Overview: Numerics of Hamilton-Jacobi-Bellman Equations in Finance

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Outline

1. General formulation of stochastic control problems
2. Two examples: illustrate basic numerical methods
   (a) Gas storage
       → semi-Lagrangian timestepping
   (b) Optimal portfolio allocation with stochastic volatility
       → Monotone discretization (wide stencil)
       → Policy iteration
3. Many HJB equations can be solved using these techniques
Basic Forms of Optimal Stochastic Control

Definitions:

- \( X(t) \): a set of state variables, usually specified by a set of SDEs
- \( q(X(t), t) \): a control, which allows us to steer \( X(t) \)
  \[ q(\cdot) \in \mathcal{Z}(X) : \] the set of admissible controls
- An objective function \( F(X(T)) \) at terminal time \( T \).
- \( C(X(t), t) \) cash flows at time \( t \)

And in addition:

\[ E_{t,x}^{q(\cdot)}[\cdot] = \text{Expectation conditional on } X(t) = x, \]
  at time \( t \), under control \( q(\cdot) \)
Value Function

Value function $V(x, t)$ defined by

$$V(x, t) = \sup_{q(\cdot)} E_{t,x}^{q(\cdot)} \left[ \int_t^T C(X(s), s) \, ds + F(X(T)) \right]$$

- Solve for value function $\rightarrow$ get optimal control $q(\cdot)$

HJB equation: standard arguments ($\tau = T - t$):

$$V_\tau = \sup_{q \in \mathcal{Z}} (\mathcal{L}^q V)$$

$\mathcal{L}^q V = \text{degenerate parabolic operator}$
Viscosity Solution

Original problem posed as a dynamic program

→ Value function defined in terms of integration
→ Value function need not be differentiable

- We seek the weak form solution of the HJB equation
  → viscosity solution

Need to guarantee numerical scheme converges to viscosity solution

- Sufficient conditions (Barles, Souganidis (1991))
  - Monotone, consistent (in the viscosity sense) and $\ell_\infty$ stable
Example I: control appears only in 1st derivative terms

Natural gas storage facilities: buy low (summer), sell high (winter)$^1$

- Long term storage: underground caverns
- Objective: determine no-arbitrage value of leasing a storage facility for a fixed term
- By-product of valuation: optimal operating strategy (i.e. when to inject, produce or do nothing)
- Control = injection strategy

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$^1$Economists describe the situation where a large deposit of natural gas crowds out other industries as the *Dutch Disease*
State Variables

Risk neutral process for spot price of gas $P$

$$dP = \alpha(K(t) - P) \, dt + \sigma P \, dZ$$

$\sigma = \text{volatility}$ ; $dZ = \text{increment of Wiener process}$

$K(t) = K_0 + \beta_{SA} \sin(4\pi(t - t_{SA}))$

Current amount of working gas inventory $I$

$I \in [0, I_{\text{max}}]$  
$c = \text{Rate of gas production from storage}$

$c \in C(I) = [c_{\text{min}}(I), c_{\text{max}}(I)]$

$c > 0 \rightarrow \text{production} ; \ c < 0 \rightarrow \text{injection}$

Gas inventory satisfies

$$\frac{dl}{dt} = -(c + a(l, c))$$

$a(l, c) = \text{production/injection losses}$
Max/min production rates are nonlinear functions of inventory $I$, e.g.

$$c_{\text{max}}(I) = k_1 \sqrt{I}; \quad k_1 = \text{const.} \quad \text{(ideal gas law)}$$

Revenue obtained from selling gas

$$\text{Revenue} = (c - b(c)) P$$

$b(c) = \text{gas loss during transportation}$

Revenue $> 0 \rightarrow$ gas released and sold

Revenue $< 0 \rightarrow$ gas purchased and stored
Value of leasing storage for $T$ years: risk neutral discounted cash flows in $[0, T]$, $(\tau = T - t)$

$$V(P, I, \tau) = \sup_{c(s) \in C(I(s))} E^{c(.)} \left[ \int_t^T e^{-r(s-t)} \left[ \underbrace{[c(s) - b(c(s))]P(s)}_{\text{cash flows}} \right] ds ight]$$

$$+ e^{-r(T-t)} \left[ \underbrace{V(P(T), I(T), T)}_{\text{lease termination penalty}} \right]$$

