Comparison between the Mean Variance optimal and the Mean Quadratic Variation optimal trading strategies

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Abstract

We compare optimal liquidation policies in continuous time in the presence of trading impact using numerical solutions of Hamilton Jacobi Bellman (HJB) partial differential equations (PDE). In particular, we compare the path dependent, time-consistent mean-quadratic-variation strategy with the path-independent, time-inconsistent (pre-commitment) mean-variance strategy. We show that the two different risk measures lead to very different strategies and liquidation profiles. In terms of the optimal trading velocities, the mean-quadratic-variation strategy is much less sensitive to changes in asset price and varies more smoothly. In terms of the liquidation profiles, the mean-variance strategy is much more variable, although the mean liquidation profiles for the two strategies are surprisingly similar. On a numerical note, we show that using an interpolation scheme along a parametric curve in conjunction with the semi-Lagrangian method results in significantly better accuracy than standard axis-aligned linear interpolation. We also demonstrate how a scaled computational grid can improve solution accuracy.

1 Introduction

Algorithmic trade execution has become a standard technique for institutional market players in recent years, particularly in the equity market where electronic trading is most prevalent. A trade execution algorithm typically seeks to execute a trade decision optimally upon receiving inputs from a human trader.

A common form of optimality criterion seeks to strike a balance between minimizing pricing impact and minimizing timing risk. For example, in the case of selling a large number of shares, a fast liquidation will cause the share price to drop, whereas a slow liquidation will expose the seller to timing risk due to the stochastic nature of the share price.

Several approaches have been suggested in the literature to quantify the minimization of pricing impact and timing risk. The first, and perhaps the most intuitive, approach maximizes the expected revenues while minimizing a risk criterion, for example, variance [14, 13], quadratic variation [2], or value-at-risk (VaR) [17]. Another approach maximizes the expected value of a utility function of revenues, for example, a power-law function or an exponential function [18, 29, 26]. The third approach minimizes the expected execution cost [7]. All these three approaches model the asset price process in the presence of pricing impact. Yet another approach, which is somewhat tangential to the above methodologies, minimizes the expected execution cost by modelling the dynamic distribution of bid and ask orders in a limit order book [27, 1].

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In this paper we focus on maximizing revenue while minimizing a risk measure. This is a popular approach in industry. More specifically, we compare the pre-commitment mean-variance strategy [14, 6, 25] with the mean-quadratic-variation strategy [15, 11]. We assume that trading takes place continuously at a finite rate, as in [4, 3]. We note that the risk-criteria in the seminal paper [2] was previously thought to be variance but is actually quadratic variation, as shown in [15]. This is also discussed in [24]. Therefore, it is interesting to investigate how suboptimal the mean-quadratic-variation strategy is in terms of mean-variance efficiency, and, conversely, the question of how suboptimal the mean-variance strategy is in terms of mean-quadratic-variation efficiency.

The main contributions of this article are:

• We show that for the same variance, the mean-quadratic-variation strategy can have a significantly suboptimal expected value compared to the mean-variance strategy. For the same quadratic-variation, the mean-variance strategy can have significantly suboptimal expected value compared to the mean-quadratic-variation strategy. If one wants to strike the middle ground of balancing both variance and quadratic-variation, the mean-variance strategy seems to be preferable.

• We show that the mean-variance strategy is much more sensitive to changes in the asset price than the mean-quadratic-variation strategy. Consequently, the trading profile of the mean-variance strategy is much more variable. The mean trading profiles of the two strategies, however, turn out to be surprisingly similar.

• We have improved our numerical method in [14] so that results for very challenging parametric cases can be computed using reasonable time and memory. In particular, we improve the method of interpolation at the foot of the characteristic in the semi-Lagrangian discretization of HJB PDEs. We also construct a scaled computational grid so that fewer grid nodes are needed to obtain accurate results. The improved method also guarantees convergence to the viscosity solution.

• The mean-variance formulation of the optimal liquidation problem is known to be ill-posed [14]. More specifically, many similar strategies can give rise to nearly the same efficient frontier. In this paper we analyze in detail this ill-posedness from both a mathematical and a computational perspective. In particular, we highlight the numerical challenges created by such ill-posedness and demonstrate that the choice of interpolation method can be critical.

2 Optimal Execution

Let

\[
S = \text{Price of the underlying risky asset},
\]

\[
B = \text{Balance of risk free bank account},
\]

\[
A = \text{Number of shares of underlying asset}.
\]

The optimal execution problem over \( t \in [0, T] \) has the initial condition

\[
S(0) = s_{\text{init}}, B(0) = 0, A(0) = \alpha_{\text{init}},
\]

(2.1)

If \( \alpha_{\text{init}} > 0 \), the trader is liquidating a long position (selling). If \( \alpha_{\text{init}} < 0 \), the trader is liquidating a short position (buying). In this article, for definiteness, we consider the selling case. At \( t = T \),

\[
S = S(T), B = B(T) = B_L, A = \alpha_{\text{end}} = 0,
\]

(2.2)

where \( B_L \) is the cash generated by selling shares and investing in the risk free bank account \( B \), with a final liquidation at \( t = T^- \) to ensure that \( \alpha_{\text{end}} = 0 \). The objective of optimal execution is to maximize \( B_L \), while at the same time minimizing a certain risk measure. The two risk measures we consider in this paper, namely variance and quadratic-variation, will be discussed in the next two sections.
We suppose that the price of the risky asset $S$ follows a geometric Brownian Motion (GBM), where the drift term is modified due to the permanent price impact of trading [4]:

$$dS = (\eta + g(v))S \, dt + \sigma S \, dZ,$$

$\eta = \text{drift rate}$,

$g(v) = \text{the permanent price impact function}$,

$\sigma = \text{volatility}$.

$dZ$ is the increment of a Wiener process. \hspace{1cm} (2.3)

Let the trading rate $v$ be

$$v = \frac{dA}{dt}. \hspace{1cm} (2.4)$$

It follows that the rate of cash flow into the bank account is

$$\frac{dB}{dt} = rB - vf(v)S,$$

where $f(v)$ is the temporary price impact function, and $r = \text{risk free rate}$. \hspace{1cm} (2.5)

Since $v < 0$ for selling, the term $(-vf(v)S)$ represents the rate of cash obtained by selling shares at price $f(v)S$ at a rate $v$.

### 2.1 Trading impact function

The temporary price impact function $f(v)$ is assumed to be of the form

$$f(v) = (1 + \kappa_s \text{sgn}(v)) \exp[\kappa_t \text{sgn}(v)|v|^\beta],$$

$\kappa_s = \text{the bid-ask spread parameter}$,

$\kappa_t = \text{the temporary price impact factor}$,

$\beta = \text{the price impact exponent}$. \hspace{1cm} (2.6)

For various studies which suggest the form (2.6), see [4, 23, 28].

The permanent price impact function $g(v)$ is assumed to be of the form

$$g(v) = \kappa_pv,$$

$\kappa_p = \text{the permanent price impact factor}$. As explained in [14], this form of permanent price impact function eliminates the possibilities of round-trip arbitrage [4, 19].

### 2.2 Definition of liquidation value

Given the state variables $(S, B, A)$ the instant before the end of trading $t = T^-$, we have one final liquidation (if necessary) so that the number of shares owned at $t = T$ is $\alpha_{\text{end}} = 0$. The liquidation value $B_L$ after this final trade is defined to be

$$B_L = B + AS \lim_{v \to -\infty} f(v). \hspace{1cm} (2.7)$$

\hspace{1cm} \footnote{In the case of liquidating a short position (buying), which is not considered in the current paper, equation (2.7) should be defined as $B_L = \max\{B + AS \lim_{v \to -\infty} f(v), B_{\min}\}$ for some large value of $|B_{\min}|$ so that $B_L$ does not become $-\infty$. In this case we are effectively penalizing the strategy if $\alpha_{\text{end}} \neq 0$.}
3 Mean-Variance Strategy

