Dynamic optimal investment strategies for benchmark outperformance with widely-used performance metrics

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Abstract

We analyze dynamic investment strategies for benchmark outperformance using two widely-used objectives of practical interest to investors: (i) maximizing the information ratio (IR), and (ii) maximizing the tracking difference (cumulative outperformance) relative to the benchmark. In the case of the tracking difference, we propose a simple and intuitive objective function based on the quadratic deviation (QD) from an elevated benchmark. First, by employing standard assumptions such as no market frictions, we solve stochastic control problems for the optimal analytical (or closed-form) investment strategies, extending known results for the IR problem to the case where jumps are allowed in the underlying process dynamics and where contributions to the portfolio are allowed. The analytical solutions to the QD problem are novel, as are the analytical results comparing the resulting IR- and QD-optimal investment strategies. Second, employing a data-driven neural network approach to solve the control problems numerically without any assumptions regarding the underlying process dynamics, we compare the resulting investment strategies under more realistic assumptions, including the discrete rebalancing of the portfolio together with multiple investment constraints, such as leverage and short-selling restrictions. Our analytical and numerical results illustrate that, compared with IR-optimal strategies with the same expected value of terminal wealth, the QD-optimal investment strategies result in comparatively more diversified asset allocations during certain periods of the investment time horizon. This ultimately results in delivering superior out-of-sample benchmark outperformance for the QD-optimal investor, and demonstrates the effectiveness of targeting benchmark outperformance via the proposed QD objective.

Keywords: Asset allocation, portfolio optimization, benchmark outperformance, neural network

JEL classification: G11, C61

1 Introduction

Despite the considerable professional talent attracted to the field of active portfolio management, where a portfolio manager (or an investment institution) brings their considerable expertise to bear on actively pursuing an investment strategy with the explicit goal of outperforming an appropriate pre-specified benchmark (Alekseev and Sokolov (2016); Kashyap et al. (2021); Korn and Lindberg (2014); Lehalle and Simon (2021); Zhao (2007)), it remains a disappointing fact that the promised outperformance hardly ever seems to materialize in practice. In fact, underperforming their benchmarks is something professional portfolio managers achieve with “surprising consistency” (Gorman et al. (2010)).

For example, S&P Global’s SPIVA 2020 research report (SPIVA (2020) and analysis by Coleman (2021)), which provides an assessment of the performance of active funds against their appropriate index benchmarks until December 2020, shows that an astonishing 86% of active domestic (US) equity funds underperformed their benchmarks over the last 20 years. In the case of active bond funds, between 82% and 97% of funds (depending on maturity and issuer type) underperformed their respective benchmarks over the last 15 years.

While the underperformance of benchmarks by active funds is of course not a recent phenomenon (see for example Kenanberry et al. (1998)), it seems to be enjoying substantial recent publicity in the popular investment literature and news articles (see for example Business Insider (2020); CNBC (2020b); New York Times (2021); Swedroe and Berkin (2020)).
Participation in an active fund also typically costs the average investor about five times more than investing in an index fund with a passive investment strategy (CNBC [2020a]), where the aim of passive investment strategies is simply the replication of the performance of some benchmark. For investors, implementation of passive investment strategies are easy and affordable due to the widespread availability of low-cost exchange-traded funds (ETFs) aimed simply at benchmark replication. While single broad market indices like the S&P 500 are often used as benchmarks (Kashyap et al. [2021]), plausible benchmarks also include (the returns of) a portfolio rebalanced according to some straightforward rebalancing rule known to yield satisfying performance historically, such as a constant proportion strategy like the 1/N rule (see for example Brightman et al. [1999a, 1999b, 2000]; Davis and Lleo [2008]; Filippi et al. [2016]; Lim and Wong [2010]; Nicolosi et al. [2018]; Oderda [2015]; Tepla [2001]; Yao et al. [2006]; Gao and Zhao [2007]); the question has taken on a new sense of urgency and remains an active area of research (for recent examples, see Aguilar and Custovic [2021]; Al-Aradi and Jaimungal [2018, 2021]; Bolshakov and Chincarini [2020]; Guastaroba et al. [2020]; Li et al. [2021]; Park et al. [2019]; Pesenti and Jaimungal [2021]; Sehgal and Melara [2019]). In addition, machine learning techniques are also increasingly used to address the problems associated with attempting to track or to outperform a given benchmark (see for example Badrania et al. [2021]; Kim [2021]; Kim and Kim [2020]; Kwak et al. [2021]; Ni et al. [2020]; Ouyang et al. [2019]; Samo and Vervuurt [2016]).

However, in surveying the literature on deriving dynamic (multi-period) investment strategies for benchmark outperformance, we observe that the objective functions and assumptions that are very popular in the academic literature often do not appear to align very well with the performance metrics used and constraints applied by investors in practice. To clarify, we briefly discuss the treatment of the benchmark outperformance problem in the relevant literature, before contrasting this with investment practice.

The objective functions used in the literature often include the use of explicit or implied utility functions (Al-Aradi and Jaimungal [2018, 2021]; Basak et al. [2006]; Basak et al. [2006]; Davis and Lleo [2008]; Lim and Wong [2010]; Nicolosi et al. [2018]; Oderda [2015]; Tepla [2001]), with the use of log utility (of outperformance) appearing to be especially popular. However, many other considerations are often included (possibly as constraints or as objective functions in their own right), for example setting maximum and minimum limits on out-/under-performance (Basak et al. [2006]), minimizing the expected time until outperformance subject to a constraint on the probability of underperformance (and some variations of this theme, see Browne [1999a, 2000]), and imposing penalties on underperforming the benchmark (Gaivoronski et al. [2005]). In addition, fairly elaborate definitions of outperformance have been proposed for the sophisticated investor, involving for example the possibility of multiple benchmarks (Al-Aradi and Jaimungal [2018, 2021]) or constraints on the correlation or more generally the dependence structure with the benchmark (Bernard and Vanduffel [2014]; Pesenti and Jaimungal [2021]).

Furthermore, an important quantity of interest in many of the objective functions in the literature is the ratio of the active portfolio wealth to the benchmark wealth (Al-Aradi and Jaimungal [2018, 2021]; Browne [1999a, 2000]; Davis and Lleo [2008]; Lim and Wong [2010]; Nicolosi et al. [2018]; Oderda [2015]). Using the wealth ratio, often in conjunction with log utility, means that contributions to or withdrawals from the portfolio cannot be included in the analysis due to analytical tractability considerations. This implies that the resulting investment strategies are potentially of limited use to many institutional investors like defined contribution pension funds, where the modelling of contributions and withdrawals are critical aspects of the problem (Forsyth and Vetzal [2019]; Forsyth et al. [2019]; Forsyth [2021]). As discussed below, we explicitly

\[1\] It is also worth noting that some ETFs aim not just to replicate the performance of an index but to offer so-called “enhanced indexation” (see for example Filippi et al. [2016]; Li et al. [2021]), according to which the aims are a combination of some level of index tracking with index outperformance. For the purposes of this paper, we consider this as simply another example of an attempt at active portfolio management to outperform a given benchmark.

\[2\] Note that we only consider dynamic investment strategies, arguably the only reasonable approach given the fairly long time horizon considered in this paper. Nevertheless, there is a large literature on applying Markowitz-inspired portfolio optimization models to derive portfolio weights for benchmark outperformance in single-period settings - see for example Filippi et al. [2016]; Gaivoronski et al. [2005]; Guastaroba et al. [2020]; Huang et al. [2018]; Jorion [2003]; Li et al. [2021]; Roll [1992].
consider contributions to the portfolio in this paper, leaving withdrawals for our future work.

Another quantity enjoying significant popularity in the objective functions considered in the literature is the tracking error, which typically measures the standard deviation or average absolute deviation of the daily, monthly or annual differences between the returns of the active portfolio and the returns of the benchmark (see for example Clarke et al. (1994); Coleman et al. (2006); Huang et al. (2018); Jorion (2003); Koll (1992)). It should be emphasized that by measuring some statistic associated with a set of individual observations of return differences, the tracking error does not include any cumulative information regarding the over/under-performance of the portfolio vs. the benchmark over some time horizon. Therefore, an active portfolio underperforming the benchmark consistently in each period (for example by similar ratios or amounts) might in fact have a very low tracking error according the definition.

Despite this observation, tracking error minimization is not only employed as expected in benchmark replication settings, but also in combination with additional objectives with the aim of obtaining outperformance and/or other desirable properties such as improved downside risk (Ammann and Zimmermann (2001); Barro and Canestrelli (2009); Beasley et al. (2003); Gaivoronski et al. (2005); Giuzio (2017); Goel et al. (2018); Guasjaroba et al. (2020); Huang et al. (2018); Jorion (2003); Kim and Kim (2020); Kwon and Wu (2017); Rossbach and Karlow (2019); Sant’Anna et al. (2017); Sehgal and Mehra (2019); Strub and Baumann (2018); Vieira et al. (2021); Wang et al. (2018); Xu et al. (2016)). The tracking error is also used by some investors for performance measurement purposes, which is reasonable provided they are primarily concerned with performance consistency rather than outperformance (Du and Overway (2021); Israelsen and Cogswell (2007); Johnson et al. (2013); Vanguard (2014)).

Finally, we note that the literature is typically concerned with obtaining closed-form solutions to the specified optimization problems, which necessarily assumes for example continuous trading, the possibility of unlimited short-selling and leverage, and usually also the possibility of continuing trading in the event of insolvency. Examples include Bajeux-Besnainou et al. (2013); Basak et al. (2006); Bo et al. (2021); Browne (1999a,b, 2000); Davis and Lleo (2008); Lim and Wong (2010); Nicolosi et al. (2018); Oderda (2015); Teplà (2001); Yao et al. (2006); Zhang and Gao (2017); Zhao (2007).

Of course, while the existing literature may be very valuable for some investors, active portfolio managers arguably have goals that are simultaneously more mundane and more ambitious than the treatment offered in the literature. On one hand, practical goals are more mundane, in the sense that outperformance is typically measured by investors and institutions alike on a far more straightforward basis (discussed below) than the aforementioned objective functions typically encountered in the literature. On the other hand, active portfolio managers simultaneously also place more ambitious demands on the resulting investment strategies compared to most of the literature, since they necessarily require dynamic investment strategies derived subject to realistic investment constraints, such as discrete rebalancing, short-selling and leverage restrictions.

In this paper, we wish to address these considerations. In terms of objective functions, we therefore limit our focus to two objectives for outperformance assessment, namely (i) the tracking difference and (ii) the information ratio, leaving additional objectives for future work.

(i) Tracking difference: In contrast to the tracking error (discussed above), the tracking difference is simply the difference between the cumulative returns of the active portfolio and that of the benchmark over a fixed time horizon, and therefore provides a straightforward and intuitive measure of the “performance gap” (Charteris and McCullough (2020)). For this reason, the tracking difference is recognized in the popular investment literature as a potentially more relevant and important metric than the tracking error for the investor, especially in the case of long-term investors (see for example Boyd (2021); ETF.com (2021); Hougan (2015); Pastant (2018); Vanguard (2014)). Its importance is also recognized by regulators such as European Securities and Markets Authority, who requires its disclosure (ESMA (2014)).

Since the tracking error does not actually convey any information regarding the actual over-/under-performance of the portfolio, it is perhaps not surprising that the relationship between the tracking difference and tracking error has been shown to be very weak, resulting in significantly different rankings of active portfolio managers (Charteris and McCullough (2020); Johnson et al. (2013)). This implies that institutional investors concerned with minimizing the tracking error, such as many pension funds, may in fact be sacrificing performance (Israelsen and Cogswell (2007)).

Despite these considerations, the tracking difference has received relatively scant attention in the academic literature (see Bonelli (2015); Charteris and McCullough (2020); Johnson et al. (2013)). By explicitly focusing on tracking difference in this paper using an intuitive objective function, we aim to address this shortcoming.
(ii) Information ratio: In one-period portfolio optimization settings, the information ratio (IR) is simply defined as a Sharpe ratio calculated using the difference in returns of the active portfolio over the benchmark instead of just the returns of the active portfolio (see Treynor and Black (1973)). However, in dynamic (or multi-period) settings where the portfolio is rebalanced either continuously or at specific discrete time intervals, the IR is typically defined (see for example Bajeux-Besnainou et al. (2013)) as the ratio of the expectation to the standard deviation of the difference between the terminal wealth of the active portfolio and the terminal wealth of the benchmark portfolio. Since this paper is concerned with dynamic investment strategies for benchmark outperformance, we also make use of this latter definition.

It is widely acknowledged that the IR is immensely popular in investment practice when measuring benchmark outperformance and for purposes of performance comparisons between funds (Bajeux-Besnainou et al. (2013); Goetzmann et al. (2002); Zhao (2007)). Instead, IR maximization is often simply employed in single-period optimization settings (see for example Bolshakov and Chincarini (2020); Hassine and Roncalli (2013); Israelsen and Cogswell (2007)), despite concerns that it could be manipulated (Goetzmann et al. (2002, 2007)). Its popularity is underscored by the fact that it is included in standard (single-period) portfolio optimization routines used in the industry (Kopman and Liu (2009)).

However, deriving dynamic investment strategies aimed at implicitly or explicitly maximizing the IR have not received significant attention in the academic literature, with the exception of Bajeux-Besnainou et al. (2013); Goetzmann et al. (2002); Zhao (2007). Instead, IR maximization is often simply employed in single-period optimization settings (see for example Bolshakov and Chincarini (2020); Gaivoronski et al. (2005); Lai et al. (2011); Park et al. (2019); Roll (1992)).

Given these observations, our contributions in this paper are as follows:

- We formulate the investment benchmark outperformance problem as a stochastic optimal control problem and consider the two above-mentioned objectives, namely the IR and the tracking difference. While the IR objective is standard in the literature (see for example Bajeux-Besnainou et al. (2013)), we propose a novel and straightforward tracking difference objective, which involves the minimization of the quadratic deviation (QD) of the wealth of the active portfolio from the wealth of an elevated benchmark portfolio over a targeted time horizon. By formulating these objectives in terms of the wealth rather than for example a wealth ratio, our treatment explicitly considers the possibility of contributions to the portfolio, making the formulation potentially more attractive from the perspective of institutional investors and active portfolio managers.

- In order to gain a theoretical understanding of the behavior of the resulting optimal investment strategies, we first solve the problems analytically using standard assumptions, including parametric underlying dynamics, continuous portfolio rebalancing and no market frictions. Initially considering only two assets (a risky and a risk-free asset), we allow for any of the commonly-used jump diffusion models in finance to be used for the risky asset (Kou (2002); Merton (1976)), thereby extending the known results of Bajeux-Besnainou et al. (2013); Goetzmann et al. (2007); Zhao (2007) in the case of the IR objective to the setting where jumps are allowed in the risky asset process. All closed-form results associated with the QD (tracking difference) objective are novel. We also present closed-form comparison results regarding certain critical aspects of the IR- and QD-optimal investment strategies. The analytical results additionally illustrate that even in the simplest case of just two underlying assets, any proposed methodology to solve these problems numerically in a realistic setting involving discrete rebalancing would require solving a partial integro-differential equation (PIDE) in four dimensions. This implies that solving these problems numerically using standard techniques (for example using numerical solutions to the corresponding PIDEs) would be very challenging.

- Next, in order to gain practical insights into the behavior of the resulting optimal dynamic investment strategies in a more realistic setting, we solve the problems numerically using a data-driven neural network approach. The formulation of both problems are now expanded to include the following: (i) multiple underlying assets, including popular factors (value and size); (ii) periodic contributions to the portfolio; (iii) waiving the requirement to specify underlying parametric process dynamics, instead relying solely on empirical market data from 1963 until the end of 2020; (iv) imposing multiple realistic investment constraints. 

\[\text{In the case of discrete rebalancing with two assets, the benchmark and the investor strategy each would be associated with two SDEs, where each SDE gives the dynamics of the amount invested according to that strategy (investor or benchmark) in an asset. Solving one of the benchmark outperformance problems as outlined in Section 3 would therefore require the solution of a four-dimensional PIDE.}\]
constraints such as short-selling and leverage restrictions; and (v) rebalancing the portfolio at discrete
time intervals. We also consider a 10-year investment time horizon, ensuring that the conclusions are
relevant to the investor concerned with long-run benchmark outperformance.

- Comparing the results using IR- and QD-optimal investment strategies obtained numerically in our more
realistic setting, we show not only how the closed-form comparison results do apply qualitatively to in-
sample investment results, but also that the associated out-of-sample implications are often surprising.
In particular, while the IR-optimal strategy retains a slightly higher probability of benchmark outperfor-
mance in-sample, the higher portfolio diversification at certain points during the investment time period
associated with the QD-optimal strategy ultimately results in superior out-of-sample benchmark outper-
mance. Our results therefore highlight the potential practical limitations of using analytical solutions
available in the literature to reach qualitative conclusions regarding the behavior of optimal strategies in
practice. Furthermore, the results demonstrate the effectiveness of targeting benchmark outperformance
via the proposed QD objective.

The remainder of the paper is organized as follows. Section 2 presents the problem formulation. Section 3
and Section 4 discuss the analytical and numerical solutions of the problems, respectively. Finally, Section 6
concludes the paper and outlines possible future work.

2 Formulation

We start by formulating the problem of outperforming a given benchmark investment strategy in general terms.
No reference will be made to the portfolio rebalancing frequency or investment constraints, which will be
addressed in subsequent sections.

Let $T > 0$ denote the fixed investment time horizon/maturity of the active portfolio manager (henceforth
simply referred to as the “investor”), and $w_0 > 0$ the initial wealth of the investor at the start of the investment
period, time $t_0 \equiv 0$.

