Variable annuities: A closer look at ratchet guarantees, hybrid contract designs, and taxation

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Preprint Alonso-Garcia et al. (2026).

International Conference on Computational Finance (ICCF) 2026, Oxford
Wednesday, September 2nd 2026

This research was supported by the Australian Research Council Centre of Excellence in Population Ageing Research (Project Number CE170100005) and Australian Research Council Discovery Project DP210101195.

Introduction

- A variable annuity (VAs) a long-term, tax-deferred* investment designed to provide a post-retirement income.
- We analyze policyholders' (PH) optimal strategies for a VA with a guaranteed minimum withdrawal benefit (GMWB) offered in tandem with a **cash fund**.
  - The cash fund serves as an intermediate repository of earnings from the VA.
  - If the PH withdraws less than the guaranteed amount, then the remainder is deposited into the cash fund.
- We consider a pre-retirement scenario where withdrawals and other proceeds from the VA are not tax exempt.
- Since withdrawals (ordinary income) and interest rate income are taxed differently, we investigate the relationship between tax rates and cash fund appreciation rates.

Related Literature

- **Basic valuation methods:** Bacinello et al. (2011); Bauer et al. (2008); Milevsky and Salisbury (2006)
- **Optimal withdrawal strategies:** Azimzadeh and Forsyth (2015); Chen et al. (2008); Cheridito and Wang (2016); Dai et al. (2008); Huang and Kwok (2014); Shevchenko and Luo (2016)
- **Financial market dynamics:** Alonso-García et al. (2018); Bacinello et al. (2016); Forsyth and Vetzal (2014); Gudkov et al. (2019); Ignatieva et al. (2016); Kang and Ziveyi (2018); Kang et al. (2022)
- **VAs and taxes:** Alonso-García et al. (2024); Bauer and Moenig (2023); Moenig and Bauer (2016); Moenig and Zhu (2018); Molent (2020); Ulm (2020)
- **Path-dependent guarantees:** Ai et al. (2023); Jeon and Kwak (2018); Shevchenko and Luo (2016)

Taxation and VAs in the Literature

- Typically, withdrawals are taxed as ordinary income and, when made prior to age 60, an additional penalty of 10% is imposed (Bauer and Moenig 2023; Horneff et al. 2015; Moenig and Bauer 2016; Moenig 2021).
- *Last-in-first-out basis:* earnings are withdrawn before the principal–uses a "tax base" to keep track of the amount that can be withdrawn from the VA tax-free (Moenig and Bauer 2016).
- Some taxation regimes allow the use of capital losses to offset capital gains from other investments (Alonso-García et al. 2024) (GMAB).
- Taxation, in general, creates a wedge between the valuation of the contract from the PH's and the provider's perspectives (Moenig and Bauer 2016; Alonso-García et al. 2024).

VA contract in a nutshell

- **Between** withdrawals: GBM mix riskless, risky (ϱ) return on $X(t)$ VA account value,
- **Just before** withdrawal: we charge the fee φ, we determine the guarantee amount:

$$X^{PF}(t_k^-) := X(t_k^-)(1 - \varphi \Delta t)$$

$$g(t_k) = \beta \hat{G}(t_k^-), \quad \text{where } \hat{G}(t_k^-) := \begin{cases} \max\{X^{PF}(t_k^-), G(t_k^-)\} & \text{(ratchet)} \\ G(t_k^-)(1 + \delta \Delta t) & \text{(roll-up)} \end{cases},$$

where $G(t_k^-)$ is the prevailing benefit base and β the proportion (typically $1/T$).

- **At** withdrawal:
    - The PH withdraws $w(t_k) \in \{w \colon 0 \leq w \leq \max\{X^{PF}(t_k^-), g(t_k)\}$,
    - If $w(t_k) < g(t_k)$, then $g(t_k) - w(t_k)$ is transferred to the cash fund.
    - Cash fund evolves (deterministically) according to cash rate $\eta > 0$.