We review here the pre-commitment mean-variance strategy, as discussed in [14]. In this case, the optimal strategy solves the following optimization problem,

\[
\mathcal{V}^{MV}(s, b, \alpha, t_0, \hat{\lambda}) = \sup_{\nu_{t_0}} \left\{ \int_{t_0}^{T} \left( \nu(t') B(t') A(t') dS(t') \right)^2 \right\},
\]

where we have used the definitions

\[
E^{s,b,\alpha,t_0}_{\nu_{t_0}}[B_L] = E[B_L | S(t_0) = s, B(t_0) = b, A(t_0) = \alpha] \quad \text{as seen at time } t_0,
\]

\[
\text{Var}^{s,b,\alpha,t_0}_{\nu_{t_0}}[B_L] = \text{Var}[B_L | S(t_0) = s, B(t_0) = b, A(t_0) = \alpha] \quad \text{as seen at time } t_0,
\]

\[\text{where we have used the definitions}\]

\[
E^{s,b,\alpha,t_0}_{\nu_{t_0}}[B_L] = E[B_L | S(t_0) = s, B(t_0) = b, A(t_0) = \alpha] \quad \text{as seen at time } t_0,
\]

\[
\text{Var}^{s,b,\alpha,t_0}_{\nu_{t_0}}[B_L] = \text{Var}[B_L | S(t_0) = s, B(t_0) = b, A(t_0) = \alpha] \quad \text{as seen at time } t_0,
\]

\[\text{where } \nu_{t_0}(\cdot) = \nu_{t_0}(S(t'), B(t'), A(t'), t') \geq t_0, \text{ is the policy controlling the stochastic processes (2.4), (2.5) and (2.3) and } Z \text{ is the set of admissible controls. The Lagrange multiplier } \hat{\lambda} \geq 0 \text{ can be interpreted as a coefficient of risk aversion. Varying } \hat{\lambda} \text{ over } [0, \infty) \text{ traces out a variance-minimizing frontier in the expected value, standard deviation plane. Since the variance-minimizing frontier is a superset of the mean-variance efficient frontier, the latter can be obtained from the former by a simple sorting procedure.}
\]

Let us denote by \( v^{\ast}_{s,b,\alpha,t_0,\hat{\lambda}}(S(t'), B(t'), A(t'), t') \geq t_0 \) the optimal policy for the problem (3.1) which optimizes mean-variance trade off as seen at time \( t_0 \). Now consider the problem

\[
\mathcal{V}^{MV}(s, b, \alpha, t_1, \hat{\lambda}) = \sup_{\nu_{t_1}} \left\{ \int_{t_1}^{T} \left( \nu(t') B(t') A(t') dS(t') \right)^2 \right\},
\]

which optimizes mean-variance takeoff as seen at time \( t_1 > t_0 \). We denote this policy by \( v^{\ast}_{s,b,\alpha,t_1,\hat{\lambda}}(S(t'), B(t'), A(t'), t') \geq t_1 \). In general

\[
v^{\ast}_{s,b,\alpha,t_0,\hat{\lambda}}(S(t'), B(t'), A(t'), t') \neq v^{\ast}_{s,b,\alpha,t_1,\hat{\lambda}}(S(t'), B(t'), A(t'), t') \quad \text{for } t' \geq t_1,
\]

(i.e. solution of problem (3.1) is time-inconsistent [6, 10]. Therefore, a dynamic programming principle cannot be directly applied to solve this problem. To tackle this, we follow the method in [9, 5, 16, 22] to pose (3.1) as a convex optimization problem. For each fixed value of \( s, b, \alpha, t_0 \) and \( \hat{\lambda} \), the optimal control \( v^{\ast}_{s,b,\alpha,t_0,\hat{\lambda}}(\cdot) \) of problem (3.1) is also the optimal control of the problem

\[
\nu_{t_0}(S(t'), B(t'), A(t'), t' \geq t_0) \in Z \quad \text{and } \left( B_L - \frac{\gamma(s, b, \alpha, t_0, \hat{\lambda})}{2} \right)^2,
\]

(3.3)

for some \( \gamma(s, b, \alpha, t_0, \hat{\lambda}) \geq 0 \). Conversely, for each fixed value of \( s, b, \alpha, t_0 \) and \( \gamma \), the optimal control \( v^{\ast}_{s,b,\alpha,t_0,\gamma}(\cdot) \) of problem

\[
\nu_{t_0}(S(t'), B(t'), A(t'), t' \geq t_0) \in Z \quad \text{and } \left( B_L - \frac{\gamma}{2} \right)^2,
\]

(3.4)

is also the optimal control of the problem (3.1) for a certain value of \( \hat{\lambda} \).

The benefit of reformulating (3.1) as (3.4) is that the dynamic programming principle can be directly applied to (3.4). We note that varying \( \gamma \) over \([0, \infty)\) traces out a variance-minimizing frontier in the expected value, standard deviation plane.

4 Mean-Quadratic-Variation Strategy

Instead of using the variance/standard deviation as the risk measure, we can use the quadratic variation [15],

\[
\int_{t_0}^{T} \left( \nu(t') A(t') dS(t') \right)^2,
\]

(4.1)
which accumulates the future value of the instantaneous risk, i.e. \((A(t') \, dS(t'))^2\), due to holding \(A(t')\) units of the risky asset \(S\). When properly normalized, quadratic variation can also be interpreted as an average standard deviation per unit time [11]. Note that we have defined the quadratic variation as the future value of the instantaneous risk to be consistent with [10].

In this case the optimal policy solves the following optimization problem,

\[
V^{MQV}(s, \alpha, t_0, \lambda) = \sup_{v_{t_0}(S(t'), A(t'), t' \geq t_0) \in \mathbb{Z}} E_{v_{t_0}(\cdot)}^{s, \alpha, t_0} \left\{ \int_{t_0}^{T} e^{r(T-t')} \left(-vf(v)S\right) dt' - \lambda \int_{t_0}^{T} \left(e^{r(T-t')} A(t') \, dS(t')\right)^2 \right\}, \tag{4.2}
\]

where \(\lambda\) is a given Lagrange multiplier, subject to the stochastic processes (2.4), (2.5) and (2.3). Let \(v^*_{s, \alpha, t_0, \lambda}(S(t'), A(t'), t'), t' \geq t_0\) be the optimal policy for problem (4.2) which minimizes quadratic variation from time \(t_0\) onwards; and \(v^*_{s, \alpha, t_1, \lambda}(S(t'), A(t'), t'), t' \geq t_1\) be the optimal policy for problem (4.2) which minimizes quadratic variation from time \(t_1\) onwards, where \(t_1 > t_0\). In this case,

\[
v^*_{s, \alpha, t_0, \lambda}(S(t'), A(t'), t') = v^*_{s, \alpha, t_1, \lambda}(S(t'), A(t'), t') \quad ; \quad t' \geq t_1, \tag{4.3}
\]

i.e. the solution of problem (4.2) is time-consistent. Hence, dynamic programming can be directly applied to this problem.

In certain special cases, it is known that strategy (4.2) is equivalent to a time consistent mean variance strategy [10, 30]. Hence (4.2) can be viewed as a natural time consistent strategy. In addition, as shown in [15], if arithmetic Brownian motion is assumed, the optimal strategy from (4.2) is actually identical to the optimal strategy in [2].

Let \(V^{MQV}(s, \alpha, \tau) = V^{MQV}(s, \alpha, t_0 = T - \tau, \lambda)\). The derivation in [15] shows that \(V^{MQV}\) satisfies the HJB equation

\[
V^{MQV}_\tau = \eta s V^{MQV}_s + \frac{\sigma^2 s^2}{2} V^{MQV}_{ss} - \lambda e^{2r \tau} \alpha^2 s^2 \sigma^2 + \sup_{v(\cdot) \in \mathbb{Z}} \left[ e^{r \tau} (-vf(v))s + g(v)s V^{MQV}_s + v V^{MQV}_\alpha \right]. \tag{4.4}
\]

We note that both the value function \(V^{MQV}\) and strategy \(v(\cdot)\) for the mean-quadratic-variation problem (4.2) is independent of the current bank account balance \(B\). In particular \(v = v(s, \alpha, \tau)\).

The reader is referred to [15] for details about the numerical method used to solve equation (4.4).

### 4.1 Exact solution for Arithmetic Brownian Motion

Under the arithmetic Brownian motion approximation and the additional assumptions of zero drift, zero interest rate, unbounded control and linear price impact function detailed in [15], the optimal trading strategy has the analytic solution

\[
v(\alpha, \tau) = -\alpha K \coth(K \tau), \tag{4.5}
\]

where \(K = \sqrt{\lambda \sigma^2 s_{init} / s_t}\). Note that the optimal strategy is independent of the spot price \(s\).

