Wealth-dependent risk-aversion in dynamic mean-variance portfolio optimization: the good, the bad, and the ugly

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Abstract

We investigate the economic challenges associated with the popular formulation of the time-consistent Mean-Variance (MV) portfolio optimization problem where the risk-aversion parameter is inversely proportional to the investor's wealth, as first proposed by T. Bjork, A. Murgoci and X.Y. Zhou, *Mathematical Finance* (2014). Specifically, compared to the corresponding strategies using a constant risk-aversion parameter, we show that the investment strategies using a wealth-dependent risk-aversion parameter exhibits some undesirable features. In order to consider the problem in a realistic setting, we extend the results of A. Bensoussan et al, *SIFIN* (2014) to allow for the case where the risky asset follows a jump-diffusion process.

Keywords: Asset allocation, constrained optimal control, time-consistent, mean-variance

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1 Introduction

Since its introduction by Markowitz (1952), mean-variance (MV) portfolio optimization has come to play a fundamental role in modern portfolio theory (see for example Elton et al. (2014)), partly due to its intuitive nature. In single-period (non-dynamic) settings, MV optimization simply involves maximizing the expected return of a portfolio given an acceptable level of risk, where risk is measured by the variance of portfolio returns.

In multi-period or dynamic settings (see for example Li and Ng (2000); Zhou and Li (2000)), MV optimization involves maximizing the expected value of the controlled terminal wealth ($\mathbb{E}[W[T]]$), while simultaneously minimizing its variance ($Var[W[T]]$), with $T > 0$ being the investment time horizon. By control, we mean the investment strategy followed by the investor over $[0, T]$. Using the standard scalarization method for multi-criteria optimization problems (Yu (1971)), the single MV objective to be maximized over a set of admissible controls (defined rigorously below), is given by

$$\mathbb{E}[W[T]] - \rho \cdot Var[W[T]], \tag{1.1}$$

where the risk-aversion (or scalarization) parameter $\rho > 0$ reflects the investor's level of risk aversion.

Since the variance term in (1.1) is not separable in the sense of dynamic programming, two main approaches for solving the stochastic optimal control problem with the MV objective (1.1) can be identified. The first approach, pre-commitment MV optimization, typically results in time-inconsistent optimal controls or investment strategies (see Basak and Chabakauri (2010),Vigna (2016)). However, pre-commitment strategies are typically time consistent under an alternative induced objective function (Strub et al. (2019)). It is the second approach, namely time-consistent MV (TCMV) optimization, that is the focus of this paper. The TCMV formulation involves maximizing the objective (1.1) subject to a time-consistency constraint, which essentially means optimization is performed only over the subset of controls which are time-consistent with respect to the objective (1.1); see for example Basak and Chabakauri (2010); Bjork et al. (2017); Bjork and Murgoci (2014); Cong and Oosterlee (2016); Wang and Forsyth (2011).

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We refer to the TCMV problem with a *constant* value $\rho > 0$ of the risk-aversion parameter in the objective (1.1) as the cMV problem. In the special case where the risky asset follows geometric Brownian motion (GBM) dynamics and no investment constraints are applicable (for example, trading continues in the event of insolvency, short selling is permitted, infinite leverage is allowed, etc.), Basak and Chabakauri (2010) solves the cMV problem to find that the resulting optimal control, or amount to be invested in the risky asset at time $t \in [0, T]$, does not depend on the investor's wealth at time t. This observation also holds for the cMV problem if the risky asset follows one of the standard jump-diffusion models for asset prices such as the Merton (1976) or the Kou (2002) models - see for example Zeng et al. (2013).

Observing that this is an economically undesirable outcome, Bjork et al. (2014) proposes replacing the constant ρ in (1.1) with a wealth-dependent risk-aversion parameter of the form

$$\rho(w) = \frac{\gamma}{2w}, \qquad \gamma > 0, \tag{1.2}$$

where $\gamma > 0$ is some constant and $w > 0$ is the investor's current wealth, and finds that the resulting optimal investment strategy depends (linearly) on the current wealth. For analytical purposes, in this paper we follow Bensoussan et al. (2014) in also considering a slightly more general formulation of (1.2), namely

$$\rho(w,t) = \frac{\gamma_t}{2w}, \qquad \gamma_t > 0, \ \forall t \in [0, T], \tag{1.3}$$

where γ_t is a positive, differentiable, non-random function of time with a bounded derivative on $[0, T]$.