VA contract in a nutshell (C'td): After a Withdrawal

At time t_k^+ : The sub-account is updated,

$$X(t_k^+) = \max\{0, \overbrace{X^{PF}(t_k^-) - g(t_k) - (w(t_k) - g(t_k))^+}^{\text{Post-withdrawal VA account value}}\},$$

and the guarantee benefit base $G(t_k)$ is updated

$$G(t_k^+) = \begin{cases} \hat{G}(t_k^-) & \text{if } w(t_k) \leq g(t_k) \\ \underbrace{\hat{G}(t_k^-) \left(\frac{X^{PF}(t_k^-) - w(t_k)}{X^{PF}(t_k^-) - g(t_k)} \right) \mathbf{1}_{\{X^{PF}(t_k^-) - g(t_k) < \hat{G}(t_k^-)\}}}_{\text{Proportional decrease}} & \text{if } w(t_k) > g(t_k) \end{cases}$$

For simplicity, we denote these update functions by $h^X(X(t_k^-), G(t_k^-), w(t_k))$ and $h^G(X(t_k^-), G(t_k^-), w(t_k))$, respectively.

Policyholder's Optimization Problem

- Without tax, the present value (PV) of all cash flows from the VA contract is

$$H_0(\mathbf{X}, \mathbf{G}, \mathbf{w}) = \sum_{k=1}^{N-1} e^{-rt_k} w(t_k) + e^{-rt_N} \sum_{k=1}^{N-1} e^{\eta(t_N - t_k)} (g(t_k) - w(t_k))^+ \\ + e^{-rt_N} \max\{X^{PF}(t_N^-), g(t_N)\}.$$

- With taxes ($\theta \in [0,1]$), the PV becomes

$$H_0(\mathbf{X}, \mathbf{G}, \mathbf{W}) = \sum_{k=1}^{N-1} e^{-rt_k} (1 - \bar{\theta}(t_k)) w(t_k) \\ + e^{-rt_N} \sum_{k=1}^{N-1} \mathbf{1}_{CF} \cdot \left([(1 - \bar{\theta}(t_N)) e^{\eta(t_N - t_k)} + \bar{\theta}(t_N)] (g(t_k) - w(t_k))^+ \right) \\ + e^{-rt_N} (1 - \bar{\theta}(t_N)) \max\{X^{PF}(t_N^-), g(t_N)\}$$

where $\bar{\theta}(t_k)$ represents whether the payment is made before preservation age a_p [liable for tax] or not [tax-free].

Assumptions and Methodology

Table: Default parameter values and mesh sizes

$P_0 = 100$	$r = 0.03$	$\varrho = 0.80$	$\sigma = 0.20$	$\beta = 0.10$
$T = 10$	$\Delta t = 1$	N_x[1] $= 80$	$N_\gamma = 80$	$N_\tau = 50$

- φ^* is found as the root of the cubic spline interpolant of VA Values (as a function of φ) for $\varphi \in [0, 400]$ (basis points).
- The conditional expectation in the first step is expressed as a PDE and numerically solved using the method of lines (MOL) algorithm (Meyer 2015).
- By way of a sensitivity analysis, we allow the values of η and θ to vary over the sets $\{2\%, 3\%, 4\%, 5\%\}$ and $\{0, 2.5\%, 5\%, 10\%, 20\%\}$, respectively.
- Here we show the case when the PH is fully liable for tax $(a_b + t_N < a_p)$. The partial taxation case $(a_b + t_N > a_p)$ is detailed in Alonso-Garcia et al. (2026).

[1] At the chosen N_x, N_γ, and N_τ, a % error of around 0.01% arises when evaluating the VA value at the fair guarantee fee.

Fair Guarantee Fees

Table: Fair guarantee fees φ^* (in basis points) under a static and dynamic withdrawal strategy for various marginal tax rates θ and cash account appreciation rates η.