As noted in [15], the arithmetic Brownian motion approximation results in an efficient frontier that is extremely close to the true mean-quadratic-variation efficient frontier computed assuming geometric Brownian motion. We also note that (4.5) is the same strategy as used in [2].

### 5 Comparison between the two strategies

#### 5.1 Risk measure

The pre-commitment mean-variance strategy is optimal in the following sense [24]. Consider trade execution in an idealized world where all our modeling assumptions are valid, and where the execution “experiments”
can be repeated many times starting from the same initial condition at $t = 0$. Suppose we measure the performance of a strategy by computing the mean and standard deviation of completed trades. The pre-commitment mean-variance strategy produces the maximum expected revenue for a given standard deviation. Any other strategy (including the time-consistent mean-variance strategy) having the same standard deviation cannot have a larger expected revenue.

The mean-quadratic-variation strategy seeks to control cumulative instantaneous risk, which arises from holding positions in the risky asset. Since holding a large position at any time during trading is penalized, this strategy is useful from a risk management perspective as it avoids taking excessive instantaneous risk. Hence, this may be an appealing strategy to risk managers faced with daily VaR reporting responsibilities.

In contrast to the strong path-dependence of quadratic-variation, the pre-commitment mean-variance objective function (3.1) is path-independent: only $B_L$ matters. Consequently, there is no control of instantaneous risk.

5.2 Uniqueness and smoothness

In mean-variance optimization, many similar strategies can give rise to almost the same efficient frontier (near ill-posedness). This can be advantageous as it permits more flexibility in executing the trade. On the other hand, this creates difficulties in obtaining a smoothly varying optimal strategy, as demonstrated and explained in [14] and the current paper. In our experience, these issues do not arise in mean-quadratic-variation optimization.

6 HJB Formulation: Mean Variance

6.1 Change of Variable

At first glance it seems necessary to solve the problem (3.4) for each value of $\gamma$ separately. Fortunately, this can be avoided by a change of variable. Define the new variable $B$ by

$$B(0) = -\frac{\gamma e^{-\tau T}}{2} \leq 0, \quad dB/dt = rB - vf(v)S$$

(6.1)

it is easy to see that

$$B(t) = B(t) - \frac{\gamma e^{-\tau(T-t)}}{2}, \quad B_L = B_L - \frac{\gamma}{2}.$$  

(6.2)

Since equation (6.1) has the same form as equation (2.5), problem (3.4) is equivalent to

$$\inf_{v_0(S(t'),B(t'),A(t'),t' \geq t_0) \in Z} E_{v_0(\cdot)}^L_B,$$

(6.3)

This change of variable is very convenient in the PDE context because the solutions corresponding to different values of $\gamma$ can be determined by examining the PDE solutions for different values of $B$ at $t = 0$. Therefore, we only need to solve the problem (6.3) once to obtain the entire variance minimizing frontier [14].

6.2 Definitions

Let $\tau = T - t$ be the backward time. Define the value functions $V$ by

$$V = V(s,b,\alpha,\tau) = \inf_{v(\cdot) \in Z} E_{v(\cdot)}^{s,b,\alpha,T-\tau}[B_L^2],$$

(6.4)

where we have simplified notation by writing the control as simply $v(\cdot)$.

Once the optimal control $v^*(\cdot)$ is obtained from solution of equation (6.4), then we need to estimate $E_{v^*(\cdot)}^{s,b,\alpha,T-\tau}[B_L]$, which can be found from

$$U = U(s,b,\alpha,\tau) = E_{v^*(\cdot)}^{s,b,\alpha,T-\tau}[B_L].$$

(6.5)
Using the estimates for $E_{v^*(\cdot)}^{s,\delta,\alpha,T-\tau}[B_2^L]$ and $E_{v^*(\cdot)}^{s,\delta,\alpha,T-\tau}[B_L]$, we can obtain the mean-variance efficient frontier as described in section B.1.

We also define the differential operator $\mathcal{L}$ by

$$\mathcal{L}V = \frac{\sigma^2 s^2}{2} V_{ss} + \eta s V_s,$$

and Lagrangian derivative $\frac{D}{D\tau}(v)$ by

$$\frac{DV}{D\tau}(v) = V_\tau - V_s g(v) s - V_\delta (r \delta - v f(v) s) - V_\alpha v,$$

which is the rate of change of $V$ along the characteristic $s = s(\tau), \delta = \delta(\tau), \alpha = \alpha(\tau)$ defined by the trading velocity $v$ through

$$\frac{ds}{d\tau} = -g(v)s, \quad \frac{d\delta}{d\tau} = -(r \delta - v f(v) s), \quad \frac{d\alpha}{d\tau} = -v.$$

### 6.3 PDE formulation

Following standard arguments, the optimal control (3.4) is given by the solution to the nonlinear HJB equation

$$\mathcal{L}V - \max_{v(\cdot) \in Z} \frac{DV}{D\tau}(v) = 0,$$

in the domain $\Omega = \{s \geq 0, \delta \in \mathbb{R}, \alpha \geq 0, \tau > 0\}$.

In view of definition (2.7), the initial condition at $\tau = 0$ is

$$V(s, \delta, \alpha, \tau = 0) = B_2^L = (\delta + as \lim_{v \to -\infty} f(v))^2.$$

Note that the set of admissible velocities $Z$ is defined such that

$$v(s, \delta, \alpha, \tau) \leq 0, \quad v(s, \delta, \alpha = 0, \tau) = 0,$$

which forbids buying or holding a short position (when liquidating stock).

At $s = 0$, equation (6.9) reduces to

$$\max_{v(\cdot) \in Z} \left\{ V_\tau - r \delta V_\delta - v V_\alpha \right\} = 0.$$

Therefore, no boundary condition at $s = 0$ is needed.

At $\alpha = 0$, (6.11) causes equation (6.9) to reduce to

$$\frac{\sigma^2 s^2}{2} V_{ss} + \eta s V_s - V_\tau + r \delta V_\delta = 0.$$

Therefore, no boundary condition at $\alpha = 0$ is needed.

After $v^*(\cdot)$ has been obtained from the above PDE solve, we can then determine the expected value by solving the linear PDE

$$\mathcal{L}U - \frac{DU}{D\tau}(v^*) = 0, \quad U(s, \delta, \alpha, \tau = 0) = B_L = \delta + as \lim_{v \to -\infty} f(v),$$

which is very inexpensive to solve, since $v^*(\cdot)$ is known.

Solving (6.9) and (6.14) gives us estimates for $E_{v^*(\cdot)}^{s,\delta,\alpha,T-\tau}[B_2^L]$ and $E_{v^*(\cdot)}^{s,\delta,\alpha,T-\tau}[B_L]$, respectively. The mean variance efficient frontier can then be obtained as described in section B.1.
7 Limiting case

For illustration purposes, consider a limiting case with extreme parameter values \( \sigma = \kappa_t = 0 \), and typical parameter values \( r = \kappa_p = \kappa_s = \eta = 0 \). Since the asset price is constant, problem (3.4) degenerates to the deterministic control problem of minimizing \( B_L^2 \). Moreover, since there is no pricing impact, \( B_L = \alpha s + \hat{b} \) with certainty. Consequently, the value function \( V \) is

\[
V(s, b, \alpha, \tau) = \inf_{v(\cdot) \in \mathcal{Z}} E^{s, b, \alpha, T-\tau} \left[ B_L^2 \right] = B_L^2 = (\alpha s + \hat{b})^2,
\]

which can also be verified by direct substitution into the HJB equation (6.9), (6.10) as follows. First, note that the initial condition (6.10) is satisfied because \( f(v) \equiv 1 \). To verify (6.9), note that the parameter values yield the simplifications

\[
\mathcal{L}V = 0, \quad \frac{DV}{D\tau}(v) = V_\tau + vsV_\hat{b} - vV_\alpha.
\]

Substituting (7.1) into (7.2) gives

\[
V_\tau = 0, V_\hat{b} = 2(\alpha s + \hat{b}), V_\alpha = 2s(\alpha s + \hat{b}) \implies \frac{DV}{D\tau}(v) \equiv 0 \text{ for all } v. \tag{7.3}
\]

Since any admissible trading velocity \( v \) is optimal in this case, the problem of determining the optimal control \( v \) is completely ill-posed.

