Group Discussion
PDEs, Quadrature, Monte Carlo methods: which formulation to use for modern financial problems?

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13:30-15:00
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What is a modern financial problem?

> *"All problems in finance are basically optimal stochastic control problems." Robert Ferstenberg, Managing Director, Morgan Stanley (retired).*

Does this statement have any truth to it?

Example: Black-Scholes option pricing
Assume that

$$
\begin{aligned}
dS &= \mu S \, dt + \sigma S \, dZ \\
&\quad S = \text{ asset price } \; ; \; \mu = \text{ real world drift} \\
dB &= rB \, dt \\
&\quad B = \text{ risk free bond } \; ; \; r = \text{ risk-free rate}
\end{aligned}
$$

Suppose that option has payoff

$$P(S, T) \; = \; \text{payoff}$$

Now form a hedging portfolio Π

$$\Pi = B - \hat{\alpha} \, S \; ; \; \hat{\alpha} = \text{ units of } S$$

Option Pricing: control formulation

Let:

$$
\begin{aligned}
& V(s, b, \alpha, t) \\
& = \min_{\bar{\alpha}(\cdot)} \left(E^{\bar{\alpha}(\cdot)} \left[\left(\Pi(S, T) - P(S, T) \right)^2 \right] \Big| S(t) = s, B(t) = b, \alpha\hat{(}t) = \alpha \right) \\
& \quad E[\cdot] = \text{ Expectation under } \mathbb{P} \qquad (1)
\end{aligned}
$$

Solve (1) for control $\hat{\alpha}(\cdot)$, using HJB PDE (Impulse Control)

$$
\begin{aligned}
B_0 &= \arg\min_b \left(V(S_0, b, \alpha = 0, t = 0) \right) \\
&= \text{Black-Scholes price}
\end{aligned}
$$

And solution is independent of μ (real drift)! No-arbitrage argument not used.

We already know this: Why bother?

But, if we think about hedging an option in this way, it is easy to generalize this to handle

- Incomplete markets, e.g. jumps.
- Discrete rebalancing
- Liquidity effects, different interest rates for borrowing and lending, position limits, CVA, DVA, FVA, XVA,...
- Model risk: uncertain parameters

Determine cost of hedging

→ May not be a unique price for long/short positions

→ Derivative *manufacturing* cost may be different for different hedgers (depends on total portfolio)

So, stochastic control formulation is more general, and we can reproduce known results in special cases (e.g. Black-Scholes equation)

More Complex Control Applications in Finance

- GMxB guarantees
  - Popular retirement income products
  - Sold by insurance companies
  - Complex optionality for holder
- Optimal trade execution
  - Sell large block of shares, minimize price impace, maximize gain
- Commodities
  - Optimal operation of a Hydroelectric plant
    - Stochastic electricity prices, rainfall
    - Environmental constraints
  - Mining
    - Optimal mine depletion strategies, stochastic prices, uncertain grades of ore
- Macroeconomic issues
  - Optimal intervention in FX markets
  - Optimal issue of government bonds

What is a modern financial problem?

- A very general approach: optimal stochastic control
- And of course, we always want to solve problems in high dimensions
  - Several stochastic factors
  - Several underlying assets
  - May need additional *path-dependent* variables

Basic algorithm :

- Dynamic programming
- Needs to work backwards in time to determine optimal control
  - simple forward looking MC won't work

Methods

HJB PDEs

- Easy to add almost any type of constraint to controls
- Need to construct monotone schemes, non-trivial
- Complexity problems if *dimension* > 3

BSDEs

- Need to compute conditional expectations
  - Longstaff-Schwartz
  - Quantization
- Accuracy: if control is not bang-bang?
- State-dependent admissible sets for controls?

Quadrature

- Direct approximation of Dynamic Programming formulation
- Sometimes this can be done efficiently (i.e. FFT)
- Optimal quantization?