		Dynamic			
Tax Rate	Static	$\eta = 2\%$	$\eta = 3\%$	$\eta = 4\%$	$\eta = 5\%$
No Tax	86.6630	126.8871	126.8871	230.1654	351.6089
$\theta = 2.5\%$	35.5875	58.4112	92.3337	198.3923	319.5487
$\theta = 5\%$	NA	5.1702	67.7839	172.2789	291.3602
$\theta = 10\%$	NA	NA	31.1690	127.1135	239.0831
$\theta = 20\%$	NA	NA	NA	48.1429	147.6102

Note: In the no-tax dynamic withdrawal case, φ^* does not decrease below 126.8871 for lower values of η.

Dynamic Withdrawals: Impact of Contract Features

Table: Fair guarantee fees φ^* (in basis points) under a dynamic withdrawal strategy in the presence or absence of taxes, in the presence or absence of a cash fund, and when the guarantee benefit base is updated by a ratchet mechanism or not.

	No Cash Fund		Cash Fund ($\eta = 4\%$)	
	No Ratchet	Ratchet	No Ratchet	Ratchet
No Tax	112.4728	126.8871	204.6804	230.1654
$\theta = 2.5\%$	27.0643	57.2017	154.0414	198.3923
$\theta = 5\%$	NA	NA	118.2448	172.2789
$\theta = 10\%$	NA	NA	67.9520	127.1135
$\theta = 20\%$	NA	NA	NA	48.1429

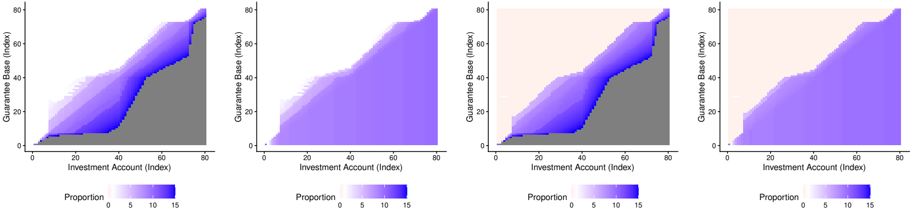
Optimal Withdrawal Strategies

- Since the guaranteed withdrawal amount $g(t_k)$ and the maximum withdrawal amount $\overline{w}(t_k)$ are state-dependent, the optimal withdrawal strategy is best understood via

$$w^*_{\mathrm{guar}}(t_k, x, \gamma) := \frac{w^*(t_k, x, \gamma)}{g(t_k)}, \qquad w^*_{\mathsf{max}}(t_k, x, \gamma) := \frac{w^*(t_k, x, \gamma)}{\overline{w}(t_k, x, \gamma)}.$$

- We use w^*_{guar} to determine whether an amount less than (< 1), equal to $(= 1)$, or greater than (> 1) the guaranteed withdrawal amount was withdrawn.
- We use $w^*_{\mathsf{max}} \in [0, 1]$ to determine when it is optimal to withdraw nothing $(= 0)$, to withdraw all available funds $(= 1)$, or any possibility in between. Figures showing w^*_{max} are in the Annex.

Effect of the Ratchet and Cash Fund (No Tax)[2]

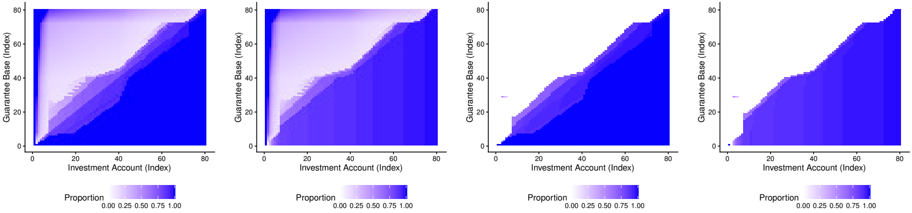


(a) no CF, no ratchet (b) no CF, with ratchet (c) CF, no ratchet (d) CF, with ratchet

Figure: The optimal withdrawal strategy for various contract specifications (cash fund vs. no cash fund, ratchet vs. no ratchet/return-to-premium) expressed in terms of $w^*_{\mathrm{guar}}(t, x, \gamma)$ for $\theta = 0$ and $t = 5$.