Although the above special case is degenerate, it provides much intuition about what happens for realistic parametric cases. Indeed, in practical parametric cases, the values of \( \sigma \sqrt{T} \) and \( \kappa_t \) are quite small and \( r \) has little effect. Therefore, the solutions for realistic parameters have much in common with the analytic solution for this special case. For example, many similar strategies can give rise to nearly the same mean-variance frontier in practice, as shown in [14].

7.1 Motivation for a parametric curve interpolation method

In using a semi-Lagrangian method [12, 8] to solve for the optimal velocity \( v \), accurate interpolation at the foot of the characteristics is essential to achieving high accuracy [12, 8]. Since a monotone discretization scheme (which allows proof of convergence to the viscosity solution) is at most first-order accurate, we will deal exclusively with linear interpolation in this paper.

Let us consider again the special case in the previous section, where the value function \( V \) has the analytic solution

\[
V(s, b, \alpha, \tau) = (\alpha s + \hat{b})^2. \tag{7.4}
\]

It is obvious that linear interpolation along each of the three coordinate axes is not exact, since the partial derivatives \( V_{ss}, V_{b\hat{b}} \) and \( V_{\alpha\alpha} \) are all non-zero.

Consider linear interpolation along the parametric curve of constant wealth \( \{\alpha s + \hat{b} = \text{constant}\} \):

\[
\frac{ds}{d\zeta} = 0, \quad \frac{d\hat{b}}{d\zeta} = vs, \quad \frac{d\alpha}{d\zeta} = -v, \tag{7.5}
\]

for any fixed trading velocity \( v \). Since \( V \) is constant along this line, linear interpolation along this parametric curve is exact. For general parametric cases, this idea can be generalized to obtain an accurate linear interpolation scheme. See section B.4.2 in the appendix for details.

8 Numerical results

Our discussion on numerical results is organized as follows. First, we explain how we arrive at our parametric cases. Then, we demonstrate convergence by numerical experiments. Finally, we compare the efficient frontiers using the mean variance strategy and the mean quadratic variation strategy.

8
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<thead>
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<th>$T$</th>
<th>$\eta$</th>
<th>$r$</th>
<th>$s_{init}$</th>
<th>$\alpha_{init}$</th>
<th>$\kappa_p$</th>
<th>$\kappa_s$</th>
<th>$\beta$</th>
<th>Action</th>
<th>$v_{\min}$</th>
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<td>100</td>
<td>1.0</td>
<td>0.0</td>
<td>0.0</td>
<td>1.0</td>
<td>Sell</td>
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Table 1: Common parameters

<table>
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<th>Case</th>
<th>$\sigma$</th>
<th>$\kappa_t$</th>
<th>Percentage of Daily Volume</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.0</td>
<td>$2 \times 10^{-6}$</td>
<td>16.7%</td>
</tr>
<tr>
<td>2</td>
<td>0.2</td>
<td>$2.4 \times 10^{-6}$</td>
<td>20.0%</td>
</tr>
<tr>
<td>3</td>
<td>0.2</td>
<td>$6 \times 10^{-6}$</td>
<td>5.0%</td>
</tr>
<tr>
<td>4</td>
<td>0.2</td>
<td>$1.2 \times 10^{-5}$</td>
<td>1.0%</td>
</tr>
<tr>
<td>5</td>
<td>0.2</td>
<td>$2.4 \times 10^{-8}$</td>
<td>0.2%</td>
</tr>
</tbody>
</table>

Table 2: Parametric cases

8.1 Parametric cases

The parametric cases we consider are listed in Table 1 and Table 2. Case 1 corresponds to a high volatility stock with low liquidity. Cases 2-5 correspond to a low volatility stock with various levels of liquidity. These parameters can be related to typical daily volume traded. As described in Appendix A, we estimate that $\kappa_t = 1.2 \times 10^{-7}$ corresponds to liquidating 1% of the daily volume traded of a typical large-cap liquid stock.

In view of our trading model and in particular the temporary trading impact function (2.6) with the choice of $\beta = 1$, we can simulate liquidating $Y\%$ of the daily volume traded by keeping $\alpha_{init}$ unchanged at unity and using $\kappa_t = (1.2 \times 10^{-7})Y$.

8.2 Convergence Analysis

To demonstrate convergence numerically, we compute the mean-variance frontier from the optimal mean-variance strategy as follows. First, we solve the HJB equation (6.9), which gives the optimal control $v^*()$. Given the optimal control, we then solve the linear PDE (6.14) to determine the mean variance frontier (see B.1). We refer to this approach as the PDE method. Alternatively, we can use the optimal $v^*()$ determined from the solution of the HJB equation (6.9), and use this as input to a Monte Carlo simulation, which is used to compute the mean and standard deviation. We refer to this approach as approach as the Monte Carlo method. We emphasize here that in both methods, the control is computed by solving the HJB equation (6.9). Only the method used to compute the mean and standard deviation is different (given the optimal control). An advantage of the Monte Carlo method is that we can compute the quadratic variation easily. Similarly, we can solve the HJB equation (4.4) to determine the optimal control for the mean quadratic variation. Given the control, either a PDE method or Monte Carlo method can be used to compute the mean, standard deviation and quadratic variation.

Tables 3 and 4 show the number of nodes and time steps used in the convergence study for the mean-variance strategy and the mean-quadratic-variation strategy, respectively. Note that only one node is needed in the $b$ direction, since this variable can be eliminated using a similarity reduction (see section B.2 and [14]).\(^2\) The $v$ node discretization is required in order to carry out a linear search to determine the optimal control.

Our parametric curve interpolation scheme (see section B.4.2 in the appendix for details) suggests that the number of $s$ nodes should be significantly more than the number of $\alpha$ nodes, a consideration that is also for the mean-variance strategy, our numerical experiments suggest that the $v$ grid needs to be fine near $t = 0$ (but not when $t$ is larger) to obtain accurate estimate of the optimal $v$ by linear search. In order to have a very fine $v$ grid near $t = 0$ but a coarse $v$ grid elsewhere, we perform 4 additional refinements to the $v$ grid in the last few backward time steps in PDE solve. There is no such concern for the mean-quadratic-variation strategy because the optimal velocity is not determined by linear search (we use a one dimensional optimization method) and hence is not restricted to values in a discrete $v$ grid.

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\(^2\)For the mean-variance strategy, our numerical experiments suggest that the $v$ grid needs to be fine near $t = 0$ (but not when $t$ is larger) to obtain accurate estimate of the optimal $v$ by linear search. In order to have a very fine $v$ grid near $t = 0$ but a coarse $v$ grid elsewhere, we perform 4 additional refinements to the $v$ grid in the last few backward time steps in PDE solve. There is no such concern for the mean-quadratic-variation strategy because the optimal velocity is not determined by linear search (we use a one dimensional optimization method) and hence is not restricted to values in a discrete $v$ grid.
Recall that the Monte Carlo computations use the optimal control determined from solving the HJB equations. We use the stored values of the control from the HJB equations (at the grid nodes) and interpolate these values for Monte Carlo computations (see section C in the appendix). Note also that the same time steps are used in both PDE calculation and Monte Carlo simulations, for each refinement level. For example, the frontiers labeled with “800 time steps” in Figure 1 use the time steps as specified as Refinement Level 2 in Table 3. Similarly for the frontiers labeled with “1600 time steps” and for the frontiers in other figures in the report.

To achieve small sampling error in Monte Carlo simulations, 400,000 simulations are performed for parametric case 1 and 100,000 simulations are performed for each of the other cases. As an example, the sampling error in Figure 1(a) can be estimated as follows. To be more specific, consider a point on the frontier with the maximum standard deviation, which equals 3. Since this standard deviation of $B_L$ is an average over 400,000 simulations, its sampling standard deviation is approximately $\frac{3}{\sqrt{400,000}} \approx 0.0047$, which is negligible in Figure 1(a). Similar calculations will show that the sampling errors are negligible in other figures as well.

As expected, Figures 1 and 2 show that the frontiers computed by both the PDE method and the Monte Carlo method converge to the same frontier as the computational grid is refined. Our numerical results demonstrate that the Monte Carlo frontiers in general converge faster to the limit solution than the PDE frontiers. This may seem counter-intuitive as the Monte Carlo simulations use the optimal trading strategies determined by the PDE method. Nevertheless, it is plausible that Monte Carlo simulations produce a better estimate of the expected value (or standard-deviation/quadratic-variation), which is what our numerical results suggests. Given their better accuracy, we will use the Monte Carlo frontiers to compare the mean-variance strategy with the mean-quadratic-variation strategy. Again, note that the optimal controls are always computed by solving the HJB PDEs.