We will subsequently refer to either (1.2) or (1.3) as simply the *wealth-dependent*[1] risk-aversion parameter ρ, and the TCMV problem using either (1.2) or (1.3) will be referred to as the dMV problem. We do not consider the additional slight generalizations $\rho(w,t) = \gamma_t / f(w)$ that has been proposed in the literature, where f is for example a linear (Hu et al. (2012); Liang et al. (2014); Peng et al. (2018); Sun et al. (2016)) or a piecewise-linear (Cui et al. (2017, 2015); Zhou et al. (2017)) function of the current wealth, since the main arguments of this paper only require ρ to be inversely proportional to wealth, which is obviously satisfied in these cases.

The wealth-dependent risk-aversion parameter formulation has proven to be very popular in the recent literature concerned with TCMV optimization. To name just a few recent examples, the formulation (1.2)-(1.3) has been described as a "suitable choice" (Bi and Cai (2019)), "more economically relevant" (Li et al. (2016)), "more realistic" (Liang et al. (2014); Zhang and Liang (2017)), "economically reasonable" (Li and Li (2013)), "intuitive and reasonable" (Wang and Chen (2018)), "reasonable and realistic from an economic perspective" (Sun et al. (2016)). Furthermore, it has also proven to be very popular in institutional settings, for example the investment-reinsurance problems faced by insurance providers (Bi and Cai (2019); Li and Li (2013)), investment strategies for pension funds (Liang et al. (2014); Sun et al. (2016); Wang and Chen (2018, 2019)), corporate international investment (Long and Zeng (2016)), and asset-liability management (Peng et al. (2018); Zhang et al. (2017)).

Considering the wealth-dependent ρ formulation on its own merits, this popularity may indeed be justified depending on the investor's perspective, since the assumption that risk-aversion is inversely proportional to wealth might be highly relevant to certain investors. However, since it is used in a TCMV setting, Bensoussan et al. (2019) astutely observes that the impact of the formulation (1.2)-(1.3) should be considered *in conjunction with* the application of the time-consistency constraint, and not on its own merits.

Unfortunately, when applying the time-consistency constraint, the wealth-dependent ρ formulation has a number of problems not associated with a constant ρ. Most criticisms in the literature narrowly focus on its most obvious challenge, first highlighted in Wu (2013), namely that it leads to irrational investor behavior if $w < 0$ since the objective (1.1) can become unbounded[2]. To address this challenge either directly or indirectly, various measures are employed in the literature, which include ruling out the short-selling of all assets to ensure $w > 0$ (Bensoussan et al. (2014), Wang and Chen (2019)), incorporating downside risk constraints (Bi and Cai (2019)), or proposing a more elaborate definitions of $\rho(w,t)$ to ensure that ρ remains non-negative even if $w < 0$ (Cui et al. (2017), Cui et al. (2015), Zhou et al. (2017)). It should be noted that in the case of many of these proposals, the primary objective is simply ensuring the non-negativity of wealth, while the actual economic reasonableness of the changes/constraints in the formulation are only of secondary importance.

In contrast, more fundamental concerns regarding the use of the wealth-dependent ρ formulation in conjunction with the time-consistency constraint are expressed relatively infrequently. For example, Cong and

[1]We note that there are other forms of the risk-aversion parameter considered in literature that are also wealth- or state-dependent, for example it can be a function of the market regime (Liang and Song (2015); Wei et al. (2013); Wu and Chen (2015)). These have not proven as popular as (1.2), and are therefore not considered in this paper.

[2]This problem does not arise in the original formulation of Bjork et al. (2014), since the optimal associated wealth process cannot attain negative values.

Oosterlee (2016) observes that (1.2) combines “easy-to-lose” with “hard-to-recover” features, in that a very small risk-aversion for high levels of wealth implies a willingness to gamble which leads to losses, and very large risk aversion for low levels of wealth result in very low investment returns. Furthermore, using numerical experiments, it is well-known that (1.2), compared to a constant ρ, appear to result in not only less MV-efficient investment outcomes (Cong and Oosterlee (2016); Van Staden et al. (2018); Wang and Forsyth (2011)), but that investment outcomes improve when investment constraints are applied (Bensoussan et al. (2014); Wang and Forsyth (2011)).