The investor’s controlled wealth process, with the control representing the investor’s investment strategy, is
denoted by $W(t), t \in [t_0, T]$. Similarly, given some benchmark investment strategy, the benchmark portfolio’s
controlled wealth process is denoted by $\hat{W}(t), t \in [t_0, T]$.

Assume that there are $N_a$ candidate assets available for investment. Let $\hat{p}_i(t, \hat{X}(t))$ denote the proportion
of the benchmark wealth $\hat{W}(t)$ invested in asset $i \in \{1, \ldots, N_a\}$ at time $t \in [t_0, T]$, where $\hat{X}(t)$ denotes the state
of the system (or informally, the information) taken into account by the benchmark entity/investor in order
to reach allocation decision $\hat{p}_i$. The vector $\hat{p}(t, \hat{X}(t)) = (\hat{p}_i(t, \hat{X}(t)) : i = 1, \ldots, N_a) \in \mathbb{R}^{N_a}$ denotes the asset
allocation of the benchmark at time $t \in [t_0, T]$.

While the benchmark strategy could of course be deterministic, in which case we would simply have the
function of time $t \rightarrow \hat{p}_i(t)$, we do not generally assume this to be the case when formulating and solving
the problem. For example, in Section 3 where we consider feedback controls in a Markovian setting, we have
$(t, \hat{W}(t)) \rightarrow \hat{p}_i(t, \hat{W}(t))$, so that $X(t) = \hat{W}(t)$. In even more generality, our numerical solution approach
discussed in Section 4 also allows for $\hat{X}(t)$ to contain additional information such as trading indicators.

Similarly, let $p_i(t, X(t))$ denote the proportion of the investor’s wealth $W(t)$ invested in asset $i \in \{1, \ldots, N_a\}$
at time $t \in [t_0, T]$, where $X(t)$ denotes the information taken into account by the investor in making the asset-allocation decision. As a concrete example, in Section 3 we consider the case where $X(t) = (W(t), \hat{W}(t), \hat{p}(t, \hat{X}(t)))$,
but more general cases incorporating additional information in $X(t)$ are also allowed in Section 4. The vector
$p(t, X(t)) = (p_i(t, X(t)) : i = 1, \ldots, N_a) \in \mathbb{R}^{N_a}$ denotes the asset allocation of the investor at time $t \in [t_0, T]$.

Define the set of rebalancing events $\mathcal{T} \subseteq [t_0, T]$, where we have $\mathcal{T} = [t_0, T]$ in the case of continuous
rebalancing, and a strict (discrete) subset $\hat{\mathcal{T}} \subset [t_0, T]$ in the case of discrete rebalancing. The investor and
benchmark investment strategies over the time horizon $[t_0, T]$ respectively, are then defined as the sets

$$
\mathcal{P} = \{p(t, X(t)) = (p_i(t, X(t)) : i = 1, \ldots, N_a) : t \in \mathcal{T}\},
$$

(2.1)

$$
\hat{\mathcal{P}} = \{\hat{p}(t, \hat{X}(t)) = (\hat{p}_i(t, \hat{X}(t)) : i = 1, \ldots, N_a) : t \in \hat{\mathcal{T}}\}.
$$

(2.2)

At this point, we make no specific assumptions regarding the investment constraints underlying $\mathcal{P}$ or $\hat{\mathcal{P}}$. However,
in general terms, for the investor’s strategy $\mathcal{P}$, we define $\mathcal{A}$ as the set of admissible controls, and $\mathcal{Z}$ as the
set of admissible values of each vector \( p(t, X(t)) \), so that \( P \in A \) if and only if \( P = \{ p(t, X(t)) \in \mathcal{Z} : t \in \mathcal{T} \} \).

Note that \( \mathcal{Z} \), and therefore by extension \( A \), encode the investment constraints faced by the investor, such as leverage constraints or short-selling restrictions.

The following assumption, applicable throughout this paper, clarifies the information regarding the benchmark strategy that is assumed to be known by the investor.

**Assumption 2.1.** *(Benchmark and investor strategies)* We assume that the investor is given the benchmark investment strategy \( \hat{P} \) as per \( [2.2] \). Implicitly, we also assume that the investor has access to at least all of the assets underlying the benchmark strategy. However, neither the investor nor the benchmark strategy is required to have a nonzero investment in all \( N_a \) candidate assets; for example, the benchmark may have \( \hat{p}_i(t, X(t)) = 0 \) for some candidate asset \( i \) at some (or all) rebalancing times \( t \in \mathcal{T} \). The investor may also apply different investment constraints than the benchmark. For convenience, the time-\( t_0 \) wealth invested in both the benchmark and investor portfolio is assumed to be \( w_0 = W(t_0) = \hat{W}(t_0) \).

Note that Assumption 2.1 is clearly stronger than just knowing the performance of the benchmark, since it requires the specification of the full benchmark asset allocation \( \hat{P} \), which is a standard assumption in the literature (see for example Bessler et al. (2017); Browne (2000); Davis and Lleo (2008); Kim and Kim (2020); Kwon and Wu (2017); Li et al. (2021); Yao et al. (2006)).

Considering concrete examples of benchmarks actually used in the literature and in practice (Alekseev and Sokolov (2016); Basak et al. (2006); Bo et al. (2021); Israelov and Tummala (2018); Zhao (2007)), Assumption 2.1 appears to be sufficiently general. For example, constant proportion strategies combining stocks and bonds in some fixed proportions, a special case of which is the investor simply trying to “beat the market” (i.e. the benchmark strategy has all wealth invested in a broad market index), all satisfy Assumption 2.1. Other pre-specified investment strategies such as glide path strategies (discussed in for example Forsyth and Vetzal (2019)) also satisfy Assumption 2.1 provided they make use of investable/tradable assets available to the investor.

Of course, Assumption 2.1 precludes the use of an arbitrary benchmark, such as for example when using the returns of a proprietary dynamic hedge fund strategy as a benchmark. This may initially appear disappointing since portfolio managers or institutional investors might also wish to outperform their peers or competing institutions (Simões et al. (2018)). However, a review of the returns of the S&P 500 and that of the HFRI Equity Hedge Fund (Total) Index shows that recent return correlations exceed 90% (Simões et al. (2018)). In addition, given the evidence presented in the Introduction regarding the historical performance of active portfolio managers, simply outperforming their respective simple index benchmarks would already place the investor (or institution) in the very top tier of their peers.

We highlight the following regarding the use of Assumption 2.1 in the analytical and numerical solutions, respectively, of the benchmark outperformance problems:

(i) In the case of the analytical solutions (Section 3 below), Assumption 2.1 is necessarily required, since we will see that the investor’s optimal outperformance strategy is defined in terms of the (known) benchmark strategy.

(ii) In the case of the numerical solutions (discussed in Section 4 below), Assumption 2.1 is not strictly necessary, since it is possible for the neural network solution approach to learn an outperformance strategy based purely on the historical performance of the benchmark (see Remark 4.2). However, for convenience and purposes of comparison with the corresponding analytical solutions, we continue to work under Assumption 2.1 in the case of the numerical results in Section 5 below.

Since the investor wishes to outperform the benchmark in some sense, we introduce two practical investment objectives to achieve this aim in the following subsections. In terms of notation, let \( E^P_{t_0, w_0} [\cdot] \) denote the expectation of some quantity taken with respect to a given initial wealth \( w_0 = W(t_0) = \hat{W}(t_0) \) at time \( t_0 = 0 \), and using control \( P \in A \) over \( [t_0, T] \). The benchmark strategy \( \hat{P} \), which is fixed and given as per Assumption 2.1, remains implicit in this notation. Similarly, we will use \( Stdev^P_{t_0, w_0} [\cdot], Var^P_{t_0, w_0} [\cdot], Cov^P_{t_0, w_0} [\cdot] \text{ and } \rho^P_{t_0, w_0} [\cdot] \) to denote the standard deviation, variance, covariance and probability, respectively, calculated under the control \( P \) and initial time and wealth given by \( (t_0, w_0) \).

However, it is worth highlighting that in the Markovian setting of Section 3 we show that the investor’s optimal allocation at a specific time \( t \in \mathcal{T} \) depends only on the benchmark allocation at that same time instant \( \hat{p}(t, X(t)) = \hat{p}(t, W(t)) \in \hat{P} \), and not on the future or past benchmark asset allocations \( \hat{p}(\hat{t}, W(\hat{t})) \), \( \hat{t} \neq t \). This implies that \( \hat{P} \) may not require full specification at time \( t_0 = 0 \) for all \( t \in \mathcal{T} \), and thus that the requirements of Assumption 2.1 could be relaxed in certain settings.
2.1 Information ratio: Problem $IR (\gamma)$

The first investment objective we consider involves maximizing the information ratio (IR), which in a dynamic setting is defined as \cite{Bajeux-Besnainouetal2013,Goetzmannetal2002}

$$IR_{P}^{t_0,w_0} = \frac{E_{P}^{t_0,w_0}[W(T) - \hat{W}(T)]}{\text{Stdev}_{P}^{t_0,w_0}[W(T) - \hat{W}(T)]}.$$ (2.3)

As discussed in \cite{Bajeux-Besnainouetal2013}, maximizing the IR (2.3) is achieved by solving the following mean-variance (MV) optimization problem with scalarization parameter $\rho$,

$$\sup_{P \in A} \left\{ E_{P}^{t_0,w_0}[W(T) - \hat{W}(T)] - \rho \cdot \text{Var}_{P}^{t_0,w_0}[W(T) - \hat{W}(T)] \right\}, \quad \rho > 0,$$ (2.4)

where $\rho$ effectively encodes the “risk appetite” of the investor for outperforming the benchmark.

In order to solve problem (2.4), we use the embedding technique of \cite{LiNg2000,ZhouLi2000}, which states that for any $\rho > 0$ and the associated control $P_{\gamma}^{\gamma} \in A$ maximizing (2.4), there exists a value of an embedding parameter $\gamma$ such that $P_{\gamma}^{\gamma} \in A$ is also optimal for the following problem,

$$(IR (\gamma)) : \inf_{P \in A} \left\{ E_{P}^{t_0,w_0} \left[ (W(T) - \hat{W}(T) + \gamma)^2 \right] \right\}, \quad \gamma > 0.$$ (2.5)

Note that (2.5) is formulated here only for the range $\gamma > 0$ in order to ensure that economically meaningful strategies for benchmark outperformance are obtained.

Associated with each value of $\gamma$ in (2.5), we therefore obtain an investment strategy that maximizes the IR (2.3) for some value of the risk appetite for benchmark outperformance that is now encoded by $\gamma$. As a result, we will subsequently refer to problem (2.5) simply as the IR (maximization) problem, abbreviated by $IR (\gamma)$. The exact relationship between $\gamma$ in (2.5) and $\rho$ in (2.4) is not important for the purposes of this paper, and it is indeed also of limited practical significance to the investor. For further clarification, the following remark highlights some practical aspects of our preference for formulation (2.5).

**Remark 2.1.** (Time-consistency of the $IR (\gamma)$-optimal control) As elaborated in \cite{Forsythetal2019,LiForsyth2019}, there appears to be some confusion in the literature regarding the time-consistency (or lack thereof) of the optimal controls associated with problems of the form (2.5). By analogy with dynamic MV optimization (see \cite{BasakChabakauri2010,BjorkMurgoci2014}), the IR-optimal control for the embedding problem (2.5) is typically time-inconsistent from the perspective of the MV formulation (2.4). This raises practical concerns as to whether the resulting IR-optimal control is in fact feasible to implement as a trading strategy. However, it should be emphasized that time-consistency is ultimately a matter of perspective, since for a fixed value of $\gamma$ in (2.5), the resulting $IR (\gamma)$-optimal control is in fact a time-consistent control from the perspective of the quadratic objective (2.5), and is therefore clearly feasible as a trading strategy \cite{Strubetal2019}. As discussed in \cite{Vigna2014} and elaborated further below, a quadratic objective such as (2.5) also allows for a straightforward interpretation in terms of a “target” (in this case, $\hat{W}(T) + \gamma$), whereas very little guidance is offered in the literature regarding the selection of $\rho$ in the case of a MV-type objectives of the form (2.4). As a result, in this paper we always view the IR-optimal control as the time-consistent investment strategy that minimizes the induced objective function (2.5), and correspondingly formulate our results in terms of the embedding parameter $\gamma$.

In contrast to the objective introduced in the next section, we emphasize that the preceding IR-related objective functions are not novel. In fact, the MV formulation (2.4) and its associated embedding formulation (2.5) clearly present reasonable objectives for benchmark outperformance in their own right, and are often used in both single-period and dynamic settings without necessarily recognizing the connections with IR maximization (see for example \cite{Gaivoronskietal2005,Jorion2003,PerrinRoncalli2020,Roll1992,Zhao2007}). Here, we explicitly highlight the link to IR maximization due to the widespread popularity of using the IR for outperformance measurement \cite{Bajeux-Besnainouetal2013,BolshakovChincarini2020,HassineRoncalli2013,IsraelsenCogswell2007} as noted in the Introduction.

The following additional observations regarding the IR objective (2.5) are relevant to the subsequent results:

(i) Since the embedding parameter $\gamma$ is a constant in problem $IR (\gamma)$, one way of interpreting (2.5) is that the investor wishes, where possible, to outperform the benchmark terminal wealth $\hat{W}(T)$ by a constant...
amount $\gamma$, regardless of the market conditions prevailing over $[t_0, T]$. The investor wishing to maximize the IR therefore effectively sets an elevated benchmark terminal wealth value, $\hat{W}(T) + \gamma$, and minimizes the (expected) quadratic deviation of the investor’s wealth $W(T)$ from this elevated target.

(ii) Re-arranging the IR problem (2.5), and using the assumption that the benchmark strategy is exogenously given (Assumption 2.1), we observe that problem $IR(\gamma)$ is equivalent to

$$\inf_{P \in A} \left\{ E^{t_0,w_0}_P \left[ (W(T) - \hat{W}(T))^2 \right] - 2\gamma \cdot E^{t_0,w_0}_P [W(T)] \right\}, \quad \gamma > 0. \quad (2.6)$$

In other words, the IR problem involves a trade-off between maximizing the performance of the investor’s portfolio as measured by $E^{t_0,w_0}_P [W(T)]$, while simultaneously minimizing the quadratic deviation of the investor’s wealth from the terminal wealth of the benchmark, $E^{t_0,w_0}_P \left[ (W(T) - \hat{W}(T))^2 \right]$, with the embedding parameter $\gamma$ encoding the relative weighting of these objectives.

(iii) In Section 3 below, we show that under some conditions, the IR problem (2.5) is equivalent to the more intuitive one-sided quadratic objective,

$$\inf_{P \in A} E^{t_0,w_0}_P \left[ \left( \min \{ W(T) - [\hat{W}(T) + \gamma] , 0 \} \right)^2 \right], \quad \gamma > 0, \quad (2.7)$$

where only the shortfall of $W(T)$ below the elevated target $\hat{W}(T) + \gamma$ is penalized. While the equivalence between (2.5) and (2.7) can only be proven analytically under certain assumptions, numerical results nevertheless suggest that the results using (2.5) and (2.7) are indistinguishable even in more general cases where the conditions for analytical equivalence do not hold.

With these observations in mind, we now consider our second objective for outperforming the benchmark.

### 2.2 Tracking difference: Problem $QD(\beta)$

As discussed in the Introduction, the tracking difference measures the cumulative performance gap between the investor’s portfolio and the benchmark portfolio over the time horizon $[t_0, T]$ (Charteris and McCullough 2020).

In a dynamic setting, we propose the following straightforward objective function based on minimizing the quadratic deviation (QD) of the investor’s terminal wealth from the terminal wealth of an elevated benchmark,

$$(QD(\beta)) : \inf_{P \in A} E^{t_0,w_0}_P \left[ (W(T) - e^\beta T \hat{W}(T))^2 \right], \quad \beta > 0. \quad (2.8)$$

We will subsequently refer to problem (2.8) as the QD problem, and we make the following observations:

(i) The strength of the formulation (2.8) lies in its simplicity, since the objective of obtaining a favorable tracking difference, widely publicized as a quantity of key interest to investors and regulators alike (Boyd 2021; Charteris and McCullough 2020; ESMA 2014; ETF.com 2021; Hougan 2015; Johnson et al. 2013; Pastant 2018; Vanguard 2014) is the central object of consideration. The simplicity of the objective function, however, does not imply a lack of sophistication of the resulting QD-optimal investment strategies, as will be evident from the results the subsequent sections.

(ii) The parameter $\beta$ in the QD problem (2.8) has a conveniently practical interpretation as the annual outperformance spread that the investor targets for the tracking difference of the active portfolio. In particular, if the investor’s strategy achieves a terminal wealth of $W(T) = e^{\beta T} \hat{W}(T)$ under some ideal scenario, the investor has achieved an (average) annual outperformance return spread $\beta$ over the benchmark.

(iii) By formulating (2.8) in terms of wealth, not only do we respect the cumulative aspect of the definition of the tracking difference, but the formulation also allows for the treatment of contributions to and withdrawals from the portfolio without difficulty (see Sections 3 and 4).

(iv) Like the IR problem (2.5), the QD problem (2.8) also formulates the outperformance objective in terms of an elevated benchmark terminal wealth value. However, in the case of the QD problem, the elevation
is applied to $\hat{W}(T)$ by the multiplicative scaling factor $e^{\beta T}$, in contrast to the IR problem where the elevation is additive (i.e. by adding a constant $\gamma$ to $\hat{W}(T)$ in (2.5)). The investor using the QD objective therefore wishes, where possible, to outperform the benchmark terminal wealth by a constant factor, and not by a constant amount as in the case of the IR problem. The implications of this observation on the resulting optimal investment strategies will be discussed in more detail in the subsequent sections.