### 8.3 Comparisons of standard deviation and quadratic variation

Figures 3 to 7 compare the mean-standard-deviation trade off and the mean-quadratic-variation trade off for both the mean variance and the mean quadratic variation strategy. For example, the left plot in Figure 3 compares the results obtained using the mean variance strategy and the mean quadratic variation strategy in terms of using standard deviation as the risk measure. Similarly, the right plot in Figure 3 compares the two strategies in terms of using quadratic variation as the risk measure.

<table>
<thead>
<tr>
<th>Refinement Level</th>
<th>Timesteps</th>
<th>$s$ nodes</th>
<th>$b$ node</th>
<th>$\alpha$ nodes</th>
<th>$v$ nodes</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>200</td>
<td>369</td>
<td>1</td>
<td>11</td>
<td>8</td>
</tr>
<tr>
<td>1</td>
<td>400</td>
<td>737</td>
<td>1</td>
<td>21</td>
<td>15</td>
</tr>
<tr>
<td>2</td>
<td>800</td>
<td>1473</td>
<td>1</td>
<td>41</td>
<td>29</td>
</tr>
<tr>
<td>3</td>
<td>1600</td>
<td>2945</td>
<td>1</td>
<td>81</td>
<td>57</td>
</tr>
</tbody>
</table>

**Table 3:** Grid and time step data for convergence studies for the mean variance strategy. The same time steps are used in both PDE calculation and Monte Carlo simulations. Note that there is only one $b$ node because of the use of similarity reduction (see section B.2 and [14]).

<table>
<thead>
<tr>
<th>Refinement Level</th>
<th>Timesteps</th>
<th>$s$ nodes</th>
<th>$\alpha$ nodes</th>
<th>$v$ nodes</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>800</td>
<td>67</td>
<td>41</td>
<td>30</td>
</tr>
<tr>
<td>1</td>
<td>1600</td>
<td>133</td>
<td>81</td>
<td>59</td>
</tr>
</tbody>
</table>

**Table 4:** Grid and time step data for convergence studies for the mean quadratic variation strategy from [15]. The same time steps are used in both PDE calculation and Monte Carlo simulations. The Monte Carlo computations interpolate the optimal control from the PDE grid values.
Figure 1: Mean variance strategy: convergence of frontiers for the PDE computation and Monte Carlo simulations. The frontiers labeled with PDE are obtained from the PDE solutions. The frontiers labeled with MC are obtained from Monte Carlo simulations. The Monte Carlo computations use the optimal controls determined by solving the HJB equation (6.9).
Figure 2: Mean quadratic variation strategy: convergence of frontiers for the PDE computation and Monte Carlo simulations. The frontiers labeled with PDE are obtained from the PDE solutions. The frontiers labeled with MC are obtained from Monte Carlo simulations. The Monte Carlo computations use the optimal controls determined by solving the HJB equation (4.4).
Several conclusions can be drawn from the comparisons. As one would expect, in terms of using standard deviation as the risk measure, the mean variance optimal strategy dominates the mean quadratic variation optimal strategy. Conversely, in terms of using quadratic variation as the risk measure, the mean quadratic variation optimal strategy dominates the mean variance optimal strategy. However, it appears that the mean variance optimal strategy performs reasonably well using either risk measure. The difference between the two strategies is most pronounced at lower risk levels.

8.3.1 Remarks

Market practitioners may consider expected implementation shortfall (the relative difference between expected value and initial stock price) of 10 basis points to be significant. To achieve small implementation shortfall, liquidation must be done slowly to reduce trading impact, at the expense of increasing timing risk. Striking a good balance is important here, as it might not be wise to aim at an expected shortfall of 10 bps if the risk (as measured by either standard deviation or mean quadratic variation) is several times larger. Our plots show that risk can be several times of a 10 bps expected shortfall in the parametric cases (a) $\sigma = 1.0$, 16.7% daily volume; (b) $\sigma = 0.2$, 20% daily volume; and (c) $\sigma = 0.2$, 5% daily volume.

The analysis above suggests that one way to choose a risk aversion level on an efficient frontier is to choose a ratio between the implementation shortfall and risk. Alternatively, a common practice among market practitioners is to pick the “corner of the frontier”. Our plots show that picking the corner can result in expected implementation shortfall much larger than 10 bps.

We also note that VWAP (Volume Weighted Average Price), i.e. the total traded value divided by the total traded quantity, is a popular benchmark used to evaluate the performance of a trading algorithm [20]. A VWAP tracking algorithm typically varies the trading rate according to a historical volume profile. For example, if trading volume is typically larger near market open, a VWAP tracking algorithm will trade faster near market open. Since trading volume can be considered as a proxy for market liquidity, a VWAP tracking algorithm minimizes overall trading impact costs in a certain sense.

In general, the volume profile used by a VWAP algorithm is not constant. In the special case of using a constant volume profile in a VWAP algorithm, the VWAP-tracking strategy is equivalent to trading at a constant rate. Recall that trading at a constant rate corresponds to the right-most points on the frontiers in Figures 3 to 7. Since the frontiers are quite flat near the right-end, our results suggest that it may not be wise to aim at achieving VWAP, as this may give rise to too much timing risk. It seems wiser to reduce some timing risk by giving up some little expected value, i.e. moving towards the “corner of the frontier”.

8.4 Comparison of strategies for similar expected values

In this section we compare the mean-variance strategy with the mean-quadratic-variation strategy when they give similar expected values. In particular, we focus on the parametric case $\sigma = 1$, $\kappa_i = 2 \times 10^{-6}$ since the differences are more apparent when volatility and pricing impacts are larger.

Figures 8 to 11 correspond to comparisons across four horizontal lines in Figure 3, with four different expected values chosen to represent the more interesting part of the frontiers. For example, in Figure 8 both strategies give an expected value of around 99.29. For the mean-variance strategy, this corresponds to $\gamma = 199.82$; for the mean-quadratic-variation strategy, this corresponds to $\lambda = 1$.

8.4.1 Common observations for each level of expected value

In each of Figures 8 to 11, the subplots labeled (a) and (b) compare the optimal trading velocities at $t = 0$, where we normalized the trading velocities so that a normalized velocity of $-1.0$ corresponds to the constant liquidation rate $-\alpha_{\text{init}}/T$. It is clear that while both strategies are aggressive in the money (sell faster as price becomes more favorable), the sensitivity in the mean-variance strategy is much more non-linear. More specifically, around the initial asset price $s_{\text{init}} = 100$, the optimal control for the mean-variance strategy is a curve with rapidly changing slope whereas that for the mean-quadratic-variation strategy is more or less
Figure 3: Comparison between mean variance strategy and mean quadratic variation strategy for the case $\sigma = 1.0, \kappa_t = 2 \times 10^{-6}$.

Figure 4: Comparison between mean variance strategy and mean quadratic variation strategy for the case $\sigma = 0.2, \kappa_t = 2.4 \times 10^{-6}$. 
Figure 5: Comparison between mean variance strategy and mean quadratic variation strategy for the case $\sigma=0.2$, $\kappa_t = 6 \times 10^{-7}$.

Figure 6: Comparison between mean variance strategy and mean quadratic variation strategy for the case $\sigma=0.2$, $\kappa_t = 1.2 \times 10^{-7}$. 
Figure 7: Comparison between mean variance strategy and mean quadratic variation strategy for the case $\sigma=0.2$, $\kappa_t = 2.4 \times 10^{-8}$.

A straight line. It is also worth noting that the optimal selling rates at $s_{\text{init}} = 100$ for the mean-variance strategy are close to but slightly larger than those for the mean-variation strategy in Figures 8 to 11.

Note that the trading velocity in Figure 8 (b) is nonsmooth for large values of asset price. This appears to be due to the near illposedness of the mean-variance formulation, as discussed in [14].