[2] $w^*_{\max}$ shown in Figure 2.

Effect of the Ratchet and Cash Fund (No Tax)



(a) no CF, no ratchet (b) no CF, with ratchet (c) CF, no ratchet (d) CF, with ratchet

Figure: The optimal withdrawal strategy for various contract specifications (cash fund vs. no cash fund, ratchet vs. no ratchet/return-to-premium) expressed in terms of $w^*_{\max}(t, x, \gamma)$ for $\theta = 0$ and $t = 4$.

- When a cash fund is available, policyholders exhibit a *bang-bang*-like strategy: they either withdraw nothing or surrender the contract entirely.
- Across all tax settings, between 85% and 98% of policyholders either refrain from withdrawals or withdraw below $\mathfrak{g}$, instead injecting funds into the cash account.
- Only 2–15% surrender, depending on θ (higher θ increasing average contract duration).
- Higher θ values intensify this trend, as many PHs prefer to defer income, avoid tax, and maximize tax-deferred growth in the appreciating cash fund.
- Notably, even without taxation, the mere availability of a cash fund shifts behavior toward non-withdrawal, illustrating its design influence beyond tax incentives.

Table: Simulation analysis of surrender (%) and average spent in the contract (in years)

	No Cash Fund		Cash Fund ($\eta = 4\%$)	
	No Ratchet	Ratchet	No Ratchet	Ratchet
No Tax	32.3% (7.61)	0% (10)	29.53% (7.65)	0% (10)
$\theta = 2.5\%$	15.06% (9.04)	0% (10)	19.91% (8.61)	0% (10)
$\theta = 5\%$	NA	NA	11.94% (9.30)	0% (10)
$\theta = 10\%$	NA	NA	1.34% (9.97)	0% (10)
$\theta = 20\%$	NA	NA	NA	0% (10)

Effect of product features on withdrawal amounts

Table: Simulation analysis of policyholder behavior: strictly smaller than $\mathfrak{g}$ $(<)$, exactly $\mathfrak{g}$ $(=)$ and above $\mathfrak{g}$ $(>)$

	No Cash Fund						Cash Fund ($\eta = 4\%$)					
	No Ratchet			Ratchet			No Ratchet			Ratchet		
θ	$<$	$=$	$>$	$<$	$=$	$>$	$<$	$=$	$>$	$<$	$=$	$>$
0%	5	56	39	0	65	35	85	0	16	89	0	11
2.5%	0	71	29	0	74	26	89	0	11	90	0	10
5%	NA	NA	NA	NA	NA	NA	90	0	10	91	0	9
10%	NA	NA	NA	NA	NA	NA	98	0	2	96	0	4
20%	NA	NA	NA	NA	NA	NA	NA	NA	NA	98	0	2

Surrender as per Table 4.

Ratcheting eliminates surrender incentives

- Recent literature has focused on managing lapse risk in variable annuities, given its implications for insurer solvency (Loisel and Milhaud 2011; Moody's Investor Service 2013).
- A range of surrender mitigation mechanisms has been proposed, including
  - *state-dependent fees* levied only if the account value falls below (Bernard et al. 2014; MacKay et al. 2017) or above (Feng et al. 2025) a specified threshold,
  - *time-dependent fees* that decline over time (Moenig and Zhu 2018; Bernard and Moenig 2019), and
  - *two-account structures* that impose charges only on a secondary, risk-free guarantee account (Alonso-García et al. 2026).
- Our results suggest that a relatively simple structural feature—the ratchet—can eliminate rational surrender behavior effectively.