(v) With the alternative formulation (2.6) of the IR problem in mind, we observe that the QD problem is equivalent to the following problem,

$$
\inf_{P \in A} \left\{ E_{P}^{\alpha, w_{0}} \left[ (W(T) - \hat{W}(T))^2 \right] - 2\hat{\gamma} \cdot E_{P}^{\alpha, w_{0}} \left[ W(T) \cdot \hat{W}(T) \right] \right\}, \quad \hat{\gamma} \equiv (e^{\beta T} - 1), \quad \beta > 0. \quad (2.9)
$$

We therefore observe that the QD objective (2.9), like the IR objective (see (2.6)), involves minimizing the quadratic deviation of the investor’s wealth from the terminal wealth of the benchmark, $E_{P}^{\alpha, w_{0}} \left[ (W(T) - \hat{W}(T))^2 \right]$. However, unlike in the case of the IR objective where this term is only combined with the simultaneous goal of maximizing the investor’s expected wealth $E_{P}^{\alpha, w_{0}} [W(T)]$, the QD objective combines this term with maximizing $E_{P}^{\alpha, w_{0}} [W(T) \cdot \hat{W}(T)]$. Of course, all else being equal, this still implies that the QD investor retains an incentive obtain higher values of $W(T)$. However, through the emphasis on covariance implied by (2.9), the role of maximizing the investor’s expected wealth is more nuanced than in the case of the IR problem (2.6).

(vi) As in the case of the IR problem (see (2.7)), we show in Section 3 that under some conditions, the QD problem (2.8) also admits the equivalent, and perhaps more intuitive, one-sided quadratic formulation,

$$
\inf_{P \in A} E_{P}^{\alpha, w_{0}} \left[ \min \left\{ W(T) - e^{\beta T} \hat{W}(T), 0 \right\} \right]^2, \quad \beta > 0, \quad (2.10)
$$

where only underperformance relative to the elevated benchmark $e^{\beta T} \hat{W}(T)$ is penalized.

In summary, we have therefore discussed two fundamentally different, yet practical and easily-understood investment objectives for outperforming a given benchmark. The following sections are devoted to explore the practical implications of the preceding observations for the investor, using both closed-form solutions (where available) and numerical solutions of the IR and QD problems.

3 Analytical (closed-form) solutions

In this section, we investigate the analytical or closed-form solutions of the IR and QD problems introduced in Section 2. Of course, analytical solutions require the standard assumptions used in the literature, such as no market frictions and no investment constraints. For simplicity, in this section only, we also limit our attention to the case of two assets ($N_{a} = 2$), namely a well-diversified stock index (the risky asset) and a risk-free asset evolving according to specified dynamics. In particular, we also allow for the modelling of jumps in the risky asset process, in contrast to the existing literature on benchmark outperformance (see for example Al-Aradi and Jaimungal (2018); Basak et al. (2006); Browne (1999a,b) (2000); Davis and Lleo (2008); Lim and Wong (2010); Oderda (2015); Tepla (2001); Yao et al. (2006); Zhang and Gao (2017); Zhao (2007)). However, we emphasize that these assumptions, including any assumptions regarding the form of the underlying dynamics, are all relaxed in Section 2.

In our analytical solutions, we explicitly allow for contributions to the portfolio, which only receives very limited treatment in the existing benchmark outperformance literature (Bo et al. (2021); Nicolosi et al. (2018)), while leaving further generalization of the results to the inclusion of withdrawals from the portfolio for our future work.

Assumption 3.1. (No market frictions, continuous rebalancing) We assume that trading continues in the event of insolvency; in other words, trading continues even if $W(t) < 0$ or $\hat{W}(t) < 0$ for some $t \in [t_{0}, T]$. No transaction costs are applicable, and no investment constraints (such as leverage or short-selling restrictions) are in effect. In addition, the portfolio is rebalanced continuously, and the investor contributes cash at a constant rate of $q \geq 0$ per year.
Since we make use of only two assets in this section, with slight abuse of notation, \( p \left( t, X(t) \right) \) and \( \hat{p} \left( t, \hat{X}(t) \right) \) will denote the proportions of wealth in the risky asset for the investor and benchmark strategies, respectively, at time \( t \in [t_0, T] \). Considering definitions \( 3.1 \) and \( 3.2 \), for the purposes of this section we will use the definitions

\[
\mathcal{P} = \left\{ \left( p \left( t, X(t) \right), 1 - p \left( t, X(t) \right) \right) : t \in [t_0, T] \right\}, \quad \text{where} \quad X(t) = \left( W(t), \hat{W}(t), \hat{p} \left( t, \hat{W}(t) \right) \right),
\]

\[
\hat{\mathcal{P}} = \left\{ \left( \hat{p} \left( t, \hat{X}(t) \right), 1 - \hat{p} \left( t, \hat{X}(t) \right) \right) : t \in [t_0, T] \right\}, \quad \text{where} \quad \hat{X}(t) = \hat{W}(t).
\]

Due to the form of \( 3.1 \), in this section we will informally refer to \( p \left( t, X(t) \right) \) and \( \hat{p} \left( t, \hat{X}(t) \right) \) as the investor and benchmark investment strategies, respectively, although the proper definition \( 3.1 \) should be kept in mind when extending the analysis to the multi-asset case presented in Section 4.

In more detail, with reference to Assumption 2.1, we assume that the investor is given \( \hat{p} \left( t, \hat{X}(t) \right) = \hat{p} \left( t, \hat{W}(t) \right), t \in [t_0, T] \), the adapted feedback control representing the fraction of the benchmark strategy wealth \( \hat{W}(t) \) invested in the risky asset at time \( t \). The investor wishes to derive \( p \left( t, X(t) \right) = p \left( t, W(t), \hat{W}(t), \hat{p} \left( t, \hat{W}(t) \right) \right) \) for \( t \in [t_0, T] \), the adapted feedback control representing the fraction of the investor’s wealth \( W(t) \) invested in the risky asset at time \( t \) according to the investor’s strategy, given benchmark strategy \( \hat{p} \left( t, \hat{W}(t) \right) \) and benchmark wealth \( \hat{W}(t) \). Given Assumption 3.1, the investor’s set of admissible controls in this section is therefore given by

\[
A_0 = \{ p \left( t, x \right) = p \left( t, w, \hat{w}, \hat{p} \right) : \mathbb{R}^3 \to \mathbb{R}, t \in [t_0, T] \}.
\]

With \( r > 0 \) denoting the continuously compounded risk-free rate, let \( B(t) \) denote the unit value of the risk-free asset at time \( t \in [t_0, T] \), with dynamics given by

\[
dB(t) = rB(t) dt. \tag{3.3}
\]

Let \( S(t) \) denote the unit value of the risky asset at time \( t \in [t_0, T] \). Since realistic modelling of \( S(t) \) requires the consideration of jumps in the risky asset process dynamics \( \left( \text{Cont and Tankov} \ 2004 \right) \), and we wish to allow for any of the commonly-encountered jump-diffusion models in finance \( \left( \text{see for example} \ \text{Kou} \ 2002; \text{Merton} \ 1976 \right) \), let \( \xi \) be a random variable denoting the jump multiplier with associated probability density function \( \text{(pdf)} f_\xi (\xi) \). For subsequent reference, we also define the following expectations associated with \( \xi \),

\[
\kappa = \mathbb{E} [\xi - 1], \quad \kappa_2 = \mathbb{E} \left[ (\xi - 1)^2 \right]. \tag{3.4}
\]

If a jump occurs at time \( t \), the risky asset unit value jumps from \( S(t^-) \) to \( S(t) = \xi S(t^-) \), where, given any functional \( \psi (t), t \in [t_0, T] \), we use the notation \( \psi (t^-) \) and \( \psi (t^+) \) as shorthand for the one-sided limits \( \psi (t^-) = \lim_{t \downarrow 0} \psi (t - \epsilon) \) and \( \psi (t^+) = \lim_{t \downarrow 0} \psi (t + \epsilon) \), respectively. The dynamics of \( S(t) \) is therefore assumed to be of the form

\[
\frac{dS(t)}{S(t^-)} = \left( \mu - \lambda \kappa \right) dt + \sigma dZ(t) + d \left( \sum_{i=1}^{\pi(t)} (\xi_i - 1) \right), \tag{3.5}
\]

where \( \mu \) and \( \sigma \) are the real world drift and volatility respectively, \( Z(t) \) denotes a standard Brownian motion, \( \pi(t) \) is a Poisson process with intensity \( \lambda \geq 0 \), and \( \xi_i \) are i.i.d. random variables with the same distribution as \( \xi \). We make the standard assumption that \( \mu > r \). In addition, we assume that \( \xi_i \), \( \pi(t) \) and \( Z(t) \) are mutually independent. Note that we can recover geometric Brownian motion (GBM) dynamics for \( S(t) \) by setting the intensity parameter \( \lambda \) to zero in \( 3.5 \). For subsequent reference, we also define the following combinations of parameters from the underlying asset dynamics,

\[
\eta = \frac{(\mu - r)^2}{\sigma^2 + \lambda \kappa_2}, \quad \alpha = \mu - \lambda \kappa - r. \tag{3.6}
\]

For investor and benchmark strategies \( \mathcal{P} \) and \( \hat{\mathcal{P}} \) as per \( 3.1 \) together with the dynamics of the underlying assets \( 3.3 \) and \( 3.5 \), the investor and benchmark controlled wealth processes therefore have the following
dynamics for $t \in (t_0, T]$, respectively,

$$
dW(t) = \left\{ W(t^-) \left[ r + \alpha \cdot p(t, X(t)) + q \right] dt + p(t, X(t)) \cdot W(t^-) \sigma dZ(t) + p(t, X(t)) \cdot \nabla V(t^-) \cdot d\pi(t) \right\},
$$

$$
d\hat{W}(t) = \left\{ \hat{W}(t^-) \left[ r + \alpha \cdot \hat{p}(t, \hat{W}(t)) + q \right] dt + \hat{p}(t, \hat{W}(t)) \cdot \hat{W}(t^-) \sigma d\hat{Z}(t) + \hat{p}(t, \hat{W}(t)) \cdot \nabla \hat{V}(t^-) \cdot d\hat{\pi}(t) \right\},
$$

(3.7) with $W(t) = \hat{W}(t) = w_0$. In dynamics (3.7), we highlight the use of the shorthand notation $X(t) = (W(t), \hat{W}(t), \hat{p}(t, \hat{W}(t)))$.

In the following subsections, we derive and compare the closed-form solutions to the IR and QD problems subject to Assumption 3.1 and wealth dynamics (3.7)-(3.8).

### 3.1 Analytical solution: IR ($\gamma$) problem

We have the following verification theorem and corresponding Hamilton-Jacobi-Bellman (HJB) equation for the IR problem (2.5).

**Theorem 3.2.** (IR problem: Verification theorem) Suppose that for all $(y, t) \in \mathbb{R} \times [t_0, T]$, there exist real-valued functions $V_{ir}(y, t)$ and $u^*_{ir}(y, t)$ with the following properties: (i) $V_{ir}$ and $u^*_{ir}$ are sufficiently smooth and solve the HJB partial integro-differential equation (PIDE) (3.9)-(3.10), and (ii) the function $u^*_{ir}$ attains the pointwise supremum in $\mathbb{R}$.

$$
\frac{\partial V_{ir}}{\partial t} + \inf_{u \in \mathbb{R}} \left\{ (r y + \alpha u) \frac{\partial V_{ir}}{\partial y} + \frac{1}{2} \sigma^2 \frac{\partial^2 V_{ir}}{\partial y^2} - \lambda V_{ir} + \lambda \int_0^\infty V_{ir}(y + u(\xi - 1), t) f_\xi(\xi) \, d\xi \right\} = 0, \quad (3.9)
$$

$$
V_{ir}(y, T) = (y - \gamma)^2. \quad (3.10)
$$

Define the auxiliary process $Y_{ir}(t)$ by

$$
Y_{ir}(t) := W(t) - \hat{W}(t), \quad \forall t \in (t_0, T], \quad \text{with} \quad Y_{ir}(t_0) = y_0 = 0. \quad (3.11)
$$

Let the auxiliary control $u(t) := u(Y_{ir}(t), t) := u(Y_{ir}(t), t; X(t))$ be given by

$$
u(t) := p(t, X(t)) \cdot W(t) - \hat{p}(t, \hat{W}(t)) \cdot \hat{W}(t), \quad \text{where} \quad X(t) = (W(t), \hat{W}(t), \hat{p}(t, \hat{W}(t))). \quad (3.12)
$$

Let $\mathcal{A}_{u,0} = \{ (u(t) = u(y, t)) | u : \mathbb{R} \times [t_0, T] \to \mathbb{R} \}$. Then under Assumption 3.1 and wealth dynamics (3.7)-(3.8), $V_{ir}$ is the value function and $u^*_{ir}$ is the optimal control for the following control problem,

$$
\inf_{u \in \mathcal{A}_{u,0}} \mathbb{E}^{t_0, y_0}_{u} \left[ (Y_{ir}(T) - \gamma)^2 \right], \quad \gamma > 0. \quad (3.13)
$$

**Proof.** Since the proof applies standard techniques (see for example Applebaum (2004); Øksendal and Sulem (2005)) to the analysis of problem (3.13), the details are omitted. For subsequent reference, we note that the auxiliary process $Y_{ir}(t)$ in (3.11) has the following dynamics in terms of auxiliary control $u(t)$ in (3.12),

$$
dY_{ir}(t) = \left[ r Y_{ir}(t) + (\alpha + \lambda \kappa) u(t) \right] dt + u(t) \sigma dZ(t) + u(t^-) \int_0^\infty (\xi - 1) \hat{N}(dt, d\xi), \quad (3.14)
$$

where $\hat{N}$ is the compensated Poisson random measure associated with the $S$-dynamics (Øksendal and Sulem (2005)).

By solving the HJB PIDE (3.9)-(3.10), the following lemma reports the IR-optimal investment strategy.

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Lemma 3.3. (IR-optimal investment strategy) Suppose that Assumption 3.1 and wealth dynamics 3.7-3.8 are applicable. Then the optimal fraction of the investor’s wealth to be invested in the risky asset for problem \( IR(\gamma) \) in 2.3) is given by \( p_{ir}^*(t, X_{ir}^*(t)) \), where

\[
p_{ir}^*(t, X_{ir}^*(t)) \cdot W_{ir}^*(t) = \frac{\mu - r}{\sigma^2 + \lambda \kappa_2} \left[ \gamma e^{-r(T-t)} - (W_{ir}^*(t) - \bar{W}(t)) \right] + \hat{p}(t, \hat{W}(t)) \cdot \hat{W}(t),
\]

with \( W_{ir}^*(t) \) denoting the investor’s wealth process 3.7 under the IR-optimal control \( p_{ir}^* \), and \( X_{ir}^*(t) = \left( W_{ir}^*(t), \hat{W}(t), \hat{p}(t, \hat{W}(t)) \right) \).

Proof. Considering the form of terminal condition 3.10, we make the ansatz that \( V_{ir}(x, t) \) is of the form \( V_{ir}(x, t) = A_{ir}(t) x^2 + B_{ir}(t) x + C_{ir}(t) \) for unknown functions of time \( A_{ir}, B_{ir} \) and \( C_{ir} \). If this is indeed the case, then the pointwise supremum in (3.9) is attained by the auxiliary control \( u_{ir}^*(t) \), where

\[
u_{ir}^*(t) = p_{ir}^*(t, X_{ir}^*(t)) \cdot W_{ir}^*(t) - \hat{p}(t, \hat{W}(t)) \cdot \hat{W}(t) = \frac{\mu - r}{\sigma^2 + \lambda \kappa_2} \left[ x + \frac{B_{ir}(t)}{2A_{ir}(t)} \right].
\]

The substitution of \( V_{ir} \) and \( u_{ir}^* \) into (3.9)-(3.10) yields three ordinary differential equations (ODEs) for \( A_{ir}, B_{ir} \) and \( C_{ir} \). Solving these equations to obtain \( A_{ir}(t) = e^{(2r - \eta)(T-t)} \) and \( B_{ir}(t) = -2\gamma e^{(r-\eta)(T-t)} \), subsequent substitution into (3.16) and simplification results in (3.15). □

Lemma 3.3 shows that under the stated assumptions, the IR-optimal control \( p_{ir}^*(t, X_{ir}^*(t)) \) only depends on the instantaneous benchmark allocation \( \hat{p}(t, \hat{W}(t)) \), and not on the future or the past of the benchmark investment strategy. It is also worth noting that the contribution rate \( q \) does not appear in the solution (3.15), which follows from the fortunate cancellation of terms in the auxiliary process X 3.14.

The following lemma reports the optimal IR that the investor can attain by implementing the IR-optimal control \( p_{ir}^* \).

Lemma 3.4. (IR: moments of \( W_{ir}^*(T) - \bar{W}(T) \) and optimal IR) Assume that Assumption 3.1 and wealth dynamics 3.7-3.8 hold, with \( t_0 = 0 \). Implementing the IR-optimal control 3.13 gives

\[
E_{p_{ir}^*}^{t_0, w_0} \left[ W_{ir}^*(T) - \bar{W}(T) \right] = \gamma \left( 1 - e^{-\eta T} \right), \quad \text{Var}_{p_{ir}^*}^{t_0, w_0} \left[ W_{ir}^*(T) - \bar{W}(T) \right] = \gamma^2 e^{-2\eta T} (e^{\eta T} - 1),
\]

so that the IR investor obtains an optimal information ratio (2.9) of

\[
IR_{p_{ir}^*}^{t_0, w_0} = (e^{\eta T} - 1)^{1/2},
\]

where \( \eta \) is given by 3.6.