In each of Figures 8 to 11, the subplots labeled (c) and (d) compare the mean and standard deviation, respectively, of the liquidation profiles $\alpha(t)$ of the mean-variance and the mean-quadratic-variation strategies over the trading horizon. It is interesting to note that while the mean profiles are very similar, the standard deviation profile of the mean-variance strategy is much larger than that of the mean-quadratic-variation strategy. This reflects the fact that the mean-variance strategy is much more sensitive to change in asset price during the liquidation, which is also suggested by the strategy subplots. We also note that the mean profiles are convex, so that the mean liquidation rate is always decreasing over time.

8.4.2 Differences among different levels of expected value

As we move from Figure 8 to 11, the expected value is increasing, and so as the standard deviation and quadratic variation. By comparing subplots (a) and (b), we see that the optimal selling rates become slower as expected value increases. Recall that the mean profiles are convex, so that the mean liquidation rate is always decreasing over time. By comparing subplots (c), we observe that the convexities of the mean liquidation profiles diminish as expected value increases, and the mean liquidation profiles approach a straight line. By comparing subplots (d), we observe that the mean-variance strategy becomes less variable as expected value increases.

9 Conclusion

In this paper, we have compared the optimal trading strategies obtained using two objective functions: mean variance and mean quadratic variation. Recall that the original strategy proposed in [2] is actually a mean quadratic variation strategy [15]. The mean quadratic variation is naturally time-consistent [10, 30]. On the other hand, the pre-commitment mean variance strategy [14, 6, 25] is not time consistent. However, the pre-commitment mean variance strategy is undoubtedly optimal if performance is measured in terms of observed post-trade mean variance data.
Figure 8: Comparison between mean variance strategy and mean quadratic variation strategy for the case \( \sigma=1.0, \kappa_t = 2 \times 10^{-6} \). The mean-variance strategy plotted has mean 99.29, standard deviation 0.68, quadratic variation 0.93, and corresponds to \( \gamma=199.82 \). The mean-quadratic-variation strategy plotted has mean 99.29, standard deviation 0.82, quadratic variation 0.84, and corresponds to \( \lambda=1 \). 1600 time steps are used to compute the results.
Figure 9: Comparison between mean variance strategy and mean quadratic variation strategy for the case $\sigma=1.0$, $\kappa_t = 2 \times 10^{-6}$. The mean-variance strategy plotted has mean 99.50, standard deviation 0.90, quadratic variation 1.05, and corresponds to $\gamma=201.30$. The mean-quadratic-variation strategy plotted has mean 99.50, standard deviation 0.98, quadratic variation 1.00, and corresponds to $\lambda=0.5$. 1600 time steps are used to compute the results.
Figure 10: Comparison between mean variance strategy and mean quadratic variation strategy for the case $\sigma=1.0$, $\kappa_t = 2 \times 10^{-6}$. The mean-variance strategy plotted has mean 99.65, standard deviation 1.13, quadratic variation 1.21, and corresponds to $\gamma=203.50$. The mean-quadratic-variation strategy plotted has mean 99.65, standard deviation 1.17, quadratic variation 1.19, and corresponds to $\lambda=0.25$. 1600 time steps are used to compute the results.
Figure 11: Comparison between mean variance strategy and mean quadratic variation strategy for the case $\sigma=1.0$, $\kappa_t = 2 \times 10^{-6}$. The mean-variance strategy plotted has mean 99.78, standard deviation 1.46, quadratic variation 1.49, and corresponds to $\gamma=209.42$. The mean-quadratic-variation strategy plotted has mean 99.78, standard deviation 1.48, quadratic variation 1.49, and corresponds to $\lambda=0.1$. 1600 time steps are used to compute the results.
The mean variance strategy is much more aggressive in the money, and is a highly nonlinear function of the asset price. By contrast, the mean quadratic variation strategy is approximately linear in the asset price. In addition, the mean variance strategy is much more variable than the mean quadratic variation strategy. Nevertheless, both strategies turn out to have very similar mean liquidation profiles.

In terms of using both standard deviation and quadratic variation as risk measures, the mean variance strategy appears to be, overall, a good strategy. The difference between the two strategies, however, are only significant at low levels of timing risk, or equivalently, high levels of implementation shortfall.

Consequently, if a highly variable strategy is acceptable, the mean-variance strategy is perhaps the better choice. Otherwise, the mean quadratic variation strategy should be chosen if less variability in the strategy is desired.

We have improved the numerical method in [14] by using a parametric curve interpolation scheme and a scaled computational grid. The parametric curve interpolation accurately approximates the foot of the semi-Lagrangian characteristics, which is essential for obtaining accurate numerical solutions for the optimal control. The scaled computation grid concentrates computational resources on region of interest in the state space so that sufficiently accurate results can be produced using few grid nodes.

### A Example Computation for the Temporary Price Impact Factor

Here we describe a realistic scenario in which the temporary price impact factor $\kappa_t = 1.2 \times 10^{-7}$ (Case 4 in Table 2) corresponds to 1% of the daily volume of a stock.

Suppose that the initial stock price $s_{\text{init}} = 100$ dollars, buy rate $= 1,000$ shares/min, corresponding temporary price impact $= 3$ dollars/min, daily trading time $= 420$ minutes, and daily volume $= 42,000,000$ shares. For such a scenario, our trading corresponds to 1% of the daily volume, and the daily market turnover for the stock is 4.2 billion dollars, corresponding to that typical of a large-cap stock.

Assuming a constant trading rate over one day ($T = 1/250$), then the total price impact is $3 \times 420$. The ratio of total price impact to total initial value of stock is then given by

$$ R = \frac{\text{total price impact}}{\text{total initial value}} = \frac{3 \times 420}{420 \times 1000 \times 100} = 3 \times 10^{-5} \quad (A.1) $$

From the trading model (2.5) and (2.6), the captured price is $s_{\text{init}} f(v) = s_{\text{init}} \exp(-\kappa_t v) \approx s_{\text{init}} (1 - \kappa_t v)$. Therefore, the ratio $R$ is approximately $\kappa_t |v|$.

Since $s_{\text{init}} = 1$ and $T = 1/250$, the constant trading rate is $v = -250$. Substituting $v = -250$ into $\kappa_t |v| = 3 \times 10^{-5}$ gives $\kappa_t = 1.2 \times 10^{-7}$.

### B Details of numerical method for solving PDEs (6.9) and (6.14)

The numerical method used in this paper for solving equations (6.9) and (6.14) is essentially the method used in [14] with two improvements: the use of a parametric curve linear interpolation scheme at the foot of the semi-Lagrangian characteristics and a scaled computational grid. In order to highlight the differences, we provide the discretization details only for these two improvements. Readers are referred to [14] for details on other aspects of the numerical method.

#### B.1 Construction of efficient frontier

Having solved (6.9) and (6.14), the variance minimizing frontier can be obtained as follows. Let $s = s_{\text{init}}$ and $\alpha = \alpha_{\text{init}}$ be the initial values of $s$ and $\alpha$ in forward time. Solving the equations

$$ V_0(b) \equiv V(s, \hat{b}, \alpha, \tau = T) = E^{s, \hat{b}, \alpha, t = 0}_{v^{\tau}(\cdot)}[B_{\gamma}^2] + \frac{\gamma^2}{4}, \quad (B.1) $$

$$ U_0(b) \equiv U(s, \hat{b}, \alpha, \tau = T) = E^{s, \hat{b}, \alpha, t = 0}_{v^{\tau}(\cdot)}[B_L] - \frac{\gamma}{2}, \quad (B.2) $$

21
where \( \dot{b} = B(0) = -\gamma e^{-\tau T}/2 \) from equation (6.1), for each value of \( \gamma \) will give us a point
\[
\left( E_{\nu(\cdot)}^{s,\dot{b},\alpha,t=0}[B_L], V_{\nu(\cdot)}^{s,\dot{b},\alpha,t=0}[B_L]\right)
\]
on the variance minimizing frontier, where
\[
E_{\nu(\cdot)}^{s,\dot{b},\alpha,t=0}[B_L] = U_0(\dot{b}) + \frac{\gamma}{2}, \tag{B.3}
\]
\[
V_{\nu(\cdot)}^{s,\dot{b},\alpha,t=0}[B_L] = E_{\nu(\cdot)}^{s,\dot{b},\alpha,t=0}[B_L] - (E_{\nu(\cdot)}^{s,\dot{b},\alpha,t=0}[B_L])^2 = V_0(\dot{b}) - (U_0(\dot{b}))^2. \tag{B.4}
\]

The whole variance minimizing frontier is then obtained by varying \( \gamma \). The mean-variance efficient frontier is then obtained by eliminating sub-optimal points.