Usual steps: HJB PDE for $V(P, I, \tau)$

$$V_{\tau} = \mathcal{L}V + \max_{c \in C(I)} \left\{ (c - b(c))P - (c + a(c))V_I \right\}$$

$$\mathcal{L}V \equiv \frac{1}{2} (\sigma P)^2 V_{PP} + \alpha (K(t) - P) V_P - rV$$
Semi-Lagrangian form

Let

\[
\frac{DV}{D\tau} = \frac{\partial V}{\partial \tau} + \frac{\partial V}{\partial I}(c + a(c))
\]

be the Lagrangian derivative along the trajectory

\[
\frac{dl}{d\tau} = c + a(c)
\]

We can then write the HJB PDE as

\[
\min_{c \in C(I)} \left\{ \frac{DV}{D\tau} - (c - b(c))P - \mathcal{L}V \right\} = 0.
\]

Define:

\[
\{P_i, l_j\}_{i=1,\ldots,N_1; j=1,\ldots,N_2}, \quad \text{Grid}
\]

\[
V_{i,j}^n = V(P_i, l_j, \tau^n) \quad \text{Discrete Solution}
\]
Discretization

The final form for the discretization is then

\[ V_{i,j}^{n+1} = \max_{c \in C_{i,j}^{n+1}} \left\{ \begin{array}{l}
\text{interpolated value} \\
\left[ \Phi^{n+1}(c) V^n \right]_{i,j} + \Delta \tau^n (c - b(c)) P_i
\end{array} \right\}
\]

at foot of characteristic

\[ + \Delta \tau^n (\mathcal{L}_h V)_{i,j}^{n+1}, \]  \hspace{1cm} (1)

Define:

\[ \Gamma_{i,j}^n = \max_{c \in C_{i,j}^{n+1}} \left\{ \left[ \Phi^{n+1}(c) V^n \right]_{i,j} + \Delta \tau^n (c - b(c)) P_i \right\} \]

\hspace{1cm} (2)

so that (1) becomes

\[ V_{i,j}^{n+1} = \Gamma_{i,j}^n + \Delta \tau^n (\mathcal{L}_h V)_{i,j}^{n+1} \]  \hspace{1cm} (3)

\[ ^2 \text{Each timestep: interpolation/optimization step (2) } \rightarrow \text{ time advance step (3). Easy to show: monotone, consistent, stable} \]
Control Surface: \( t = 0 \)

- Injection \( c < 0 \)
- Production \( c > 0 \)
- Note region where optimal to do nothing
- Option value of waiting
Example II: control appearing in the 2nd derivative terms

Continuous time mean variance portfolio allocation: stochastic volatility

Long term investor can allocate wealth into two assets:

*Amount* in Risk-free bond $B$

$$dB = r B \ dt$$

$r = \text{risk-free rate}$

*Amount* in risky-asset $S$

$$\frac{dS}{S} = (r + \xi V) \ dt + \sqrt{V} \ dZ_1$$

$\xi V = \text{market price of volatility risk}$

Variance process $V$

$$dV(t) = \kappa(\theta - V(t)) \ dt + \sigma \sqrt{V(t)} \ dZ_2$$

$\sigma = \text{vol of vol} ; \; \kappa = \text{Mean-reversion speed}$

$\theta = \text{mean variance} ; \; \rho dt = dZ_1 \ dZ_2$
SDE for Total Wealth

The investor’s total wealth \( W = S + B \) follows the process

\[
dW(t) = (r + p\xi V(t)) W(t) \, dt + p\sqrt{V} W(t) \, dZ_1.
\]

\[
p = \left( \frac{S}{W} \right) = \text{fraction invested in risky asset}
\]

Constraints on control \( p \)