Proof. After substituting (3.15) in 3.14 to obtain the dynamics of \( Y_{ir}^*(t) := W_{ir}^*(t) - \bar{W}(t) \), standard techniques (Oksendal and Sulem 2005) give results (3.17), so that (3.18) follows from the definition (2.3). □

Lemma 3.4 extends the known information ratio results of Goetzmann et al. (2007) to the case of jumps in the risky asset process. Specifically, in the case of no jumps (i.e. setting \( \lambda = 0 \), we simply have \( \eta = (\mu - r)^2 / \sigma^2 \), in which case the optimal IR (3.18) reduces to the result reported in Goetzmann et al. (2002, 2007).

The following lemma presents an important property of the IR-optimal strategy 3.15, whereby the IR-optimal investor will simply match the benchmark in terms of the amount invested in the risky asset once sufficient outperformance can be assured.

Lemma 3.5. (IR: Matching the benchmark risky asset amount) Given Assumption 3.1 and wealth dynamics 3.7-3.8, suppose that at some time \( T \in (t_0, T] \), the IR-optimal investor observes a wealth value \( W_{ir}^*(T) \) of

\[
W_{ir}^*(T) = \gamma e^{-r(T-T)} + \bar{W}(T).
\]

Then for the remainder of the investment time horizon \( t \in [T, T] \), the IR-optimal investor (using strategy 3.13) will simply match the benchmark strategy in terms of the amount invested in the risky asset; in other words,

\[
p_{ir}^*(t, X_{ir}(t)) \cdot W_{ir}^*(t) = \hat{p}(t, \hat{W}(t)) \cdot \hat{W}(t), \quad \forall t \in [T, T].
\]

(3.20)
Lemma 3.6. (IR: equivalence with only penalizing underperformance) If Assumption 3.1 and wealth dynamics (3.7)-(3.8) apply with no jumps (i.e. \( \lambda = 0 \) in the risky asset process (3.3)), then

\[
W^*_{ir} (t) < \gamma e^{-(T-t)} + \hat{W} (t), \quad \forall t \in [t_0,T].
\]

As a result, the IR optimization problem (2.5) is equivalent to the one-sided quadratic problem (2.4), where only the underperformance of the investor’s portfolio (compared to the elevated benchmark) is penalized.

Proof. The equivalence assertion follows from the results of Di Giacinto et al. (2010), provided that (3.22) holds. Since in the case of no jumps, \( Q^*_{ir} (t) \) in (3.21) is a GBM with initial value \( Q^*_{ir} (t_0) = \gamma e^{-(T-t_0)} > 0 \), we have \( Q^*_{ir} (t) > 0 \) for all \( t \in (t_0,T) \), which is (3.22).

In general, even if the assumptions of this section are violated, it should be emphasized that both Lemma 3.5 and the more restrictive Lemma 3.6 provide very valuable intuition for understanding the behavior of the IR-optimal investment strategies presented in Section 5.

The following lemma shows that if we apply the assumption of no jumps in Lemma 3.6, then the probability of the IR investor underperforming the benchmark admits a simple analytical expression. Note that we prefer formulating the result in the negative sense of underperformance, since it directly expresses a key quantity of concern for the active investor.

Lemma 3.7. (IR: probability of underperformance) If Assumption 3.1 and wealth dynamics (3.7)-(3.8) apply with no jumps (i.e. \( \lambda = 0 \) in the risky asset process (3.3)), the probability of the IR-optimal wealth falling below the benchmark wealth at any \( t \in (t_0,T) \) is given by

\[
P^{0,\text{no}}_{P^*_{ir}} \left( W^*_{ir} (t) \leq \hat{W} (t) \right) = \Phi \left( -\frac{3}{2} \sqrt{\eta t} \right), \quad \forall t \in (t_0,T],
\]

where \( \Phi \) denotes the standard normal cumulative distribution function (CDF), and \( \eta \) is given by (3.6).

Proof. As noted in the proof of Lemma 3.6, in the case of no jumps, \( Q^*_{ir} (t) \) (3.21) is a GBM, with (3.23) being equal to \( P^{0,\text{no}}_{P^*_{ir}} \left[ Q^*_{ir} (t) \geq \gamma e^{-(T-t)} \right] \).

While the result of Lemma 3.7 does not depend on contributions to the portfolio (perhaps as expected, given the preceding results such as (3.15)), it is nevertheless remarkable that under the stated assumptions, the IR-optimal probability of underperformance (3.23) does not depend on the value of \( \gamma \) nor on the specific form of the benchmark strategy \( \hat{W} \). Even if the assumptions of Lemma 3.7 are violated, for example when multiple realistic investment constraints are applicable and the investor trades in multiple assets, numerical results in Section 5 show that the dependence of the IR-optimal probability of underperformance on the value of \( \gamma \) remains very weak.

---

5The proof of Lemma 3.6 uses the results of Di Giacinto et al. (2010), which hold only when there are no jumps in the risky asset process. However, even in the case where there are jumps, the behavior of the optimal strategy typically satisfies (3.22), but this can only be verified numerically.

6We do not offer a conjecture as to why (3.7) does not depend on \( \gamma \). However, it is worth noting that this quantity corresponds to simply one point of the CDF \( P^{0,\text{no}}_{P^*_{ir}} \left[ W^*_{ir} (t)/\hat{W} (t) \leq \hat{k} \right] \), and the moments and tail behavior of this CDF is indeed significantly affected by the choice of \( \gamma \).
Although it is a quantity of relevance to the subsequent results, we do not report any analytical results for the IR-optimal probability of insolvency, defined for our purposes (see Assumption 3.1) simply as $p^{	ext{IR}-0}_{q_0}$, $W^*_t (t) \leq 0$.

The reason for the absence of a closed-form expression is that this probability remains analytically intractable even if we introduce restrictive assumptions such as those of Lemma 3.7. However, it is straightforward to assess the IR-optimal probability of insolvency numerically using the (joint) Monte Carlo simulation of the $Q^*_t (t)$ and $\bar{W} (t)$ dynamics (3.21) and (3.8), since

$$p^{	ext{IR}-0}_{q_0} [W^*_t (t) \leq 0] = p^{	ext{IR}-0}_{q_0} [Q^*_t (t) - \bar{W} (t) \geq \gamma e^{-r(T-t)}, \quad t \in (t_0, T).$$

(3.24)

3.2 Analytical solution: QD ($\beta$) problem

We now discuss the closed-form solution of the tracking difference problem using our proposed objective function (2.8). The following verification theorem reports the HJB equation satisfied in the case of the QD problem.

**Theorem 3.8.** (QD: Verification theorem) Suppose that for all $(y, t) \in \mathbb{R} \times [t_0, T]$, there exist real-valued functions $V_{qd} (y, t)$ and $v^*_q (y, t)$ with the following two properties. (i) $V_{qd}$ and $v^*_q$ are sufficiently smooth and solve the HJB PIDE (3.25)-(3.26), and (ii) the function $v^*_q (y, t)$ attains the pointwise supremum in (3.25).

Define the auxiliary process $Y_{qd} (t)$ by

$$Y_{qd} (t) := W (t) - e^{\beta T} \bar{W} (t), \quad \forall t \in (t_0, T), \quad \text{with} \quad Y_{qd} (t_0) = y_0 = w_0 (1 - e^{\beta T}) .$$

(3.27)

Let the auxiliary control $v (t) := v (Y_{qd} (t), t) := v (Y_{qd} (t), t; X (t))$ be given by

$$v (t) := p \left( t, X (t) \right) \cdot W (t) - e^{\beta T} \cdot \tilde{p} \left( t, \bar{W} (t) \right) \cdot \bar{W} (t), \quad \text{where} \quad X (t) = \left( W (t), \bar{W} (t), \tilde{p} \left( t, \bar{W} (t) \right) \right).$$

(3.28)

Let $A_{u,0}$ be as defined in Theorem 3.3. Then under Assumption 3.7 and wealth dynamics (3.7)-(3.8), $V_{qd}$ is the value function and $v^*_q$ is the optimal control for the following control problem,

$$\inf_{v \in A_{u,0}} E^{t_0, y_0} \left[ \left( V_{qd} (T) \right)^2 \right].$$

(3.29)

**Proof.** As in the case of Theorem 3.2, the result follows the application of standard techniques (Oksendal and Sulem (2005)). For subsequent reference, we note that the dynamics of the auxiliary process $Y_{qd} (t)$ can be written in terms of the auxiliary control $v (t)$ as follows,

$$dV_{qd} (t) = \left[ h_\beta (t) - r \cdot (h_\beta (t) - Y_{qd} (t)) + v (t) (\mu - r) \right] dt + v (t) \sigma dZ (t) + v (t) \int_0^\infty (\xi - 1) \tilde{N} (dt, d\xi),$$

(3.30)

where for a fixed value of the parameter $\beta$ and the contribution rate $q$, $h_\beta (t)$ is the following function of time,

$$h_\beta (t) := (e^{\beta T} - 1) \cdot \int_t^T q e^{-r(T-z)} dz = \frac{q}{r} (e^{\beta T} - 1) \left( 1 - e^{-r(T-t)} \right), \quad t \in [t_0, T],$$

(3.31)

while $h_\beta (t) = \frac{d}{dt} h_\beta (t)$.

Solving the HJB PIDE (3.25)-(3.26), we obtain the QD-optimal control as reported by the following lemma.

**Lemma 3.9.** (QD-optimal control) Suppose that Assumption 3.7 and wealth dynamics (3.7)-(3.8) are applicable. Then the optimal fraction of the investor’s wealth to be invested in the risky asset for problem QD ($\beta$) in (2.8)
is given by $p^{*}_{q\delta}(t, X^{*}_{q\delta}(t))$, where

$$
    p^{*}_{q\delta}(t, X^{*}_{q\delta}(t)) \cdot W^{*}_{q\delta}(t) = \frac{\left(\mu - r\right)}{\sigma^2 + \lambda \kappa^2} \left[ h\delta(t) - \left(W^{*}_{q\delta}(t) - e^{\gamma T} \hat{W}(t)\right)\right] + e^{\gamma T} \hat{p}(t, \hat{W}(t)) \cdot \hat{W}(t),
$$

(3.32)

with $W^{*}_{q\delta}(t)$ denoting the investor’s wealth process under the QD-optimal control $p^{*}_{q\delta}$, and $X^{*}_{q\delta}(t) = (W^{*}_{q\delta}(t), \hat{W}(t), \hat{p}(t, \hat{W}(t)))$.

Proof. As in the case of Lemma 3.3, the terminal condition (3.26) suggests an ansatz for $V^{*}_{q\delta}$ that is quadratic in $y$, in other words $V^{*}_{q\delta}(y, t) = A^{q\delta}(t) y^2 + B^{q\delta}(t) y + C^{q\delta}(t)$. In this case, the pointwise supremum in (3.25) is attained by the auxiliary control $v^{*}_{q\delta}$ with a qualitatively similar form in terms of $(y, t)$ as the result reported in (3.16). Substituting $V^{*}_{q\delta}$ and $v^{*}_{q\delta}$ into (3.33) yields ODEs for $A^{q\delta}, B^{q\delta}$ and $C^{q\delta}$, which are solved to obtain $A^{q\delta}(t) = e^{(2r-\gamma)(T-t)}$ and

$$
    B^{q\delta}(t) = \frac{2q}{r} \left(1 - e^{\gamma T}\right),
$$

(3.33)

The necessary substitution and simplification yields (3.32).

Lemma 3.9 shows that in the case of Lemma 3.3, the QD-optimal control $p^{*}_{q\delta}(t, X^{*}_{q\delta}(t))$ also only depends on the instantaneous benchmark allocation $\hat{p}(t, \hat{W}(t))$ and not on its past or future. However, in contrast to the IR-optimal control (3.15), the contribution rate $q$ does affect the QD-optimal control (3.32) through the term $h\delta(t)$ (3.31).

The following lemma reports the first two moments of the difference between the investor’s wealth and that of the elevated benchmark, quantities which are useful subsequently when comparing investment outcomes.

**Lemma 3.10. (QD: moments of $W^{*}_{q\delta}(T) - e^{\gamma T} \hat{W}(T)$)** Assume that Assumption 3.1 and wealth dynamics (3.7), (3.8) hold, with $t_0 = 0$. Implementing the QD-optimal control (3.32) gives

$$
    E^{t_0, w_0}_{q\delta} \left[W^{*}_{q\delta}(T) - e^{\gamma T} \hat{W}(T)\right] = - \left[ h\delta(0) + (e^{\gamma T} - 1) w_0\right] e^{(r-\gamma)T},
$$

(3.34)

$$
    \text{Var}^{t_0, w_0}_{q\delta} \left[W^{*}_{q\delta}(T) - e^{\gamma T} \hat{W}(T)\right] = \left[ h\delta(0) + (e^{\gamma T} - 1) w_0\right]^2 e^{2(r-\gamma)T} (e^{\gamma T} - 1).
$$

(3.35)

Proof. Substituting (3.32) in (3.30) gives the dynamics of $Y^{*}_{q\delta}(t) := W^{*}_{q\delta}(t) - e^{\gamma T} \hat{W}(t)$. Standard techniques (Oksendal and Sulem [2005]) give results (3.34)–(3.35).

The following lemma shows that the QD-optimal strategy will, like the IR-optimal strategy (see Lemma 3.5), match the benchmark in terms of the amount invested in the risky asset once the investor is assured of the desired outperformance.

**Lemma 3.11. (QD: Matching the benchmark risky asset amount)** Given Assumption 3.1 and wealth dynamics (3.7), (3.8), suppose that at some time $\hat{t} \in (t_0, T]$, the QD-optimal investor observes a wealth value $W^{*}_{q\delta}(\hat{t})$ satisfying

$$
    W^{*}_{q\delta}(\hat{t}) = e^{\gamma T} \hat{W}(\hat{t}) + h\delta(\hat{t}),
$$

(3.36)

Then for the remainder of the investment time horizon $t \in [\hat{t}, T]$, the QD-optimal investor (using strategy (3.32)) will match the benchmark strategy in terms of the amount invested in the risky asset, so that

$$
    p^{*}_{q\delta}(t, X^{*}_{q\delta}(t)) \cdot W^{*}_{q\delta}(t) = \hat{p}(t, \hat{W}(t)) \cdot \hat{W}(t), \quad \forall t \in [\hat{t}, T].
$$

(3.37)

Proof. Substituting (3.31) into (3.36), note that condition (3.36) can equivalently be written as

$$
    W^{*}_{q\delta}(\hat{t}) + \int_{\hat{t}}^{T} q e^{-r(T-z)}dz = e^{\gamma T} \left[ \hat{W}(\hat{t}) + \int_{\hat{t}}^{T} q e^{-r(T-z)}dz\right],
$$

(3.38)

which provides intuition as to why result (3.37) should hold. The proof proceeds along the same lines as in the
case of Lemma 3.5 except that (3.37) can be established using the properties of the auxiliary process

\[ Q_{qd}^* (t) := h_\beta (t) - \left[ W_{qd}^* (t) - e^{\gamma T} \hat{W} (t) \right], \]  

(3.39)   

which has dynamics that are formally the same as the dynamics of \( Q_{ir}^* \) in (3.21). □

By analogy with Lemma 3.6, the following lemma establishes some conditions under which the equivalence of problems (2.8) and (2.10) can be established analytically.

**Lemma 3.12.** (QD: equivalence with only penalizing underperformance) If Assumption 3.1 and wealth dynamics (3.7)-(3.8) apply with no jumps (\( \lambda = 0 \)) in the risky asset process (3.5), then

\[ W_{qd}^* (t) < h_\beta (t) + e^{\gamma T} \hat{W} (t), \quad \forall t \in [t_0, T]. \]  

(3.40)

As a result, the QD optimization problem (2.8) is equivalent to the one-sided quadratic problem (2.10), where only the underperformance of the investor’s portfolio (compared to the elevated benchmark) is penalized.

**Proof.** The proof proceeds along the same lines as for Lemma 3.6 but follows from analyzing the properties of \( Q_{qd}^* \) in (3.39) with \( \lambda = 0 \). □

As in the case of the IR problem, Lemma 3.11 and Lemma 3.12 provides intuition for the behavior of the QD-optimal investment strategies even if the assumptions of this section are relaxed.

For the QD problem, it appears unlikely that the probability of underperforming the benchmark can be established analytically for an arbitrary benchmark strategy \( \hat{p} \left( t, \hat{W} (t) \right) \) as in the case of the IR problem (see Lemma 3.7). However, when a constant proportion benchmark \( \hat{p} \left( t, \hat{W} (t) \right) \equiv \hat{p} > 0 \) for all \( t \) is used, both the QD-optimal probability of underperforming the benchmark and the QD-optimal probability of insolvency (i.e. probability of \( W_{qd}^* (t) \leq 0 \)) can be obtained analytically under some conditions, as the following lemma shows.