**B.2 Similarity Reduction**

The assumption of Geometric Brownian Motion (2.3), the form of the price impact functions (2.6), (2.7), and the initial conditions (6.9), (6.14) imply the homogeneity properties
\[
V(\xi s, \xi \dot{b}, \alpha, \tau) = \xi^2 V(s, \dot{b}, \alpha, \tau),
\]
\[
U(\xi s, \xi \dot{b}, \alpha, \tau) = \xi U(s, \dot{b}, \alpha, \tau),
\]
\[
v^*(\xi s, \xi \dot{b}, \alpha, \tau) = v^*(s, \dot{b}, \alpha, \tau). \tag{B.5}
\]
Therefore we can use similarity reduction to reduce the original three dimensional problem to a two dimensional problem, in which we only need to solve for one fixed value of \( \dot{b} \).

**B.3 Semi-Lagrangian discretization**

In this section we demonstrate how equation (6.9) can be discretized by a semi-Lagrangian method. Equation (6.14) can be discretized in a similar fashion. For more details concerning semi-Lagrangian methods for HJB equations, the reader is referred to the references in [12].

Define a set of nodes \( \{s_i\}, \{b_j\}, \{\alpha_k\} \) and \( \{\tau^n\} \), where \( 0 \leq i \leq i_{\text{max}}, \ b_j \equiv \dot{b}^* < 0, \ 0 \leq k \leq k_{\text{max}}, \) and \( 0 \leq n \leq n_{\text{max}} \). We order the nodes in ascending order and make \( s_0 = 0, \ a_0 = 0, \ a_{k_{\text{max}}} = \alpha_{\text{init}}, \tau_0 = 0, \) and \( \tau_{n_{\text{max}}} = T, \) Note that there is only one node in the \( \dot{b} \) grid because of the use of similarity reduction. We denote the discrete approximation to \( V \) at the point \( (s_i, b_j, \alpha_k, \tau^n) \) by \( V_{i,j,k}^n \) to distinguish it from the exact value \( V(s_i, b_j, \alpha_k, \tau^n) \). We also specify that the set of admissible control \( Z \) is of the form \( [v_{\text{min}}, 0] \), where \( v_{\text{min}} < 0 \) is the fastest liquidation rate allowed.

Since the Lagrangian derivative \( \frac{DV}{D\tau}(v_{i,j,k}^{n+1}) \) at the node \( (s_i, b_j, \alpha_k, \tau_{n+1}) \) is the derivative of \( V \) along the trajectory defined by (6.8). Solving equations (6.8) backwards in time from \( \tau_{n+1} \) to \( \tau^n \), for a fixed \( v_{i,j,k}^{n+1} \), gives the the foot of the characteristics \( (s_i, b_j, \alpha_k, \tau^n) \), which in general is not on the PDE mesh. We use the notation \( V_{i,j,k}^{n+1} \) to denote an approximation of \( V(s_i, b_j, \alpha_k, \tau^n) \) obtained by interpolation.

**B.3.1 Local optimization**

Denote the discrete form of \( \mathcal{L} \) by \( \mathcal{L}_h \). By using an implicit discretization of \( \mathcal{L} V \) and the semi-Lagrangian discretization on equation (6.9), we obtain
\[
V_{i,j,k}^{n+1} = \min_{v_{i,j,k} \in Z_{i,j,k}^{n+1}} V_{i,j,k}^n + (\tau_{n+1} - \tau^n)(\mathcal{L}_h V)_{i,j,k}^{n+1}, \tag{B.6}
\]
with the initial condition
\[
V_{i,j,k}^0 = \dot{b}_j^2, \tag{B.7}
\]
where we restrict the admissible velocities to \( Z_{i,j,k}^{n+1} \) so that \( \alpha_k \geq 0 \).

Once the optimal control \( (v^*)_{i,j,k}^{n+1} \) is determined, equation (6.14) can be solved by
\[
U_{i,j,k}^{n+1} = U_{i,j,k}^n |_{v = (v^*)_{i,j,k}^{n+1}} + (\tau_{n+1} - \tau^n)(\mathcal{L}_h V)_{i,j,k}^{n+1}, \tag{B.8}
\]
with the initial condition
\[ U_{i,j,k}^0 = b_j. \] (B.9)

Since no analytical expression is available for the local objective function, we find the optimal \( v_{i,j,k}^{n+1} \) by discretizing the control space \( Z_{n+1}^{i,j,k} \) and look for the optimal value using a linear search. This has the advantage of not making any assumptions about the local objective function, at the expense of a higher computational cost. Numerical experiments demonstrate that accurate results can be obtained by a rather coarse discretization of the control space.

**B.4 Computational challenges and solutions**

**B.4.1 Difficulties in determining optimal velocity numerically**

Recall that in the special case considered in section 7, the Lagrangian derivative is identically zero for any admissible trading velocities. In terms of equation (B.6), this means that \( V_{i,j,k}^n (v_{i,j,k}^{n+1}) \) as a function of \( v_{i,j,k}^{n+1} \) is completely flat. In the parametric cases we consider, both \( \sigma \sqrt{T} \) and \( \kappa \) are quite small, therefore these realistic cases are indeed similar to the completely ill-posed special case. Consequently \( V_{i,j,k}^n (v_{i,j,k}^{n+1}) \) as a function of \( v_{i,j,k}^{n+1} \) can be very flat, which means determining the true minimizer demands extremely high accuracy. It is also obvious that even small interpolation error can significantly alter the estimated trading velocities.

Similar computational issues also arise when ordinary finite-differencing is used instead of the semi-Lagrangian method: since the optimal velocity \( v \) will be a function that depends on the ratio of the partial derivatives in \( \frac{DV}{D\tau} \). Any error in approximating the partial derivatives can also significantly alter the estimated trading velocities.

**B.4.2 Parametric curve linear interpolation**

The previous section has explained the importance of accurate interpolation at the foot of the semi-Lagrangian characteristics. In [14], a standard axis-aligned linear interpolation is used, which turns out to be too inaccurate. This is not surprising given the quadratic nature of the value function \( V \) and the analysis in section 7.

In section 7 we have shown the benefit of performing a parametric curve linear interpolation for the special case considered. Here we extend the idea to general cases. In essence, the parametric line (7.5) is generalized to the line \( L \) defined by

\[
L = (s_i, b_j, \alpha_k) + \zeta \left( \frac{ds}{d\zeta}, \frac{db}{d\zeta}, \frac{d\alpha}{d\zeta} \right),
\]

\[
\frac{ds}{d\zeta} = -g(v_{i,j,k}^{n+1})s_i, \quad \frac{db}{d\zeta} = -(rb_j - v_{i,j,k}^{n+1}f(v_{i,j,k}^{n+1})s_i), \quad \frac{d\alpha}{d\zeta} = -v_{i,j,k}^{n+1},
\] (B.10)

where \( v_{i,j,k}^{n+1} \) is a candidate control value.

Since equations (B.10) express how changes in \( \alpha \) lead to changes in \( s \) and \( b \) through both trading impact and pricing impact (through the terms \( g(v_{i,j,k}^{n+1}) \) and \( f(v_{i,j,k}^{n+1}) \)), interpolating along \( L \) can be seen as an extension to interpolation along (7.5), which takes into account trading but not pricing impact.

Figure 12 compares the standard axis-aligned linear interpolation and the parametric curve linear interpolation along the line \( L \). For simplicity in illustration, linear interpolation along the \( s \) coordinate axis is not shown, and there is a actual \( b \) grid, i.e. no similarity reduction.

Note that the parametric curve linear interpolation as shown in Figure 12 does not require interpolation along the \( \alpha \) coordinate axis but still requires linear interpolation along the other axes. When linear interpolation along the \( s \) direction or the \( b \) direction is performed, a fine grid is still needed to reduce interpolation error. In other words, when we treat \( \alpha \) specially as in Figure 12, we avoid the need of a fine \( \alpha \) grid, but a fine \( s \) grid (and a fine \( b \) grid when no similarity reduction is used) is still necessary.

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We also note that the parametric curve linear interpolation is similar to the edge-directed interpolation method [21] in the image processing literature. Our method is similar in the sense that the direction of the line $L$ is not fixed but adapts to the candidate control velocity $v_{i,j,k}^{n+1}$. Our method is different from that in [21] in that the parametric curve linear interpolation method does not necessarily use the neighboring grid nodes (See Figure 12(b)).