- Trading must stop if \( W = 0 \)
- Leverage is constrained: \( p \leq p_{\text{max}} \)

Objective: determine optimal control \( p(W, t) \) which generates points on the efficient frontier

\[
\sup_p \left\{ E^{p(\cdot)}[W(T)] - \lambda \text{Var}^{p(\cdot)}[W(T)] \right\}
\]

- Varying \( \lambda \in [0, \infty) \) traces out the efficient frontier
Reformulate MV Problem $\Rightarrow$ Dynamic Programming

Embedding technique$^3$ for fixed $\lambda$, if $p^*(\cdot)$ maximizes

$$\sup_{p(\cdot) \in \mathcal{Z}} \left\{ \underbrace{E^p[W(T)]}_{\text{Expected Value}} - \lambda \underbrace{\text{Var}^p[W(T)]}_{\text{Variance}} \right\},$$

$\mathcal{Z}$ is the set of admissible controls

$\rightarrow \exists \gamma$ such that $p^*(\cdot)$ minimizes

$$\inf_{p(\cdot) \in \mathcal{Z}} E^p(\cdot) \left[ \left( W(T) - \frac{\gamma}{2} \right)^2 \right].$$

$^3$Zhou and Li (2000), Li and Ng (2000)
Value Function $\mathcal{U}(w, v, \tau)^4$

$$\mathcal{U}(w, v, \tau) = \inf_{p(\cdot) \in \mathbb{Z}} E^{p(\cdot)}_{\tau, w, v} \left[ \left( W(T) - \frac{\gamma}{2} \right)^2 \right]$$

$w = \text{wealth} ; \ v = \text{local variance} ; \ \tau = T - t$

HJB PDE for optimal allocation strategy $p(\cdot)$:

$$\mathcal{U}_{\tau} = \inf_{p \in \mathbb{Z}} \left\{ (r + p\xi v)w \ \mathcal{U}_w + \kappa(\theta - v) \ \mathcal{U}_v \right.$$ 

$$+ \left( \frac{(p\sqrt{v}w)^2}{2} \right) \mathcal{U}_{ww} + \left( p\rho\sigma\sqrt{v}w \right) \mathcal{U}_{wv} + \left( \frac{\sigma^2 v}{2} \right) \mathcal{U}_{vv} \right\},$$

$$\mathcal{U}(w, v, 0) = \left( w - \frac{\gamma}{2} \right)^2.$$ 

Given $p(\cdot)$, compute $E^{p(\cdot)}[W_T], \ Var^{p(\cdot)}[W_T]$

$^4$For a fixed $\gamma$, this gives one point on the efficient frontier.
Main Problem: cross derivative term

Construct finite difference grid

\[ \{ w_i, v_j \} i=1,...,N_1 ; j=1,...,N_2 \]

We need to construct a monotone scheme
  - Control appears in 2nd derivative terms

Solution: Wide stencil\(^5\)
  - Grid spacing \(O(h)\)
  - At each node, do virtual rotation, eliminate x-derivative term, finite difference on rotated grid
  - Values are interpolated from real grid
  - Size of virtual stencil \(O(\sqrt{h})\)

\(^5\)Debrebant and Jakobsen (2013) factor the diffusion tensor
Local Rotation

Note: local rotation angle $\theta_{i,j}$ depends on

- Node location, i.e. $(w_i, v_j)$
- Control $p$ at this node
Discretization: fully implicit timestepping

\[ U_{i,j}^n = U(w_i, v_j, \tau^n) \]

\[ U^n = (U_{1,1}^n, U_{2,1}^n, \ldots, U_{N_1,1}^n, \ldots, U_{1,N_2}^n, \ldots, U_{N_1,N_2}^n) \]

\[ U^\ell_n = U_{i,j}^n, \quad \ell = i + (j - 1)N_1. \]

Similarly the vector of optimal controls is

\[ P = (p_{1,1}, \ldots, p_{N_1 N_2}) \]