**Lemma 3.13.** (QD: probability of underperformance and insolvency) Suppose the following assumptions hold:

(i) Assumption 3.1 and wealth dynamics (3.7)-(3.8) with no jumps (i.e. \( \lambda = 0 \)) in the risky asset process (3.5); (ii) contributions are zero (\( q = 0 \)), and (iii) the benchmark strategy is a constant proportion strategy with \( \hat{p} \left( t, \hat{W} (t) \right) \equiv \hat{p} > 0 \) for \( t \in [t_0, T] \). Then the probability of the QD-optimal wealth underperforming the benchmark wealth at any \( t \in (t_0, T] \) is given by

\[ P_{P_{qd}}^{\lambda_0, \nu_0} \left( W_{qd}^* (t) \leq \hat{W} (t) \right) = \Phi \left( \frac{-\left[ \frac{3}{2} \gamma + (\mu - \hat{p}) \hat{p} - \frac{1}{2} \hat{p}^2 \sigma^2 \right]}{\sqrt{\eta + \hat{p}^2 \sigma^2}} \sqrt{t} \right), \quad t \in (t_0, T], \]  

(3.41)

while the probability of insolvency at any time is given by

\[ P_{P_{qd}}^{\lambda_0, \nu_0} \left( W_{qd}^* (t) \leq 0 \right) = \Phi \left( \frac{\log \left( 1 - e^{-\beta T} \right) - \left[ \frac{3}{2} \gamma + (\mu - \hat{p}) \hat{p} - \frac{1}{2} \hat{p}^2 \sigma^2 \right] T}{\sqrt{\eta + \hat{p}^2 \sigma^2} \sqrt{T}} \right), \quad t \in (t_0, T]. \]  

(3.42)

**Proof.** The results follow from analyzing the properties of \( Q_{qd}^* \) in (3.39) under the stated conditions. □

We emphasize that, in contrast to the IR-optimal probability of underperformance (see (3.23)), the closed-form expression (3.41) can only be obtained if we assume a constant proportion benchmark strategy and zero contributions. Under these assumptions, we observe that (3.41) does not depend on the targeted outperformance spread \( \beta \). In addition, (3.42) shows that we can obtain a closed-form expression for the probability of insolvency in the case of the QD problem (under the stated assumptions), unlike in the case of the IR problem (see (3.24) and associated discussion).

Lemma 3.14 below presents a simple but interesting comparison result for the ultimate probability of benchmark underperformance associated with the IR- and QD-optimal investment strategies, in the sense that it holds regardless of the values of the parameters \( \gamma \) and \( \beta \) in the IR (\( \gamma \)) and QD (\( \beta \)) problems, respectively.

**Lemma 3.14.** (QD vs IR: Probability of underperformance) Suppose that the assumptions of Lemma 3.13 hold.

Then the probability that the QD-optimal strategy underperforms the benchmark always exceeds the corresponding probability associated with the IR-optimal strategy, in other words

\[ P_{P_{qd}}^{\lambda_0, \nu_0} \left( W_{qd}^* (t) \leq \hat{W} (t) \right) \geq P_{P_{ir}}^{\lambda_0, \nu_0} \left( W_{ir}^* (t) \leq \hat{W} (t) \right), \quad \forall t \in [t_0, T]. \]  

(3.43)
**Proof.** Note that the assumptions of Lemma 3.13 are required to hold since the proof requires the analytical result (3.41) for the left-hand side of (3.43). Since these (more restrictive) assumptions also imply that the assumptions of Lemma 3.7 are satisfied, the right-hand side of (3.43) is given by (3.23). Using the fact that the CDF is non-decreasing, (3.43) holds if

\[
-\left[\frac{3}{2}\eta + (\mu - r) \hat{p} - \frac{1}{2} \hat{p}^2 \sigma^2\right] + \sqrt{\eta + \hat{p}^2} 
\geq \frac{3}{2} \sqrt{\eta},
\]  

(3.44)

where we recall that \(\eta\) is given by (3.6). Therefore, for a fixed value of \(t\), and recalling that we have assumed \(\mu > r\), (3.44) holds if \(\hat{p} \sqrt{\eta + \hat{p}^2} \geq 0\), which is clearly satisfied if \(\hat{p} > 0\).

In our numerical tests, we observe that (3.43) appears to remain true provided Assumption 3.1 holds, even if we allow for contributions \((q > 0)\) and jumps \((\lambda > 0)\).

While it is an interesting result, it should be emphasized that Lemma 3.14 only considers a single point of a cumulative distribution function, namely \(P_{\hat{p},q} \left[ W_{q,t} (t) / \hat{W}(t) \leq 1 \right]\). As the results of Section 5 and Appendix B show, this is a very unreliable basis for the practical evaluation and comparison of investment strategies, especially since no mention is made of tail behavior (upside or downside) of the different strategies.

Before proceeding to the numerical solutions of the problems under more realistic assumptions (Section 4), we conclude this section by briefly highlighting two qualitative observations regarding the analytical results.

(i) Implications for diversification: We recall from Section 2 that the objective functions suggest that the QD investor wishes (where possible) to outperform the benchmark terminal wealth by a constant factor, whereas the IR investor hopes to achieve the benchmark terminal wealth by a constant amount irrespective of the underlying market scenario.

The results of Lemmas 3.3, 3.6, 3.9 and 3.12 rigorously confirms this intuition not only holds at time \(T\), but also for all \(t < T\). Specifically, we see that at time \(t < T\), the IR-optimal strategy can be interpreted as being associated with an implicit target of \(\gamma e^{-r(T-t)} + \hat{W}(t)\) for \(W_{q,t}^*(t)\) (see (3.15), (3.19) and (3.22)).

Similarly, ignoring contributions for the moment, the QD-optimal strategy can be interpreted as having an implicit target of \(e^{\beta T \hat{W}}(t)\) for \(W_{q,t}^*(t)\) (see (3.32), (3.36) and (3.40)). By “implicit target”, we mean that in the case of both the IR and QD strategies, the risky asset exposure is increased in direct proportion with the extent to which the investor’s wealth is underperforming the above-mentioned target values at time \(t\).

This observation has significant implications for the diversification of the resulting investment strategies. In particular, in adverse market scenarios (which of course also affects the benchmark), the IR strategy effectively aims to outperform the benchmark by a larger factor than in “typical” market scenarios due to the constant amount of specified outperformance, and is thus required to take on more extreme positions in the risky (or riskiest) asset compared to the QD strategy. This is clearly demonstrated in the numerical results (Section 5, see for example Figures 5.2 and 5.3), regardless of whether realistic investment constraints are applicable or not.

(ii) Investment risk profile over time: The preceding observation also implies that early in the investment time horizon, when the investor’s wealth is expected to be small relative to wealth at later stages, the IR strategy is therefore expected to take on significantly more risk (i.e. investing more in the riskiest asset) than the QD strategy due to its higher relative target implied by the constant amount of outperformance. These statements can be made rigorous in the case of 2 assets under Assumption 3.1 (see Appendix A, in particular Theorem A.4), but the numerical results in Section 5 show that this observation remains true in even more general cases.

4 Numerical solutions

In this section, we consider the IR and QD problems under more realistic assumptions, and as a result we discuss the numerical solutions of the problems in more detail. In particular, we do not apply the standard assumptions in the literature (see for example Bajeux-Besnainou et al. (2013); Basak et al. (2006); Bo et al. 2021; Browne (1999); 2000; Davis and Lleo (2008); Lim and Wong (2010); Nicolosi et al. (2018); Oderda 2015; Tepla (2001); Yao et al. (2006); Zhang and Gao (2017); Zhao (2007)) which were also applied in Section 3 as Assumption 3.1. More fundamentally, we do not make any parametric process assumptions (such as 3.3)
and \((3.5)\), but instead implement the data-driven neural network approach of \cite{ni2020} to solve problems \((2.5)\) and \((2.8)\).

While a short overview of the underlying methodology of \cite{ni2020} is given below, it should be emphasized that we now allow for the following: (i) investment in multiple assets; (ii) the application of multiple realistic investment constraints, specifically restricting short-selling and leverage; and (iii) the rebalancing of the portfolio at discrete time intervals. Note that transaction costs can also be incorporated without difficulty in this framework (see \cite{van2021}).

### 4.1 Discrete rebalancing with investment constraints

Suppose that the investor rebalances the portfolio at each of \(N_{rb}\) equally-spaced rebalancing times in \([t_0 = 0, T]\). Therefore, we now define the set \(\mathcal{T}\) of rebalancing times (see Section 2) as

\[
\mathcal{T} = \{ t_n = n \Delta t | n = 0, \ldots, N_{rb} - 1 \}, \quad \Delta t = T / N_{rb}. \tag{4.1}
\]

Instead of assuming that contributions to the investor and benchmark portfolio occur at a constant rate as in Section 3, we now assume a given cash contribution schedule \(\{q(t_n) : n = 0, \ldots, N_{rb} - 1\}\), where \(q(t_n)\) denotes the amount of cash contributed to each portfolio (investor and benchmark portfolios) at rebalancing time \(t_n \in \mathcal{T}\). Note that the last rebalancing event occurs at time \(t_n = (N_{rb} - 1) \Delta t = T - \Delta t\).

The basic aspects of the formulation is as in Section 2, including the use of \(N_a\) assets. In particular, the investor and benchmark strategies \((2.1)-(2.2)\) are respectively given by

\[
\mathcal{P} = \{ p \left( t_n, X(t_n) \right) = \left( p_i \left( t_n, X(t_n) \right) : i = 1, \ldots, N_a \right) : t_n \in \mathcal{T} \}, \tag{4.2}
\]

\[
\mathcal{P}^* = \{ \hat{p} \left( t_n, \hat{X}(t_n) \right) = \left( \hat{p}_i \left( t_n, \hat{X}(t_n) \right) : i = 1, \ldots, N_a \right) : t_n \in \mathcal{T} \}, \tag{4.3}
\]

where \(\mathcal{T}\) is now being given by \((4.1)\).

As per Assumption 2.1, we assume that \(\mathcal{P}^* (4.3)\) is given. The investor is assumed to be subject to the investment constraints of (i) no shorting and (ii) no leverage, so that we require \(\mathcal{P} = \{ p \left( t_n, X(t_n) \right) \in \mathcal{Z} : t_n \in \mathcal{T} \}\),

where \(\mathcal{Z}\) is given by

\[
\mathcal{Z} = \left\{ (y_1, \ldots, y_{N_a}) \in \mathbb{R}^{N_a} : \sum_{i=1}^{N_a} y_i = 1 \text{ and } y_i \geq 0 \text{ for all } i = 1, \ldots, N_a \right\}. \tag{4.4}
\]

To describe the wealth dynamics in this discrete setting, we assume for the moment that we fix investment strategies \((4.2)-(4.3)\), and consider the events at an arbitrary rebalancing time \(t_n \in \mathcal{T}\). The investor observes the information \(X(t_n)\), the contribution \(q(t_n)\) is made to each portfolio, and investor and benchmark portfolios are rebalanced to the appropriate proportions of wealth in each asset given by the vectors \(p \left( t_n, X(t_n) \right)\) in \((4.2)\) and \(\hat{p} \left( t_n, \hat{X}(t_n) \right)\) in \((4.3)\), respectively.

Moving forward in time by \(\Delta t\), a return on investment of \(R_i(t_n)\) is observed for asset \(i\) over the time interval \([t_n, t_{n+1}].\) Of course, this return is observed by both the investor and the benchmark, but since their respective allocations to asset \(i\) is expected to differ, so does the wealth impact. Therefore, the investor and benchmark wealth dynamics in this discrete setting can be written respectively as

\[
W \left( t_{n+1}^- \right) = \left[ W \left( t_n^- \right) + q(t_n) \right] \cdot \sum_{i=1}^{N_a} p_i \left( t_n, X(t_n) \right) \cdot [1 + R_i(t_n)], \tag{4.5}
\]

\[
\hat{W} \left( t_{n+1}^- \right) = \left[ \hat{W} \left( t_n^- \right) + q(t_n) \right] \cdot \sum_{i=1}^{N_a} \hat{p}_i \left( t_n, \hat{X}(t_n) \right) \cdot [1 + R_i(t_n)], \tag{4.6}
\]

for \(n = 0, \ldots, N_{rb} - 1.\) As per Assumption 2.1, we have \(W \left( t_0^- \right) = \hat{W} \left( t_0^- \right) = w_0.\) Since there is no rebalancing or contributions at the terminal time, the terminal wealth for each strategy is obtained by setting \(n = N_{rb} - 1\) in \((4.5)\) and \((4.6)\), respectively. In other words, we simply have \(W(T) := W \left( t_{N_{rb}}^- \right)\) and \(\hat{W}(T) := \hat{W} \left( t_{N_{rb}}^- \right).\)
4.2 Neural network (NN) approach

As noted above, since we follow the approach of Ni et al. (2020) (which builds on Li and Forsyth (2019)) in using a neural network (NN) to model the investor’s investment strategy \( \mathcal{P} = \{ p(t_n, X(t_n)) \in \mathcal{Z} : t_n \in \mathcal{T} \} \) in the context of a stochastic benchmark outperformance problem, only a brief outline is provided here to ensure that this discussion is reasonably self-contained. We start by making the following remark, which places the chosen methodology within the context of the machine learning (ML) literature.

**Remark 4.1.** (NN approach in the context of the ML literature) The data-driven NN approach of Ni et al. (2020) differs from other popular approaches in the ML literature to solve dynamic programming problems of the form (2.5) or (2.8) in a number of ways. However, the key difference is that we solve a single optimization problem, which does not rely on the dynamic programming principle, to obtain the optimal investment strategy (as a function of time and state) applicable at any rebalancing time \( t_n \in \mathcal{T} \). Note that the time \( t_n \) is used as a feature (input) into the NN, while the neural network weights matrices and bias vectors do not depend on \( t_n \). Since only a single optimization problem is solved, this approach stands in contrast with for example the Q-learning algorithm, arguably the most popular data-driven Reinforcement Learning algorithm to solve dynamic programming problems (see for example Dixon et al. (2020); Gao et al. (2020); Lucarelli and Borrotti (2020); Park et al. (2020)), where the reliance on value iteration to obtain the optimal investment strategy effectively implies an optimization problem has to be solved to determine the value function at each rebalancing time \( t_n \in \mathcal{T} \).

Consider a fully-connected, feed-forward NN with \( \mathcal{L} \) hidden layers and parameter vector \( \theta \in \mathbb{R}^{n_{\theta}} \), where \( n_{\theta} \in \mathbb{N} \) denotes the total number of NN parameters (all weights and biases). The NN layers are indexed by \( \ell \in \{0, \ldots, \mathcal{L} + 1\} \), where \( \ell = 0 \) and \( \ell = \mathcal{L} + 1 \) denote the input and output layers, respectively. Let \( n_{\ell} \in \mathbb{N} \) denote the number of nodes in layer \( \ell \), with the number nodes in the NN output layer (\( \ell = \mathcal{L} + 1 \)) equal to the number of assets, i.e. \( n_{\mathcal{L} + 1} = N_a \). The output layer of the NN uses a softmax activation function \( a^{[\mathcal{L}+1]} : \mathbb{R}^{N_a} \to \mathbb{R}^{N_a} \), so that it automatically generates vectors \( a^{[\mathcal{L}+1]} = [a_1^{[\mathcal{L}+1]}, \ldots, a_{N_a}^{[\mathcal{L}+1]}] \in \mathbb{Z} \) as per (4.1).

The training data of the NN consists of the set \( \mathcal{Y} \subset \mathbb{R}^{N_a \times N_b \times N_a} \) for asset \( i \in \{1, \ldots, N_a\} \) over the time period \( [t_n, t_{n+1}] \), where \( n \in \{0, \ldots, N_b - 1\} \). A sample path \( j \in \{1, \ldots, N_d\} \) of joint returns is simply the subset \( \mathcal{Y}(j) \subset \mathcal{Y} \), where

\[
\mathcal{Y}(j) = \{ Y_i^{(j)}(t_n) : n = 0, \ldots, N_b - 1, \ i = 1, \ldots, N_a \} \in \mathbb{R}^{N_a \times N_a}, \quad j \in \{1, \ldots, N_d\}.
\]

For illustrative purposes, we fix a sample path \( j \in \{1, \ldots, N_d\} \) and a rebalancing event \( n \in \{0, \ldots, N_b - 1\} \), and use the superscript (j) and argument \( (t_n) \) to highlight the dependence of the NN inputs and outputs, as well as other quantities like the controlled wealth, on \( j \) and \( n \).

The number of nodes in the input layer, \( n_{\theta} \), corresponds to the number of elements in the feature (input) vector \( \phi \in \mathbb{R}^{n_{\theta}} \). Specifically, in the case of the benchmark outperformance problems considered here, the feature vector at rebalancing time \( t_n \) along sample path \( j \) is given by \( \phi^{(j)}(t_n) = (t_n, X^{(j)}(t_n; \theta)) \in \mathbb{R}^{n_{\theta}} \). As suggested by the analytical results in the Markovian setting in Section 3, \( X^{(j)}(t_n; \theta) \) includes at a minimum \( X^{(j)}(t_n; \theta) = \{ W^{(j)}(t_n; \theta), W^{(j)}(t_n; \theta), \ldots \} \), with \( W^{(j)}(t_n; \theta) \) denoting the investor’s wealth along sample path \( j \) at time \( t_n \) obtained using NN parameter vector \( \theta \). In addition to these features, the investor could also augment \( X^{(j)}(t_n; \theta) \) with additional information deemed to be of relevance.