**B.4.3 Remark on convergence proof**

Having changed the interpolation scheme in the semi-Lagrangian method, it is important that the convergence proof in [14] is still valid. Linear interpolation is obviously consistent. To demonstrate stability, we need the following easy observation:

$V_{i,j,k}^n$ as approximated in the new scheme takes the form

$$V_{i,j,k}^n = \sum_p w_p V_p^n,$$

where $w_p \geq 0$, $\sum_p w_p = 1$, and $V_p^n$ are grid node values. Therefore, we have $|V_{i,j,k}^n| \leq ||V^n||$. This property allows the proof of stability in [14] to go through without change. In addition, since $U_{i,j,k}^n$ as approximated in the scheme also takes the form

$$U_{i,j,k}^n = \sum_p w_p U_p^n,$$

where the $w_p$’s are the same as those in equation (B.11). Therefore, the proof in [14] which shows $(U_{i,j,k}^n)^2 \leq V_{i,j,k}^n$ is also valid for our scheme.

**B.4.4 Computational grid consideration**

The more interesting part of the efficient frontier is made up by points with moderate (neither too small nor too large) values of standard deviation compared to $\delta_{\text{max}}$, which is defined as the maximum standard deviation achieved by a constant liquidation rate. Since $\delta_{\text{max}}$ is quite small for typical parameter values (between
Figure 13: The scaled computational grid in which the $s$ grids are different for different values of $\alpha$ such that $s \propto 1/\alpha$. The basic $s$ grid is used at $\alpha = \alpha_{\text{init}} = 1$ and $\alpha = 0$, which correspond to the dots at $\alpha = 1$ and the dots at the $\alpha = 0$. The other dots correspond to the scaled $s$ grid used at that particular $\alpha$ value.

0.15 and 3 in our parametric cases), equation (B.4) implies that the value function $V_0(b) = E^{s,b,\alpha,t=0}_v[B_L^2]$ is approximately $\text{Var}^{s,b,\alpha,t=0}_v[B_L] \leq \delta_{\text{max}}^2$, which is also quite small. Consequently, the interesting part of the efficient frontier corresponds to the region in the state space where $V \approx 0$.

Consider again the analytical solution for the special case derived in section 7:

$$V(s, b, \alpha, \tau) = (\alpha s + b)^2. \tag{B.13}$$

As we explain above, the region of interest is where $V \approx 0$. For the special case, this corresponds to $(\alpha s + b) \approx 0$, which is a small subset of the state space. Therefore, in the standard construction of a PDE computational grid (tensor product of one-dimensional grids), many of the grid nodes will be far away from the interesting region, wasting both time and space. To better utilize computational resources, the computational grid should be constructed so that most of the grid nodes are concentrated in the region where $V \approx 0$.

Because of the use of similarity reduction, there is only one $b$ grid node, which we denote by $b^*$. It then follows that the region where $V \approx 0$ is the region where $\alpha s \approx -b^*$. This analysis therefore suggests scaling the $s$ grid for different values of $\alpha$ grid nodes, i.e. $s = -b^*/\alpha$. Note that the $s$ grid corresponding to $\alpha = 0$ is not scaled. See Figure 13 for an illustration.

C Details of Monte Carlo simulations

Numerically solving the mean-variance HJB equation (6.9) gives us the optimal strategy on the discrete computational mesh, i.e. $v_{\text{MV}}(s_i, b^*, \alpha_k, t^n)^3$. By using $v_{\text{MV}}(s_i, b^*, \alpha_k, t^n)$ as input for Monte Carlo simulations, we can obtain information about the trading strategy which is not necessarily available in the PDE solutions. For example, we can estimate the probability distribution of $B_L$, the quadratic variation, and the mean and the standard deviation of the liquidation profile (plot of $A(t)$ against $t$).

---

3 Note the use of forward time notation.
Similarly, numerically solving the mean-quadratic-variation HJB equation (4.4), for each fixed value of \( \lambda \), gives us the discrete optimal strategy \( v_{MV}(s_i, \bar{b}, \bar{t}^n; \lambda) \), which can be used as input for Monte Carlo simulations.

In particular, Monte Carlo simulations enable us to compute the quadratic variation of the mean-variance strategy, and conversely, the variance of the mean-quadratic-variation strategy. These allow us to compare the two strategies in terms of either variance or quadratic variation, given the same level of expected return.

The Monte Carlo simulations also provide a verification of the PDE solutions, in the sense that given the optimal control, we can obtain independent estimates of mean, variance and quadratic variation.

In the following we detail how the Monte Carlo method is conducted for the mean-variance strategy. Simulations of the mean-quadratic-variation strategy are performed in the same way, except that the mean quadratic variation strategy \( v_{MV}(s_i, \bar{b}, \bar{t}^n; \lambda) \) is explicitly indexed by \( \lambda \), and standard axis-aligned linear interpolation suffices.

C.1 Change of variable

Suppose that the optimal strategy \( v_{MV}(s_i, \bar{b}, \bar{t}^n) \) is obtained from the PDE solve. For a fixed value of \( \gamma \), each Monte Carlo simulation starts with the initial values \( S(0), B(0), A(0) \) at time \( t = 0 \) and is updated at the discrete times \( \{t^n\} \), i.e. the time grid nodes in the PDE solve. Below we give a full specification of the simulation procedure by detailing the simulation from time point \( t_{old} \) to the immediate next time point \( t_{new} \).

At \( t_{old} \), the state is \( (S_{old}, B_{old}, A_{old}) \). To look up the optimal trading velocity, we first need to change the variable from \( B \) to \( B \). For the fixed value of \( \gamma \), we have \( B_{old} = B_{old} - \gamma e^{-r(t-t_{old})}/2 \) from equation (6.2). Now the optimal trading velocity \( v(S_{old}, B_{old}, A_{old}, t_{old}) \) needs to be interpolated from the discrete \( v_{MV}(s_i, \bar{b}, \bar{t}^n) \).

C.1.1 Interpolation

Our numerical study shows that it is more accurate to linearly interpolate \( v_{MV}(s_i, \bar{b}, \bar{t}^n) \) along a constant line of wealth \( \{\alpha s + \bar{b} = \text{constant}\} \) than along the coordinate axes. Therefore, we interpolate \( v_{MV}(s_i, \bar{b}, \bar{t}^n) \) as in Figure 12(b) with \( L \) given by the constant line of wealth \( \{\alpha s + \bar{b} = \text{constant}\} \). This is the same as the interpolation for the limiting parametric case in section 7.1. Note that the form of the line \( L \) as defined by equation (B.10) is not applicable in the current context because there is no candidate control \( v_{i,j,k}^{n+1} \).

C.1.2 Updating state variables

Let \( \Delta t = t_{new} - t_{old} \), we update the state variable as follows:

\[
\begin{align*}
A_{opt} & = A_{old} + v(S_{old}, B_{old}, A_{old}, t_{old}) \Delta t, \quad \text{(C.1)} \\
A_{new} & = \max(A_{opt}, 0), \quad \text{(C.2)} \\
v_{opt} & = (A_{new} - A_{old})/\Delta t, \quad \text{(C.3)} \\
S_{new} & = S_{old} \exp\left( (\eta + g(v_{opt}) - \frac{1}{2}\sigma^2)\Delta t + \sigma \sqrt{\Delta t} \mathcal{N}(0,1) \right), \quad \text{(C.4)} \\
B_{new} & = B_{old} \exp\left( r\Delta t - v_{opt} f(v_{opt}) S_{old} \Delta t \right), \quad \text{(C.5)} \\
QV_{new} & = QV_{old} + (A_{old}(S_{new} - S_{old}))^2, \quad \text{(C.6)}
\end{align*}
\]

where \( \mathcal{N}(0,1) \) is a standard normal variate and \( QV_{new} \) is an approximation of \( \int_0^{t_{new}} (A(t') dS(t'))^2 \).

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\footnote{We use the notations \( i \) and \( k \) to emphasize that the \( s \) grid and \( a \) grid for solving the mean-quadratic-variation HJB equation (4.4) is not necessarily the same as that for solving the mean-variance HJB equation (6.9). The time grid \( \{t^n\} \), however, is chosen to be the same.}
References