Key Findings

- Depending on the prevailing guarantee benefit base and the sub-accout value, optimal withdrawal strategies are on a continuum (in contrast to optimal bang-bang-like strategies).
- When there are no taxes, the VA becomes more valuable as the cash rate increases.
- For a sufficiently high tax rate and a sufficiently low cash rate, the contract becomes unattractive to the PH, even if there are no fees.
- Taxation substantially impacts the withdrawal strategy–it is sometimes optimal for the policyholder to withdraw nothing when the tax rate is high.
- When taxes are charged, depending on the cash rate, the PH may be better off withdrawing at the guaranteed amount (i.e. no injections to the cash fund).

- Guaranteed benefit base upgrades (e.g. a ratchet) are sometimes necessary to make the product worthwhile to the PH in the presence of taxation.
- If a cash fund is included in the contract, then the PH will typically never withdraw the guaranteed amount.
- If the cash fund performs better than the risk-free asset and the tax rate is high, it is optimal for the PH to never withdraw anything for most of the life of the contract.
- When there are no taxes, the fair guarantee fee does not fall below a certain level as the cash rate decreases.

Thank you for your attention

Questions & Discussion

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Fee calculation

- We assume the PH is a risk-neutral agent who seeks to maximize the total discounted cash flows from the contract through their choice of withdrawal strategy:
$$J_0(x, \gamma; \mathbf{w}^*) = \sup_{\mathbf{w} \in \mathcal{W}} \mathbb{E}^{\mathbb{Q}}[H_0(\mathbf{X}, \mathbf{G}, \mathbf{w})] =: V_0(x, \gamma).$$
- The fair guarantee fee is the fee ϕ^* at which the valuation of the contract at inception t_0 is equal to the initial premium paid by the PH;
$$P_0 = V_{t_0}(P_0, P_0; \varphi^*).$$
- The fair guarantee fee is nonnegative, as the insurer uses the revenue to finance the guarantees in the contract.
- However, in some cases (to be shown in the numerical illustration), no fair guarantee fee exists.

Backward Dynamic Programming

- Using standard agreements in stochastic optimal control this problem may be solved through backward induction via the Bellman equation

$$V_{t_k}(x,\gamma) = \sup_{w(t_k)\in\mathcal{W}(t_k)} \Big\{ C(t_k,x,\gamma,w(t_k)) \\ + \mathbb{E}^{\mathbb{Q}}_{w(t_k)}\Big[e^{-r\Delta t} V_{t_{k+1}}(X(t^-_{k+1}),G(t^-_{k+1}))|X(t^-_k)=x,G(t^-_k)=\gamma\Big]\Big\},$$

  with terminal condition $V_{t_N}(x,\gamma) = D(x,\gamma)$.
- Here, $C(\cdot,\cdot,\cdot,\cdot)$ and $D(\cdot,\cdot)$ are the running and terminal cash flow functions corresponding to the VA contract (defined from $H_0(\mathbf{X},\mathbf{G},\mathbf{W})$).

Two-Step Solution

For $k = N-1, N-2, \ldots, 2, 1$ (Shevchenko and Luo 2016):

1. For fixed γ, compute the conditional expectation

$$V_{t_k^+}(x,\gamma) = \mathbb{E}^{\mathbb{Q}}\left[e^{-r\Delta t} V_{t_{k+1}^-}(X(t_{k+1}^-),\gamma) | X(t_k^+) = x, G(t_k^+) = \gamma\right].$$

2. Apply the *jump condition* to solve $V_{t_k^-}(x,\gamma)$,

$$V_{t_k^-}(x,\gamma) = \sup_{w(t_k)\in\mathcal{W}(t_k)} \left\{C(t_k, x, \gamma, w(t_k)) + V_{t_k^+}(h^X(x,\gamma,w(t_k)), h^G(x,\gamma,w(t_k)))\right\}.$$

Note: The quantity $V_{t_k^+}(h^X(x,\gamma,w(t_k)), h^G(x,\gamma,w(t_k)))$ is post-withdrawal value function $V_{t_k^+}(x,\gamma)$ evaluated at the **updated** sub-account and guarantee benefit base after the policyholder makes a withdrawal of $w(t_k)$.