**Remark 4.2.** (Numerical solutions: Information required about the benchmark strategy) Numerical experiments show (see for example Appendix B.3) that there is now no need to include the benchmark strategy \( \hat{P} \) as per (4.3) explicitly in the specification of \( X^{(j)}(t_n; \theta) \). The reason is that the benchmark wealth outcomes \( W^{(j)}(t_n; \theta) \) with dynamics (4.6) constitute sufficient information for the neural network to learn the appropriate outperformance strategy using the specified \( N_a \) underlying assets. In more detail, given only the historical returns of an arbitrary benchmark strategy, the chosen market data generator (e.g. bootstrapping, as in Ni
et al. (2020)) could be used to augment the neural network’s training data set $Y$ with the benchmark strategy returns without knowing any further details regarding $\mathcal{P}$. However, for purposes of convenience and comparison with the analytical solutions presented in Section 5 below, we continue working under Assumption 2.1. In the subsequent results, we therefore use a given strategy $\mathcal{P}$ to calculate the benchmark wealth outcomes (4.6).

As noted in Remark 4.1, the neural network weights matrices and bias vectors do not depend on the rebalancing time $t_n$, and also not on the sample path $j$. At rebalancing time $t_n$ along sample path $j$, let $a_i^{(L+1)(j)}(t_n; \theta)$ denote the output of the $i$th node in the NN output layer using parameter vector $\theta \in \mathbb{R}^\theta$. The investor and benchmark wealth dynamics (4.5)-(4.6) along sample path $j$ (including constraints) can then be written as

$$W^{(j)}(t_{n+1}; \theta) = \left[W^{(j)}(t_n; \theta) + q(t_n)\right] \cdot \sum_{i=1}^{N_\theta} a_i^{(L+1)(j)}(t_n; \theta) \cdot Y_i^{(j)}(t_n), \tag{4.9}$$

$$\hat{W}^{(j)}(t_{n+1}) = \left[\hat{W}^{(j)}(t_n) + q(t_n)\right] \cdot \sum_{i=1}^{N_\theta} \hat{p}_i^{(j)}(t_n, \hat{X}(t_n)) \cdot Y_i^{(j)}(t_n), \tag{4.10}$$

where we observe that the output of the neural network’s $i$th output node is interpreted in (4.9) as the fraction of wealth invested in the $i$th asset, i.e. $a_i^{(L+1)(j)}(t_n; \theta) \equiv p_i^{(j)}(t_n, X^{(j)}(t_n; \theta))$. As noted above, since a softmax activation function is used in the output layer, the investment constraints (no short selling, and no leverage) are therefore automatically satisfied.

### 4.3 Training and testing the NN

Since the neural network gives the investor’s investment strategy $\mathcal{P}$ (see (4.2)), solving the IR- and QD-problems is equivalent to training the neural network - see Li and Forsyth (2019); Ni et al. (2020) for more detail. Specifically, since the investment constraints are automatically satisfied given the NN structure, the IR- and QD-problems can be approximated by the following unconstrained optimization problems,

$$\begin{equation}
(\text{IR} (\gamma)) : \min_{\theta \in \mathbb{R}^\theta} \left\{ \frac{1}{N_d} \sum_{j=1}^{N_d} f_\gamma \left(W^{(j)}(T; \theta), \hat{W}^{(j)}(T)\right) \right\}, \tag{4.11}
\end{equation}$$

$$\begin{equation}
(\text{QD} (\beta)) : \min_{\theta \in \mathbb{R}^\theta} \left\{ \frac{1}{N_d} \sum_{j=1}^{N_d} f_\beta \left(W^{(j)}(T; \theta), \hat{W}^{(j)}(T)\right) \right\}. \tag{4.12}
\end{equation}$$

Here, $f_\gamma$ and $f_\beta$ are the following functions appearing in the objectives of the IR problem (2.5) and QD problem (2.8), respectively,

$$f_\gamma(w, \hat{w}) = (w - [\hat{w} + \gamma])^2, \quad \text{and} \quad f_\beta(w, \hat{w}) = (w - e^{\beta T} \hat{w})^2. \tag{13}$$

In contrast to Li and Forsyth (2019), we use a stochastic gradient descent (SGD) algorithm to train the neural networks and obtain the optimal NN parameter vectors $\theta^{*\gamma}$ and $\theta^{*\beta}$ for (4.11) and (4.12), respectively. In particular, we use the Gadam algorithm of Granziol et al. (2020), which combines the Adam algorithm (Kingma and Ba (2015)) with tail iterate averaging for improved convergence properties, including variance reduction (Mucke et al. (2019); Neu and Rosasco (2018); Polyak and Juditsky (1992)).

Gradient calculations in problems of the form (4.11)-(4.12) using wealth dynamics of the form (4.5)-(4.6) are discussed in detail in Li and Forsyth (2019). Here we simply note that the gradient of the investor’s terminal wealth with respect to neural network parameters, $\nabla_\theta W^{(j)}(T; \theta)$, can be obtained via iterative computation (timestepping) over $t_n \in T$. However, in contrast to Li and Forsyth (2019); Ni et al. (2020), we use backpropagation for the calculation of gradients at each timestep, since we use deeper neural networks to obtain the results described in Section 5 below.

Following the training of the neural networks to obtain $\theta^{*\gamma}$ and $\theta^{*\beta}$, the resulting IR- and QD-optimal investment strategies can be tested by implementing the resulting optimal controls (neural networks) on a testing data set, $Y^{\text{test}}$, with $N^{\text{test}}_d$ sample paths of returns. The contents of $Y^{\text{test}}$ is expected to differ from that of the training dataset $Y$, since it might be based on a different historical time period or different data generation assumptions, but it is assumed to have a similar structure to the training dataset (see Li and Forsyth (2019); Ni et al. (2020)).
Finally, it should be emphasized that under some conditions (for example when investments constraints are not binding), solving (4.11)-(4.12) results in numerical solutions that indeed converge to the analytical solutions of problems (2.5) and (2.8) given in Section 3. More information on these ground truth comparison results can be found in Appendix B.

5 Illustration of investment results

In this section, we illustrate the results from investing according to the optimal strategies associated with problems (2.5) and (2.8), using both analytical solutions (Section 3), as well as numerical solutions (Section 4).

We formulate a realistic investment scenario, where the investor wishes to outperform reasonable and popular benchmarks over the long term using both “standard assets” (a broad stock market index, Treasury bills and bonds) as well as two popular investment “factors” from the factor investing literature (see for example Ang (2014)).

5.1 Investment scenario

In this subsection, we briefly outline the investment scenario details on which all the subsequent results are based. Table 5.1 summarizes the general assumptions. Note that we choose an investment time horizon of $T = 10$ years since we assume the investor is primarily concerned with long-run benchmark outperformance. The case of continuous rebalancing is approximated using 3,600 time steps in $[0,T]$, while the more realistic discrete rebalancing scenario assumes the annual rebalancing of the portfolio.

Table 5.1: Key investment scenario assumptions

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Analytical solutions (No constraints)</th>
<th>Numerical solutions (Realistic constraints)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Investment constraints</td>
<td>None</td>
<td>No short-selling, no leverage allowed</td>
</tr>
<tr>
<td>$T$</td>
<td>10 years</td>
<td>10 years</td>
</tr>
<tr>
<td>$w_0$</td>
<td>120</td>
<td>120</td>
</tr>
<tr>
<td>Contributions</td>
<td>$q = 12$ (rate per year)</td>
<td>$q(t_n) = 12, \forall n$ (annual contribution)</td>
</tr>
<tr>
<td>$N_{rb}$</td>
<td>3600 (continuous rebalancing)</td>
<td>10 (annual rebalancing), $\Delta t = 1$</td>
</tr>
</tbody>
</table>

In the subsequent analyses, we will compare the results from investing according to the IR- and QD-optimal investment strategies. Since there are many possibilities for the basis of comparison, we make the practical assumption that the investor wishes to achieve an expected terminal wealth of $E$ regardless of whether the IR or QD strategy is followed. In more detail, if the benchmark investment strategy $\hat{P}$ results in an expected value of benchmark terminal wealth $E_{\hat{P}}^{t_0, w_0} [\hat{W}(T)] = \mathcal{K}$, we assume the investor chooses some value of $\hat{\beta} > 0$ in (5.1) below to achieve an expected terminal wealth of $E$:

$$E_{P^{\xi_x}_{ir}}^{t_0, w_0} [W^{\xi_x}(T)] = E_{P^{\xi_x}_{qd}}^{t_0, w_0} [W^{\xi_x}(T)] := E := e^{\hat{\beta}T} \cdot \mathcal{K} = e^{\hat{\beta}T} \cdot E_{\hat{P}}^{t_0, w_0} [\hat{W}(T)].$$  (5.1)

The desired target expectation $[5.1]$ can be achieved by solving numerically (or in some cases, analytically - see Appendix A) for values of $\gamma = \gamma^{\xi_x}_{ir}$ in the IR ($\gamma$) problem and $\beta = \beta^{\xi_x}_{qd}$ in the QD ($\beta$) problem such that the associated IR- and QD-optimal strategies $P^{\xi_x}_{ir}$ and $P^{\xi_x}_{qd}$, respectively, each result in the desired expected value of terminal wealth $E$. Note that (5.1) implies that we always have the strict inequality $E > \mathcal{K}$ (since $\hat{\beta} > 0$), which is required since if $E = \mathcal{K}$, then the IR- and QD-optimal strategies will be identical to the benchmark strategy.

5.2 Underlying assets and source data

Table 5.2 summarizes the combinations of candidate assets considered by the investor for investment (combinations are identified by the label “P$x^x$; $x \in \{0,1\}$), as well as the benchmarks under consideration (benchmarks

\begin{footnote}{While intuitive, the fact that $E = \mathcal{K}$ implies $P^{\xi_x}_{ir} = P^{\xi_x}_{qd} = \hat{P}$ can also be shown analytically by setting $E = \mathcal{K}$ in the expressions for $\gamma^{\xi_x}_{ir}$ and $\beta^{\xi_x}_{qd}$ in Lemma A.2 in Appendix A, and then substituting the resulting values into the optimal controls 3.15 and 3.9.}

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are identified by the label “BMx”, \( x \in \{0, 1\} \). Both benchmarks portfolios are equally-weighted between stocks and bonds.

As per Assumption 2.1, the investor is not necessarily limited to investing in the same assets as underlying the benchmark. For example, if the investor considers the underlying assets in portfolio P1 (\( N_a = 5 \)) in order to outperform the benchmark BM1 (3 assets with nonzero investment), the benchmark asset allocation (2.2) for modelling purposes is encoded as \( \hat{p}(t, \hat{W}(t)) = (T10, B10, \text{Market}, \text{Size}, \text{Value}) = (0.25, 0.25, 0.5, 0.0, 0.0) \) for all \( t \in T \), corresponding to the allocations for T10, B10, Market, Size and Value, respectively. While more details are provided in subsequent sections, for now we simply note that we will assume that the investor will construct portfolio P0 to outperform benchmark BM0, and portfolio P1 to outperform benchmark BM1.

Table 5.2 also indicates the data sources used to obtain the underlying data. In summary, data for the basic assets such as the T-bills/bonds and the broad market index were obtained from the CRSP \(^8\) whereas factor data for Size and Value (see Fama and French \(1992, 2015\)) were obtained from Kenneth French’s data library \(^9\). All data was obtained for the period from 1963:07 to 2020:12, which of course includes the period of significant market volatility experienced during 2020.

As a result of the reasonably long time horizon (Table 5.1), we will assume as in for example Forsyth and Vetzal \(2019\); Forsyth et al. \(2019\) that the investor is primarily interested in the real (or inflation-adjusted) performance of the portfolio. Prior to calculations, all time series were inflation-adjusted using data from the US Bureau of Labor Statistics \(10\).

Table 5.2: Underlying assets and data sources for the investor portfolios and the benchmarks. Portfolios of candidate assets considered by the investor are abbreviated by “P\(x\)”, \( x \in \{0, 1\} \). The tick mark “✓” indicates the inclusion of the asset in the portfolio optimization problem, with \( N_a \) denoting the total number of candidate assets. Two constant proportion benchmarks are considered, abbreviated by “BM\(x\)”, \( x \in \{0, 1\} \), with asset holdings as a percentage of wealth \( \hat{p}_i \) as indicated. CRSP refers to the Center for Research in Security Prices, and KFDL refers to Kenneth R. French’s Data Library.

<table>
<thead>
<tr>
<th>Assets</th>
<th>Investor portfolios</th>
<th>Benchmarks</th>
</tr>
</thead>
<tbody>
<tr>
<td>Label</td>
<td>Data source and definition</td>
<td>P0</td>
</tr>
<tr>
<td>T30</td>
<td>CRSP: Monthly returns for 30-day Treasury bill.</td>
<td>✓</td>
</tr>
<tr>
<td>B10</td>
<td>CRSP: Monthly returns for 10-year Treasury bond.</td>
<td>✓</td>
</tr>
<tr>
<td>Market (broad equity market index)</td>
<td>CRSP: Monthly returns, including dividends and distributions, for a capitalization-weighted index (the VWD index) consisting of all domestic stocks trading on major US exchanges.</td>
<td>✓</td>
</tr>
<tr>
<td>Size (portfolio of small stocks)</td>
<td>KFDL, “Portfolios Formed on Size”: Monthly returns on a capitalization-weighted index consisting of the firms (listed on major US exchanges) with market value of equity, or market capitalization, at or below the 30th percentile (i.e. smallest 30%) of market capitalization values of NYSE-listed firms.</td>
<td>✓</td>
</tr>
<tr>
<td>Value (portfolio of value stocks)</td>
<td>KFDL, “Portfolios Formed on Book-to-Market”: Monthly returns on a capitalization-weighted index of the firms (listed on major US exchanges) with book-to-market value of equity ratios at or above the 70th percentile (i.e. highest 30%) of book-to-market ratios of NYSE-listed firms.</td>
<td>✓</td>
</tr>
</tbody>
</table>

Number of candidate assets (\( N_a \)): 2 5 2 3

\(^8\)Calculations were based on data from the Historical Indexes 2020©, Center for Research in Security Prices (CRSP), The University of Chicago Booth School of Business. Wharton Research Data Services was used in preparing this article. This service and the data available thereon constitute valuable intellectual property and trade secrets of WRDS and/or its third party suppliers.

\(^9\)See https://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html

\(^10\)The annual average CPI-U index, which is based on inflation data for urban consumers, was used - see http://www.bls.gov/cpi
5.3 Illustration of analytical solutions

For the illustration of the analytical results of Section 3 which are limited to $N_a = 2$ assets, we assume that the investor portfolio $P_0$ is constructed to outperform benchmark BM0 as per Table 5.2. In the terminology of Section 3, T10 and Market (Table 5.2) are therefore associated with the risk-free and risky assets, respectively.

In order to parameterize (3.3) and (3.5), we use the same calibration methodology as outlined in Dang and Forsyth (2016). For illustrative purposes, we assume the risky asset evolves according to the dynamics of the Kou (2002) model, with log $\xi$ having an asymmetric double-exponential distribution,

$$f_\xi(\xi) = \nu \xi^{\frac{\zeta_1 - 1}{\zeta_1}} I_{[\xi \geq 1]}(\xi) + (1 - \nu) \zeta_2 \xi^{\frac{\zeta_2 - 1}{\zeta_2}} I_{[0 \leq \xi < 1]}(\xi), \quad v \in [0, 1] \text{ and } \zeta_1 > 1, \zeta_2 > 0,$$

where $\nu$ denotes the probability of an upward jump given that a jump occurs. Table 5.3 summarizes the resulting parameters.

Table 5.3: Analytical solutions: Calibrated, inflation-adjusted parameters for asset dynamics (3.3) and (3.5), with $f_\xi(\xi)$ given by (5.2). For calibration purposes, a jump threshold equal to 3 has been used in the methodology of Dang and Forsyth (2016).

<table>
<thead>
<tr>
<th>Parameter</th>
<th>$\tau$</th>
<th>$\mu$</th>
<th>$\sigma$</th>
<th>$\lambda$</th>
<th>$\nu$</th>
<th>$\zeta_1$</th>
<th>$\zeta_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Value</td>
<td>0.0074</td>
<td>0.0749</td>
<td>0.1392</td>
<td>0.2090</td>
<td>0.2500</td>
<td>7.7830</td>
<td>6.1074</td>
</tr>
</tbody>
</table>

We now compare analytical investment results on the basis of (5.1), using $10^6$ Monte Carlo simulations of asset dynamics (3.3), (3.5), and (5.2) with parameters as in Table 5.3.

Table 5.4 presents the key results using two expected value targets as in (5.1), namely $E = 370$ ($\hat{\beta} \simeq 1\%$ in (5.1)) and $E = 400$ ($\hat{\beta} \simeq 2\%$). Results are shown for both the terminal wealth (absolute performance) and the wealth ratio (relative performance).

Since the results in Table 5.4 include both jumps (see Table 5.3) and nonzero contributions (see Table 5.1), two key assumptions underlying Lemma 3.14 are violated, but it is interesting that the probability of underperformance (i) is larger for the QD strategy than for the IR strategy, and (ii) remains insensitive to the aggressiveness of the outperformance target. Note that these results are obtained via an implementation of optimal investment strategies (3.15) and (3.32) in a Monte Carlo simulation, and thus the probabilities of underperformance are expected to exhibit some variability.

Table 5.4 shows that as the outperformance target becomes more aggressive, differences between the outcomes associated with the IR and QD strategies are magnified, with the IR strategy resulting in improved downside wealth outcomes, as well as improved target outperformance (as illustrated by the wealth ratio results).