The nonlinear algebraic equations are then

\[ \inf_{P \in \mathbb{Z}} \left\{ -A(P)U^{n+1} + C(P) \right\} = 0, \]

\[ A = \text{matrix of discretized equations} \quad (4) \]

\[ ^6 \text{Row } \ell \text{ of } A, C \text{ depends only on } p_{\ell} \]
Algorithm 1 Policy Iteration

1: Let \((\hat{U})^0 = \text{Initial estimate for } U^{n+1}\)
2: for \(k = 0, 1, 2, \ldots \) until converge do
3: \(\mathcal{P}_k^\ell = \arg \min_{\mathcal{P}_\ell \in Z} \left\{ -[A(\mathcal{P})]\hat{U}_k + C(\mathcal{P}) \right\}_\ell \)
4: Solve \([A(\mathcal{P}_k)]\hat{U}_k^{k+1} = C(\mathcal{P}_k)\)
5: if converged then
6: break from the iteration
7: end if
8: end for

Theorem (Convergence of Policy Iteration)

If \([A(\mathcal{P})]\) is an M matrix, Policy iteration converges to the unique solution of equation (4).

\(^7\)Use ILU-PCG method to solve matrix, complexity = \(O((N_1 N_2)^{5/4})\).
Numerical Example

<table>
<thead>
<tr>
<th>$\kappa$</th>
<th>$\theta$</th>
<th>$\sigma$</th>
<th>$\rho$</th>
<th>$\xi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.04</td>
<td>0.0457</td>
<td>0.48</td>
<td>$-0.767$</td>
<td>1.605</td>
</tr>
</tbody>
</table>

**Table:** $\mathbb{P}$ measure Heston parameters

<table>
<thead>
<tr>
<th>Investment Horizon $T$</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>The risk free rate $r$</td>
<td>0.03</td>
</tr>
<tr>
<td>Leverage constraint $p_{\text{max}}$</td>
<td>2</td>
</tr>
<tr>
<td>Initial wealth $w_0$</td>
<td>100</td>
</tr>
<tr>
<td>Initial variance $v_0$</td>
<td>0.0457</td>
</tr>
</tbody>
</table>

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Efficient Frontier (vary speed of mean reversion $\kappa$)

Figure: $T = 10$ years. $w_0 = 100$. $\kappa = 5 \approx 2.5$ months mean reversion time. Curves for $\kappa = 5, 20$ very close.
Long Term Investment: Stochastic Volatility Unimportant?

Fix $\gamma$ (parameter that traces out efficient frontier)

A Assume constant volatility GBM$^9$, compute and store optimal strategy

B Assume stochastic volatility$^{10}$, compute and store optimal strategy

Assume real world follows stochastic volatility, compute result using MC simulations, for both A and B

<table>
<thead>
<tr>
<th></th>
<th>$\gamma = 540$</th>
<th></th>
<th>$\gamma = 1350$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Mean</td>
<td>Stndrd Dev</td>
<td>Mean</td>
</tr>
<tr>
<td>GBM Control A</td>
<td>212.68</td>
<td>58.42</td>
<td>329.13</td>
</tr>
<tr>
<td>Stoch Vol Control B</td>
<td>213.99</td>
<td>58.53</td>
<td>331.28</td>
</tr>
</tbody>
</table>

Table: $\kappa T > 20$, stochastic vol well approximated by GBM.

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$^9$Constant volatility = mean value from stochastic vol model

$^{10}\kappa \approx 5$, $T = 10$ years
Conclusions

- If the control appears only in the first derivative term
  → Semi-Lagrangian timestepping simple and effective
- Similar timestepping method can be used for impulse control.
- Control appearing in 2nd derivative terms
  → For non-zero correlation, need monotone discretization (wide stencil)
  → Non-linear algebraic equations easily solved using Policy iteration
- Low accuracy control (e.g. GBM for stoch vol, coarse control set discretization)
  → Accurate value function. Why?
- Challenges:
  - Higher dimensions
  - Wide stencil only 1st order
  - Solution of local optimization problem at each node (need global optimum)