta\tau}-B^*, B^*, \tau^{n+1}) \\ &\geq \mathcal{G}_h(s'+be^{r\Delta\tau}-B^*, B^*, \tau^{n+1}) \\ &\geq \min_{B\in\mathcal{Z}_{semi}} \mathcal{G}_h(s'+be^{r\Delta\tau}-B, B, \tau^{n+1}) \end{aligned} \tag{C.57}$$

Combining equations (C.56) and (C.57), and using Lemma C.7 and equation (C.51), we obtain

$$V_h(s,b,\tau_+^{n+1}) - V_h(s',b,\tau_+^{n+1}) \geq 0 \ \ ; \ \ \forall (s,b),(s',b) \in \mathcal{S}_{loc} \ \ . \tag{C.58}$$

From equation (C.10), we note that

$$V_h(s,b,\tau_+^{n+1}) = (\gamma/2 - (s+b)e^{r\tau^{n+1}})^2 \ \ ; \ \ (s,b) \in \Omega_{loc} \backslash \mathcal{S}_{loc} \tag{C.59}$$

is nonincreasing in $s$ for fixed $b$. Finally note that if $(s,b) \in \Omega_{loc}\backslash\mathcal{S}_{loc}$ and $(s',b) \in \mathcal{S}_{loc}$ then

$$V_h(s,b,\tau_+^{n+1}) \geq (\gamma/2)^2 \geq V_h(s',b,\tau_+^{n+1}) \ , \tag{C.60}$$

where we have used Lemma C.3, hence

$$V_h(s,b,\tau_+^{n+1}) \geq V_h(s',b,\tau_+^{n+1}) \ ; \ s' > s \ \ ; \ \ \forall (s,b),(s',b) \in \Omega_{loc}. \tag{C.61}$$

□

**Theorem C.1** (Embedded MV non-optimal withdrawal)**.** *Assuming conditions C.1, and that the conditions required for Lemma C.3 are satisfied, with the discrete equations and boundary conditions as given in equation (C.20), and if the initial condition (C.8) is imposed, then there exist embedded MV optimal strategies which move* $V_h(\cdot, \tau^{n+1}) \rightarrow V_h(\cdot, \tau_+^{n+1}), \forall n$ *such that* $d \equiv 0$.

*Proof.* Initial condition (C.8) satisfies condition (C.51) of Lemma C.8 at $\tau = 0$, hence this follows from Lemma C.8. □

**Remark C.2** (Embedded MV non-optimality of withdrawing)**.** *Theorem C.1 states that an embedded MV optimal strategy has no withdrawal after the initial withdrawal at $\tau = 0$. However, Theorem 3.1 states that an embedded MV optimal strategy is to withdraw cash whenever $W_c(t) > W_{opt}(t)$. There is no contradiction here. In terms of pre-commitment MV, the optimal strategies are non-unique. These different strategies amount to doing different things with the wealth which exceeds $W_{opt}(t)$. See discussion in Section 5.3.*

**Remark C.3** (Extension of Theorem C.1)**.** *Using similar steps, it is straightforward to extend the results of Theorem C.1. An optimal strategy is no withdrawal (after the initial withdrawal at $\tau = 0$), for the cases*

- *Continuous re-balancing, jumps and leverage possible ($q_{\max} \geq 1$)*
- *Discrete re-balancing, jumps but no leverage ($q_{\max} = 1$)*
- *Discrete and continuous re-balancing, leverage possible ($q_{\max} \geq 1$), no jumps*

*Note that the discrete re-balancing case with jump and leverage s is noticeably absent. In fact, it is not clear that Theorem C.1 can be extended for this case in general.*

**Remark C.4** (A posteriori check of embedded MV non-optimal withdrawal)**.** *It is easy to check (computationally) if it is ever embedded MV optimal to withdraw if $W_c(t) < W_{opt}(t)$. The first step is to compute $V_h(s_i, b_j, \tau_+^n)$ assuming $d = 0$ if $W_c(t) < W_{opt}(t)$. Then, if it is embedded MV non-optimal to withdraw, the following condition must hold*

$$V_h(0, b_{j+1}, \tau_+^n) \leq V_h(0, b_j, \tau_+^n) \quad ; \quad 0 \leq b_j \leq W_{opt}(t) \ . \tag{C.62}$$

*In all our numerical experiments, even if we violate some of the conditions required for Theorem C.1, we have observed that condition (C.62) always holds.*

**Theorem C.2.** *Provided Conditions C.1 are satisfied, and the initial condition is given by (C.8), for the cases listed in Remark C.4, then it an optimal strategy to not withdraw for equations (B.5-B.9), assuming equations (B.5-B.9) satisfy a strong comparison principle.*

*Proof.* From Dang and Forsyth (2014), the discretization (C.20) satisfies all the conditions required for convergence to the viscosity solution of equations (B.5-B.9). From Theorem C.1, the optimality of not withdrawing holds for any $h$, we take the limit as $h \to 0$. □

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