Table 5.4: Analytical solutions, no constraints, investor portfolio $P_0$, benchmark BM0: Selected quantities associated with the distributions of the investor’s target terminal wealth $W_j^i(T)$ and ratio $W_j^i(T)/\hat{W}(T)$, for $j \in \{i, qd\}$. $10^6$ Monte Carlo simulations, with $E = 370$ and $E = 400$ in (5.1). “CExp 5%” refers to the average of the lowest 5% of outcomes, and “Prob. underp.” is the probability of underperformance, $P\left[ W_j^i(T)/\hat{W}(T) \leq 1 \right]$, for $j \in \{i, qd\}$.

<table>
<thead>
<tr>
<th>Quantity</th>
<th>BM0</th>
<th>P0, $E = 370$</th>
<th>P0, $E = 400$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\hat{W}(T)$</td>
<td>$W_j^i(T)$</td>
<td>$W_j^i(T)/\hat{W}(T)$</td>
</tr>
<tr>
<td>Mean</td>
<td>330</td>
<td>370</td>
<td>370</td>
</tr>
<tr>
<td>CExp 5%</td>
<td>208</td>
<td>193</td>
<td>201</td>
</tr>
<tr>
<td>5th pctile</td>
<td>228</td>
<td>244</td>
<td>236</td>
</tr>
<tr>
<td>Median</td>
<td>323</td>
<td>368</td>
<td>365</td>
</tr>
<tr>
<td>95th pctile</td>
<td>454</td>
<td>504</td>
<td>518</td>
</tr>
<tr>
<td>Prob. underp.</td>
<td>2.62%</td>
<td>3.35%</td>
<td>2.61%</td>
</tr>
</tbody>
</table>

Figure 5.1 illustrates the simulated probability density functions (PDFs) associated with the benchmark and investor wealth outcomes using the analytical solutions. The corresponding CDFs are shown in Appendix B. Figure 5.2 shows the IR- and QD-optimal strategies share many (qualitative) similarities in terms of the wealth PDFs when contrasted with the benchmark, while Figure 5.1(b) illustrates the fat left tails of the ratio PDF for both strategies. This has the result that although the 5th percentile of the ratio
distribution is better/higher for the IR strategy than for the QD strategy as per Table 5.4, the average of the lowest 5% outcomes of the ratio results (see “CExp 5%” in Table 5.4) are identical for both strategies.

Figure 5.1: Analytical solutions, no constraints, investor portfolio $P_0$, benchmark $B_0$: Simulated PDFs of benchmark and investor’s target terminal wealth $\hat{W}(T)$ and $W_j^{e^*}(T)$, respectively, as well as the ratio $W_j^{e^*}(T)/\hat{W}(T)$, for $j \in \{ir, qd\}$. $10^6$ Monte Carlo simulations, $\mathcal{E} = 400$ in (5.1). The corresponding CDFs are shown in Figure B.2 in Appendix B.

Considering the investment strategy in more detail, Figure 5.2(a) shows the relatively larger reliance placed by the IR strategy on the risky asset early in the investment time horizon, which has the effect (Figure 5.2(b)) that the IR strategy relies more heavily on trading in bankruptcy (allowed in this case as per Assumption 3.1) to achieve the desired benchmark outperformance.

For both strategies, Figure 5.2(a) also illustrates that as time passes, the risky asset holdings of both the IR- and QD-optimal investment strategies trend closer to the benchmark holdings, which is (qualitatively) to be expected given the results of Lemma 3.5 and Lemma 5.11.

Figure 5.2: Analytical solutions, no constraints, investor portfolio $P_0$, benchmark $B_0$: 80th percentiles of $P_j^{e^*}(t)$ and probability of insolvency as a function of time $t \rightarrow P_{jir}^{e^*,\text{ir}}[W_j^{e^*}(t) \leq 0]$, for $j \in \{ir, qd\}$. $10^6$ Monte Carlo simulations, $\mathcal{E} = 400$ in (5.1).

Note that the qualitative aspects of the relative behavior of the optimal investment strategies observed in Figure 5.2(a) is in fact to be expected, as we show rigorously in Appendix A (see Theorem A.4).

5.4 Illustration of numerical solutions

We now consider the more realistic scenario of multiple investment constraints and discrete rebalancing (see Table 5.1). In contrast to the preceding results, the problems are now solved as outlined in Section 4 but the results are still compared on the basis of (5.1). In particular, note that we now require the targeted expected value $\mathcal{E} = e^{\gamma T} K$ (see (5.1)) to be achieved on the neural network’s training data set $Y$. This means that the values $\gamma = \gamma^{e^*}_i$ and $\beta = \beta^{e^*}_{irj}$ achieving (5.1) can be no longer be derived analytically as in the preceding section (see Lemma A.2 in Appendix A), but are obtained numerically using iterative solutions of problems 4.11–4.12.

In terms of the NN structure, all subsequent results are obtained using 2 hidden layers, each with $N_a + 2$ nodes. Numerical results and ground truth solutions (see Appendix B) indicated that this structure captured sufficient complexity to ensure that reliable results were obtained. For the NN input layer, the three minimal
features as discussed in Section 3 were used (investor wealth, benchmark wealth, and time-to-go), which ensures that the results are realistic and robust against possible overfitting.

To construct both the training and testing data sets for the neural network, $Y$ and $Y^{test}$ respectively, we use stationary block bootstrap resampling. For a detailed discussion of the rationale as well as a theoretical analysis, see Ni et al. (2020). Note that all subsequent results were also tested using various different assumptions for expected blocksize, and since qualitatively similar results were obtained (as expected based on the robustness assessments presented in Li and Forsyth (2019); Ni et al. (2020)), only results for the data sets as outlined in Table 5.5 are presented.

Note that Table 5.5 also includes one training data set (DS0) based on a simulation of the parametric asset dynamics used for the results of Subsection 5.3 above, and therefore does not rely on block bootstrap resampling. This data set has been included to isolate the impact of discrete rebalancing and investment constraints on the results of Subsection 5.3 without confounding the impact based on data considerations.

Table 5.5 also provides the mean benchmark terminal wealth $\kappa$ obtained on each training data set. In the case of DS0 and benchmark BM0, note that the mean $\kappa = 334$ is slightly higher than the mean reported in Table 5.4; this difference is entirely due to the effects of discrete rebalancing. For data sets DS1 and DS2, the relatively shorter expected blocksize used for the testing data is due to the relatively shorter historical time period (11 years) of source data used for out-of-sample testing. However, we emphasize again that our conclusions remain very robust to expected blocksize assumptions.

Table 5.5: Data set combinations, labelled DS$x$, $x \in \{0, 1, 2\}$, used for training and testing the neural network. “SBBR” refers to stationary block bootstrap resampling, with expected blocksize reported in brackets.

<table>
<thead>
<tr>
<th>Label</th>
<th>Training data set $Y$ ($N_y = 10^6$)</th>
<th>Testing data set $Y^{test}$ ($N_y^{test} = 5 \times 10^5$)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Source data</td>
<td>Data set generation</td>
</tr>
<tr>
<td>DS0</td>
<td>$10^6$ Monte Carlo simulations of asset dynamics ${1.7, 1.7, 1.7}$ with parameters as in Table 5.3</td>
<td>BM0: $\kappa = 334$</td>
</tr>
<tr>
<td>DS1</td>
<td>Historical data, 1963:07 - 2009:12 (6 months)</td>
<td>SBBR</td>
</tr>
<tr>
<td>DS2</td>
<td>Historical data, 1963:07 - 1999:12 (6 months)</td>
<td>SBBR</td>
</tr>
</tbody>
</table>

Table 5.6 illustrates the combinations of investor portfolios and benchmarks used in the subsequent results, as well as the targeted level of outperformance. In the case of portfolio P0, benchmark BM0 and data set DS0, we choose $\mathcal{E} = 370$ to ensure alignment with the analytical solutions presented in Table 5.4 but note that this still implies that $\hat{\beta} \simeq 1.0\%$. In the case of using portfolio P1 (5 assets) to outperform BM1 (3 assets), we use a slightly more ambitious value of $\hat{\beta} \simeq 1.7\%$, since the investor has more opportunities for outperformance given that factors are available for investment (see Van Staden et al. (2021b)). Note that the $\mathcal{E}$ values reported for DS1 and DS2 are different due to different values of $\kappa$ (see Table 5.5).

<table>
<thead>
<tr>
<th>Investor portfolio</th>
<th>To outperform benchmark:</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>BM0 (2 assets)</td>
</tr>
<tr>
<td>P0 (2 assets)</td>
<td>DS0: $\mathcal{E} = 370$ ($\hat{\beta} \simeq 1.0%$)</td>
</tr>
<tr>
<td>P1 (5 assets)</td>
<td>N/a</td>
</tr>
</tbody>
</table>

Table 5.7: Based on using portfolio P0 to outperform benchmark BM0 on training data set DS0, shows the impact of applying realistic investment constraints and discrete rebalancing to the results of Subsection 5.3. For ease of reference, the results for $\mathcal{E} = 370$ from Table 5.4 are repeated in the “No constraints” columns, while the BM0 performance changes slightly in the “With constraints” case due to discrete rebalancing being used (as noted above).

The key observations regarding Table 5.7 are the following: (i) with constraints, QD-optimal probability of underperformance is now lower than the corresponding IR-optimal value, and thus the results of Lemma 3.14.
no longer qualitatively hold; (ii) the QD-optimal strategy results in better downside performance than the IR strategy for both the wealth and the wealth ratio when constraints are applied.

Table 5.7: Effect of constraints: analytical solutions vs. numerical solutions, investor portfolio P0, benchmark BM0. “No constraints” and “With constraints” columns are based on the assumptions for the analytical solutions and numerical solutions, respectively, as per Table 5.4 NN trained on data set DS0. Since no out-of-sample testing is conducted for DS0 (see Table 5.5), the “With constraints” results are obtained on the training data set.

<table>
<thead>
<tr>
<th>Quantity</th>
<th>No constraints: P0, $\mathcal{E} = 370$ (Table 5.4)</th>
<th>With constraints: P0, $\mathcal{E} = 370$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>BM0</td>
<td>IR</td>
</tr>
<tr>
<td>Mean</td>
<td>330</td>
<td>370</td>
</tr>
<tr>
<td>CExp 5%</td>
<td>208</td>
<td>193</td>
</tr>
<tr>
<td>5th pctile</td>
<td>228</td>
<td>244</td>
</tr>
<tr>
<td>Median</td>
<td>323</td>
<td>368</td>
</tr>
<tr>
<td>95th pctile</td>
<td>454</td>
<td>504</td>
</tr>
<tr>
<td>Prob. underp.</td>
<td>2.62%</td>
<td>3.35%</td>
</tr>
</tbody>
</table>

Table 5.8 focuses only on the numerical results obtained when applying realistic investment constraints, and compares the training and testing (out-of-sample) results for portfolio P1 constructed to outperform benchmark BM1 on data set DS1. While the training data results are (qualitatively) as expected given the results of Table 5.4, it is the out-of-sample results in Table 5.8 that are the most significant. Specifically, we see that the QD strategy outperforms the IR strategy in the out-of-sample results regardless of whether we consider the wealth, the wealth ratio or the probability of underperformance. Qualitatively similar results are also obtained for a different data set (DS2), as shown in Appendix B (see Table B.1), so it should be emphasized that the reported out-of-sample performance is robust to the choice of underlying data sets.

We also note the potential risk of underperforming the benchmark is significantly larger out-of-sample (both in terms of probability and the downside statistics of the wealth ratio) than for the training data. This is to be expected, since with the actual market data used here the true underlying data generating process is not known, and thus the training results are not expected to generalize perfectly.

Table 5.8: Numerical solutions, with constraints, investor portfolio P1, benchmark B1, data set DS1: Training and testing results for mean terminal wealth $\mathcal{E} = 400$ ($\beta \approx 1.7%$ in (5.1)) on the training data.

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>BM1</td>
<td>IR</td>
</tr>
<tr>
<td>Mean</td>
<td>338</td>
<td>400</td>
</tr>
<tr>
<td>CExp 5%</td>
<td>199</td>
<td>213</td>
</tr>
<tr>
<td>5th pctile</td>
<td>219</td>
<td>254</td>
</tr>
<tr>
<td>Median</td>
<td>328</td>
<td>394</td>
</tr>
<tr>
<td>95th pctile</td>
<td>490</td>
<td>563</td>
</tr>
<tr>
<td>Prob. underp.</td>
<td>3.07%</td>
<td>2.77%</td>
</tr>
</tbody>
</table>

In order to assist in the explanation of the results of Table 5.8, Figure 5.5 presents the 80th percentiles of the proportion of wealth invested in each candidate asset in P1 over time according to the IR- and QD-optimal investment strategies, on the training data set of DS1. While a detailed analysis of the risk/return analysis of these underlying assets using historical data is provided in Van Staden et al. (2021b), here we simply note that the zero investment in Size is to be expected given its historical performance, and that Value and T30 could be viewed as the assets with the most and the least risk in P1, respectively.

With regards to Figure 5.5, we observe that the key qualitative observations regarding the analytical solutions discussed in Subsection 5.3 (see also Theorem A.3 in Appendix A) hold even if investment constraints are applied. Specifically, compared to the QD strategy, Figure 5.5 shows that the IR strategy maintains a larger stake in both the riskiest asset (Value) as well as the asset with the least risk (T30). In this sense, the IR strategy is less diversified than the QD strategy, in the sense that it takes more extreme positions in the assets with the most extreme risk/return trade-offs.
Figure 5.3: Numerical solutions, with constraints, investor portfolio P1, benchmark BM1, data set DS1, $X = 400$: 80th percentile of the proportion of wealth invested in each asset over time on the training data set (DS1). Zero investment in Size, thus it is omitted. Note the same scale on the y-axis, and that the last rebalancing event is at $t = T - \Delta t = 9$ years.

Figure 5.4 illustrates the consequences of applying the investment strategies illustrated in Figure 5.3 on the out-of-sample (testing) data of DS1. Note that the corresponding CDFs are illustrated in Appendix B, Figure B.3. While the wealth distribution of the QD strategy is clearly preferable (Figure 5.3(a)), the different wealth ratio distributions (Figure 5.4(b)) provide a dramatic visual illustration of the underlying results giving rise to the selected statistics in Table 5.8 with the QD strategy resulting in a much more desirable outperformance profile than the IR strategy.

The preceding results presented for data set DS1 (Table 5.8 and Figure 5.4) also hold for other data sets, for example data set DS2 (see Appendix B).

Finally, Table 5.9 presents the performance on the (single) historical path of the QD and IR strategies implemented starting the month indicated by the first column and continuing until the maturity $T = 10 + t_0$ years is reached. Results are shown for strategies trained on the training data of both DS1 and DS2. Note that there is significant overlap (5 years) between the underlying data of each pair of adjacent rows. It should be emphasized that Table 5.9 presents out-of-sample results, since the probability that the actual historical path appears in the training data set can be shown to be vanishingly small (Ni et al. (2020)).

With the exception of single investment time period $[t_0, T + t_0]$ commencing in January 2000, Table 5.9 shows that the QD strategy consistently outperforms the IR strategy on the historical path, which is perhaps to be expected given the preceding out-of-sample performance results of the QD strategy. Regardless, Table 5.9 highlights the attractiveness in terms of historical performance of directly targeting the tracking difference using the proposed QD objective.

The preceding numerical results also provide a vivid illustration of the potential challenges in broadly assuming (as is often implicitly assumed in the literature) that the qualitative aspects of the analytical solutions will also hold in more realistic settings. Specifically, we highlight the following observations.

- Reliance on trading in insolvency: We observed that the inequality (3.43) of Lemma 3.14 reverses in a
In this paper, we derived and compared two dynamics investment strategies for outperforming a benchmark using two widely-used objectives of practical interest to the investor, namely (i) maximizing information ratio (IR) and (ii) maximizing the tracking difference (cumulative outperformance). In the case of the tracking difference, we introduced a simple and intuitive objective function (the QD objective) for achieving this goal. Numerical evaluations using \( (3.24) \) and \( (3.42) \) showed that the IR-optimal strategy relies to a far greater extent than the QD strategy on the trading in insolvency (see Figure 5.2(b)) which is permitted under Assumption 3.1. Once this is no longer allowed, the risk profile of the IR strategy (see Figure 5.2 and Theorem A.4, as well as Figure 5.3) can in fact become a liability, since the investor can no longer rely on trading in insolvency if the larger risky asset exposure of the IR strategy early in the investment time horizon results in poor performance.

- Out-of-sample benchmark outperformance: Arguably the most fundamental assumption underlying the analytical results is that the underlying process dynamics \( (3.7)-(3.8) \) are fully specified. In reality, lack of full knowledge regarding the underlying data generating process creates challenges for the out-of-sample and historical performance of the IR strategy, as our numerical results showed. The relatively lower reliance on the riskiest asset by the QD strategy early in the investment time horizon (Figures 5.2 and 5.3) improves its out-of-sample performance. In this sense, the IR strategy expresses much stronger convictions regarding the expected risk/return performance of the underlying assets than the QD strategy. The IR-strategy therefore retains some resemblance to the results from MV optimization, where “high conviction” strategies typically result in out-of-sample performance challenges (see for example Lehalle and Simon (2021)).

6 Conclusion

In this paper, we derived and compared two dynamics investment strategies for outperforming a benchmark allowing for jumps in the risky asset process, thereby extending known results for IR-optimal investment strategies, whereas all results associated with the QD objective are novel. We also presented closed-form comparison results of the strategies which generalize very effectively (in a qualitative sense) to settings where the assumptions underpinning the analytical results no longer hold. Specifically, by applying leverage and short-selling restrictions as well as discrete rebalancing to portfolios of multiple assets, we solved both problems numerically using a data-driven neural network approach. This also enabled us to compare the results on a more realistic basis in terms of what performance might be expected by an investor in practice.

We demonstrated that compared to the IR-optimal strategies, the resulting QD-optimal strategies are typically associated with less extreme positions in the assets with the most and the least risk, respectively, leading to improved benchmark outperformance in out-of-sample testing. Our results therefore demonstrate the attractiveness directly targeting the tracking difference using the proposed QD objective.

As noted in the Introduction, fairly complex objective functions have been formulated in the literature for benchmark outperformance. As discussed, in this paper we have made the deliberate choice to focus instead on objective functions targeting metrics which are valued by investors in practice (see the Introduction for
A natural question is how the benchmark outperformance of the IR- and QD-optimal strategies presented here compare out-of-sample to that of the optimal strategies associated with more complex objective functions. We leave this for our future work.

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Appendix A: Additional analytical results

In this appendix, we present additional analytical comparison results which assist in rigorously understanding some of the observations regarding the numerical results of Section 4.

As noted in Section 4, there are many possibilities for comparing the IR- and QD-optimal results. Since the strategies are compared in Section 4 on the basis of equal expectation of terminal wealth (see (5.1)), we formally introduce Assumption A.1 outlining the basis of the comparison of the subsequent analytical results. Note that these results are again all derived within the setting of Section 3 and in particular require that Assumption 3.1 and wealth dynamics (3.7)-(3.8) hold.

**Assumption A.1. (Expected value target for terminal wealth)**  Assume that Assumption 3.1 and wealth dynamics (3.7)-(3.8) hold. Suppose that the given benchmark investment strategy, given by the fraction of wealth in the risky asset \( \hat{p}(t, \hat{W}(t)) \), results in an expected value of benchmark terminal wealth satisfying

\[
E^{\hat{p}_{ir,wo}}_{\hat{W}} [\hat{W}(T)] := \mathcal{K}, \quad \text{where } \mathcal{K} > w_0 e^{rT}, \quad (A.1)
\]

We assume that, regardless of whether the IR- or QD-optimal strategy is implemented, the investor wishes to achieve a given multiple of the benchmark expected wealth (A.1). In other words, the investor chooses the parameters \( \gamma = \gamma_{ir}^{\hat{p}} \) in the IR (\( \gamma \)) problem and \( \beta = \beta_{qd}^{\hat{p}} \) in the QD (\( \beta \)) problem such that the associated IR- and QD-optimal strategies \( \hat{p}_{ir}^{\hat{p}} \) and \( \hat{p}_{qd}^{\hat{p}} \), respectively, each result in the desired expected value of terminal wealth,

\[
E^{\hat{p}_{ir,wo}}_{\hat{W}_{ir}} [W_{ir}^{\hat{p}}(T)] = E^{\hat{p}_{qd,wo}}_{\hat{W}_{qd}} [W_{qd}^{\hat{p}}(T)] = \mathcal{E} = e^{\hat{\beta}T} \mathcal{K}, \quad \text{for some } \hat{\beta} > 0. \quad (A.2)
\]

The value of \( \mathcal{E} \) (A.2) will be referred to as the expected value target for terminal wealth.

Note that in this setting, where we do have closed-form solutions, the values of \( \gamma = \gamma_{ir}^{\hat{p}} \) and \( \beta = \beta_{qd}^{\hat{p}} \) achieving (A.2) can be derived analytically, as the following lemma shows.

**Lemma A.2. (Analytical values \( \gamma \) and \( \beta \) achieving expected value target).**  Suppose that Assumption 3.1 and wealth dynamics (3.7)-(3.8) hold. The optimal controls of problems IR (\( \gamma = \gamma_{ir}^{\hat{p}} \)) and QD (\( \beta = \beta_{qd}^{\hat{p}} \)) achieve the required expected value target \( E^{\hat{p}_{ir,wo}}_{\hat{W}_{ir}} [W_{ir}^{\hat{p}}(T)] \equiv \mathcal{E}, \ j \in \{ir, qd\} \), provided \( \gamma_{ir}^{\hat{p}} \) and \( \beta_{qd}^{\hat{p}} \) are given respectively by

\[
\gamma_{ir}^{\hat{p}} = \frac{(\mathcal{E} - \mathcal{K})}{(1 - e^{-\eta r})}, \quad \text{and} \quad \beta_{qd}^{\hat{p}} = \frac{1}{T} \log \left( \frac{\mathcal{E} - \frac{\mathcal{K}}{r} (1 - e^{-T}) + w_0}{\frac{\mathcal{K}}{r} (1 - e^{-T}) + w_0 e^{(r-\eta)T}} \right), \quad (A.3)
\]

where \( \eta \) is given by \( (3.6) \).

**Proof.**  Follows from re-arranging (3.17) and (3.34), and using (A.2). □

Note that the results of Lemma A.2 allows for jumps in the risky asset process, as does the following lemma which uses a constant proportion benchmark strategy to achieve (A.1).

**Lemma A.3. (Constant proportion benchmark, \( N_a = 2 \))**  Suppose that Assumption 3.1 and wealth dynamics (3.8) hold. If the benchmark strategy is a constant proportion strategy with \( \hat{p}(t, \hat{W}(t)) \equiv \bar{p} \) for \( t \in [t_0, T], \) then the benchmark terminal wealth has expectation \( E^{\hat{p}_{ir,wo}}_{\hat{W}} [\hat{W}(T)] \equiv \mathcal{K} = K(T) \), where \( t \to K(t) \) is given in terms \( \bar{p} \) by

\[
K(t) := E^{\hat{p}_{ir,wo}}_{\hat{W}} [\hat{W}(t)] = w_0 e^{r+(\mu - r)\bar{p}t} + \frac{q}{r + (\mu - r)\bar{p}} \cdot (e^{r+(\mu - r)\bar{p}t} - 1), \ t \in [t_0, 0, T]. \quad (A.4)
\]

In the particular case where contributions are zero \( (q = 0) \), a given value of \( E^{\hat{p}_{ir,wo}}_{\hat{W}} [\hat{W}(T)] \equiv \mathcal{K} \) can therefore be achieved by choosing the constant \( \bar{p} \) according to

\[
\bar{p} = \frac{1}{(\mu - r)T} \log \left( \frac{\mathcal{K}}{w_0 e^{rT}} \right), \quad (if \ q = 0). \quad (A.5)
\]

**Proof.**  Follows from applying standard analysis techniques to (3.8) after setting \( \hat{p}(t, \hat{W}(t)) \equiv \bar{p} > 0. \) □
The preceding results enable the main comparison result, given in Theorem A.4 below, which allows for jumps in the risky asset process.

**Theorem A.4.** (QD-optimal vs. IR-optimal strategies: Risky asset exposure over time) Suppose the following assumptions hold: (i) Assumption A.1 and wealth dynamics (3.7)-(3.8); (ii) the investor compares investment strategies on the basis of Assumption A.2; (iii) contributions are zero (q = 0); (iv) the benchmark strategy is a constant proportion strategy with \( \hat{p}(t, W(t)) \) a constant proportion strategy with \( \hat{p}(t, W(t)) \) and \( \hat{p} > 0 \) for \( t \in [t_0, T] \). Note that \( X_{t_0}^q(t) = (W_{t_0}^q(t), W(t), \hat{p}(t, W(t))) \) and \( X_{t_0}^q(T) = (W_{t_0}^q(t), W(t), \hat{p}(t, W(t))) \).

Then, at inception \( t = t_0 \), the IR-optimal strategy requires a larger investment in the risky asset than the QD-optimal strategy,

\[
p_{t_0}^{E_q}(t_0, X_{t_0}^q(t_0)) > p_{t_0}^{E_q}(t_0; X_{t_0}^q(t_0)). \tag{A.6}
\]

At maturity \( t = T \), the IR-optimal strategy is expected to invest less wealth in the risky asset than the QD-optimal strategy,

\[
E_{t_0}^{t_0, w_0} \left[ p_{t_0}^{E_q} \left( t, X_{t_0}^q(T) \right) \cdot W_{t_0}^q(T) \right] < E_{t_0}^{t_0, w_0} \left[ p_{t_0}^{E_q} \left( T, X_{t_0}^q(T) \right) \cdot W_{t_0}^q(T) \right]. \tag{A.7}
\]

Furthermore, if it is additionally assumed that \( \eta \) (see (3.6)) satisfies \( \eta > r \), then the function

\[
t \to f(t) := E_{t_0}^{t_0, w_0} \left[ p_{t_0}^{E_q} \left( t, X_{t_0}^q(t) \right) \cdot W_{t_0}^q(t) \right] - E_{t_0}^{t_0, w_0} \left[ p_{t_0}^{E_q} \left( t, X_{t_0}^q(t) \right) \cdot W_{t_0}^q(t) \right]
\]

is monotonically decreasing on \( t \in [t_0, T] \).

**Proof.** Since we are assuming that \( K > w_0e^{rT} \) (see (A.1)), the assumption that \( \mu > r \) (see Section 3) together with (A.5) implies that \( \hat{p} > 0 \). Combining, under the stated assumptions, the results (3.15), (3.32), (A.2), (A.3) and (A.5), tedious algebra results in the function \( f(t) \) in (A.8) given by the following expression on \( t \in [t_0 = 0, T] \),

\[
f(t) = \frac{(\xi - K)w_0}{(\mu - r)(K - w_0e^{(r - \eta)t})} \cdot \left( \frac{\eta}{1 - e^{-\eta t}} \right) \cdot \left( \frac{K}{w_0e^{rT}} - 1 \right) \cdot e^{(r - \eta)t} - \frac{1}{T} \log \left( \frac{K}{w_0e^{rT}} \right) \cdot \left( \frac{K}{w_0} \right)^{t/T}. \tag{A.9}
\]

The results (A.6), (A.7) and (A.8) follow from an analysis of the properties of the function \( f \) (A.9).

Note that the additional requirement \( \eta > r \) leading to (A.8) is indeed satisfied in the case of typical process parameters, including by the parameters in Table 5.3.

Theorem A.4 suggests that in order to achieve the same expected value of terminal wealth, the IR-optimal strategy requires comparatively more extreme positions in the risky (riskiest) asset than the QD strategy, and in that qualitative sense the IR strategy is expected to be less diversified than the QD strategy at some points during the investment time horizon. Specifically, compared to the QD strategy, the IR strategy relies on a larger investment in the riskiest asset early in the investment time horizon, and once the desired outperformance becomes increasingly likely, the exposure to the riskiest asset is expected to be reduced to a level below that of the QD strategy.

The results of Theorem A.4 are illustrated in Figure A.1. Specifically, Figure A.1(a) shows the expected (average) amount invested in the risky asset over time according to each strategy, while Figure A.1(b) shows that the difference \( f(t) \) as per (A.8) remains monotonically decreasing in this case where there are nonzero contributions to the portfolio (in contrast to the assumptions of Theorem A.4).

Note that the qualitative implications of Theorem A.4 hold even if the underlying assumptions are relaxed and the problems are considered in a more realistic setting (see Section 5), such as when multiple investment constraints are applied and contributions are nonzero. As a result, Theorem A.4 is valuable both for confirming the numerical results of Section 4.

**Appendix B: Additional numerical results**

In this appendix, additional numerical results are presented which relate to the various sections of the paper as indicated.
\[ E^{ir,qd}_j(t) = E^{ir,qd}_j(t, X_j^*(t)) \cdot W_j^X(T), \quad j \in \{ir, qd\} \]

**Figure A.1**: Analytical solutions, no constraints, investor portfolio P0, benchmark BM0: Expected amount \( t \rightarrow E^{ir,qd}_j(t, X_j^*(t)) \cdot W_j^X(T), \quad j \in \{ir, qd\} \) in risky asset, as well as the resulting difference \( f(t) \) as per (A.8). 10⁶ Monte Carlo simulations, \( \mathcal{E} = 400 \) in \( \mathcal{A.2} \).

### B.1 Cumulative distribution functions (CDFs)

As a supplement to the results in Subsection 5.3, Figure B.2 illustrates CDFs corresponding to the PDFs presented in Figure 5.1. Recall that Lemma 3.14 focused on just one point of the CDF, whereas Figure B.2(b) illustrates the complete CDFs. We observe that Figure B.2 appears to show a form of (partial) stochastic dominance of IR over QD for wealth outcomes below the mean \( \mathcal{E} \) (see [Van Staden et al. (2021a)] for a definition and discussion).

However, the situation changes when realistic investment constraints are applied, as is the case in Figure B.3, which illustrates the corresponding CDFs to the PDFs presented in Figure 5.4 (Subsection 5.4). In this case, it appears that QD effectively achieves stochastic dominance over IR (and not just partial stochastic dominance for downside outcomes) regardless of whether wealth or the wealth ratio is considered.

**Figure B.2**: Analytical solutions, no constraints, investor portfolio P0, benchmark BM0: Simulated CDFs of benchmark and investor’s target terminal wealth \( \hat{W}(T) \) and \( W_j^X(T), \) respectively, as well as the ratio \( W_j^X(T) / \hat{W}(T), \) for \( j \in \{ir, qd\}. \) 10⁶ Monte Carlo simulations, \( \mathcal{E} = 400 \) in \( \mathcal{A.1} \).

**Figure B.3**: Out-of-sample (testing) results for DS1 using numerical solutions, with constraints, investor portfolio P1, benchmark BM1: Simulated CDFs of benchmark and investor’s target terminal wealth \( \hat{W}(T) \) and \( W_j^X(T), \) respectively, as well as the ratio \( W_j^X(T) / \hat{W}(T), \) for \( j \in \{ir, qd\}. \)
B.2 Portfolio P1, benchmark BM1, data set DS2

As a supplement to the results of Subsection 5.4, Table B.1 presents results for using investor portfolio P1 to outperform benchmark BM1 on data set DS2. Compared to Table 5.8 which is based on DS1, we observe that the qualitative aspects of the comparative performance of the IR and QD-optimal strategies also hold on data set DS2.

Table B.1: Numerical solutions, with constraints, investor portfolio P1, benchmark B1, data set DS2: Training and testing results for mean terminal wealth $E = 430$ ($\hat{\beta} \approx 1.7\%$ in (5.1)) on the training data.

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$W(T)$</td>
<td>$W^*_j(T)$</td>
</tr>
<tr>
<td>----------------</td>
<td>--------</td>
<td>------------</td>
</tr>
<tr>
<td>BM1 IR QD</td>
<td>364</td>
<td>430</td>
</tr>
<tr>
<td></td>
<td>212</td>
<td>249</td>
</tr>
<tr>
<td>5th pctile</td>
<td>235</td>
<td>286</td>
</tr>
<tr>
<td>95th pctile</td>
<td>354</td>
<td>422</td>
</tr>
<tr>
<td>Prob. underp.</td>
<td>531</td>
<td>601</td>
</tr>
</tbody>
</table>

B.3 Ground truth results

In order to show that the numerical solutions obtained as described in Section 4 can converge under suitable conditions to the closed-form solutions as described in Section 3, we immediately encounter the problem that the numerical solutions are explicitly constructed (via the NN output layer activation function) to enforce the desired investment constraints. While a different output layer activation function could be implemented, the treatment of trading in the case of insolvency (i.e when wealth crosses zero into the negative domain) needs to be carefully addressed in any numerical solution.

Instead of modifying the methodology used to obtain numerical solutions, we observe that if a relatively short time horizon (e.g. $T = 1$ year) is combined with a reasonable outperformance target (e.g. $\hat{\beta} \approx 1.0\%$ in (5.1)), then the probability of insolvency is negligible, as is the need for leverage or short-selling in the closed-form solutions. This allows us to use the numerical solutions (with constraints) to approximate the closed-form solutions (no constraints), provided the underlying data is the same. This latter requirement is readily achieved by using parametric models with parameters as in Table 5.3 to simulate paths of the underlying assets. Analytical investment strategies are calculated based on these parameters, while the numerical approach uses the sample paths based on these parameters as training data for the neural network.

The results, obtained using $10^6$ Monte Carlo simulations, are compared in Table B.2 for investor portfolio P0, benchmark BM0. Note that we assume contributions are zero, $q = q(t_n) = 0$ to avoid discrete approximation errors when comparing a continuous contribution rate to discrete contribution amounts made at rebalancing times.

As for the NN parameters to obtain these results, we used two hidden layers, each with $N_a + 2$ nodes. For training the NN, 64,000 stochastic gradient steps (using the Gadam algorithm, see Section 4) were used, each implementing a mini-batch size of 100 paths. Numerical tests showed that results with this configuration are very stable and reliable, for example nearly identical results are obtained when the NN is trained multiple times on the same underlying data.

Table B.2 shows that in this scenario, the numerical results using the data-driven NN approach as described in Section 4 indeed recover the analytical results obtained as per Section 3.
Table B.2: Ground truth comparison, investor portfolio P0, benchmark BM0, and data set DS0 used for NN training data: \( w_0 = 100, q = q(t_n) = 0, T = 1 \) year. Since BM0 results in an expected terminal wealth \( K = 104.20 \), a value of \( E = 105.25 \) implies \( \beta \approx 1.0\% \). Analytical solutions based on 360 rebalancing events approximating continuous rebalancing. Numerical results are based on only 36 discrete rebalancing events to ensure that computation times remain reasonable.

<table>
<thead>
<tr>
<th>Quantity</th>
<th>Analytical solutions: P0</th>
<th>Numerical solutions (using NN): P0</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>BM0</td>
<td>( W^*_T )</td>
</tr>
<tr>
<td>W (T)</td>
<td></td>
<td></td>
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<tr>
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<td>105.2</td>
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<td>117.9</td>
<td>121.9</td>
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</table>