A systematic and rigorous analysis of the latter phenomenon is presented by Bensoussan et al. (2019) for the case of GBM dynamics for the risky asset in combination with a specific set of investment constraints. Specifically, Bensoussan et al. (2019) show how the time-consistency constraint in connection with the wealth-dependent ρ results in some economically unreasonable results when no shorting of either asset and no leverage is allowed.

The main objective of this paper is to present a general overview of the economic challenges of using a wealth-dependent ρ in TCMV optimization, and show that these challenges do not arise when a constant ρ is used in conjunction with the time-consistency constraint. The main contributions of this paper are as follows.

- We analytically solve the dMV problem subject to short-selling prohibitions applicable to both the risky and risk-free assets, extending the results of Bensoussan et al. (2014) to allow for the use of any of the commonly used jump-diffusion models in finance as a model of the risky asset process.

- After discussing the arguments presented in Bjork et al. (2014); Landriault et al. (2018) in favor of a wealth-dependent ρ of the form (1.2)-(1.3), we investigate some of the practical economic implications of using this formulation in conjunction with a time-consistency constraint. Our investigation incorporates the available analytical solutions, and where not available, employs numerical solutions of the problem using the algorithm of Van Staden et al. (2018), which allow us to investigate different combinations of investment constraints and portfolio rebalancing frequencies. In all our numerical results, we use model parameters calibrated to inflation-adjusted, long-term US market data (89 years), ensuring that realistic conclusions can be drawn from the results. Taken together, this enables us to make observations regarding some of the economic challenges associated with the wealth-dependent ρ formulation that are wider in scope than the challenges highlighted in for example Bensoussan et al. (2019); Cong and Oosterlee (2016); Van Staden et al. (2018); Wang and Forsyth (2011).

- Our investigation leads to the conclusion that the wealth-dependent ρ of the form (1.2)-(1.3), when used in conjunction with the time-consistency constraint in a dynamic MV optimization setting, leads to potentially undesirable economic outcomes which are not associated with a constant ρ. We therefore conclude that a TCMV investor, by necessity facing realistic investment constraints such as leverage constraints and the need to avoid insolvency, might prefer the constant ρ formulation over the wealth-dependent ρ formulation in the objective (1.1).

The remainder of the paper is organized as follows. Section 2 formulates the various optimization problems as well as the investment constraints under consideration. Section 3 presents the known analytical solutions to the cMV and dMV problems, and presents analytical results for the case where the risky asset follows a jump-diffusion process. In Section 4, the economic challenges of using a wealth-dependent ρ together with a time-consistency constraint are presented and contrasted with the outcomes when using a constant ρ in this setting. Finally, Section 5 concludes the paper.

2 Formulation

Let $T > 0$ denote the fixed investment time horizon/maturity. For any functional f, let $f(t^-) = \lim_{\epsilon \downarrow 0} f(t-\epsilon)$ and $f(t^+) = \lim_{\epsilon \downarrow 0} f(t+\epsilon)$. Informally, t^- and t^+ denotes the instants of time immediately before and after the forward time $t \in [0, T]$, respectively.

We consider portfolios consisting of two assets only, namely a risky asset and a risk-free asset. Since we consider the risky asset to be a well-diversified stock index (see Appendix B) instead of a single stock, this treatment allows us to focus on the primary question of the stocks vs bonds allocation of the portfolio welth, rather than secondary questions relating to risky asset basket compositions[3].

[3]In the available analytical solutions for multi-asset time-consistent MV problems (see, for example, Li and Ng (2000); Zeng and Li (2011)), the composition of the risky asset basket remains relatively stable over time, which suggests that the primary question remains the overall risky asset basket vs. the risk-free asset composition of the portfolio, instead of the exact composition of the risky asset basket.

2.1 Discrete portfolio rebalancing

To model the discrete rebalancing of the portfolio (continuous rebalancing is described in Subsection 2.2 below), let $S(t)$ and $B(t)$ denote the *amounts* invested at time $t \in [0,T]$ in the risky and risk-free asset, respectively. Furthermore, let $X(t) = (S(t), B(t)), t \in [0,T]$ denote the multi-dimensional controlled underlying process, and $x = (s,b)$ the state of the system. The controlled portfolio wealth, denoted by $W(t)$, is given by

$$W(t) = W(S(t), B(t)) \quad = \quad B(t) + S(t), \qquad t \in [0,T]. \tag{2.1}$$

Define $\mathcal{T}_m$ as the set of m predetermined, equally spaced rebalancing times in $[0,T]$,

$$\mathcal{T}_m \quad = \quad \{ t_n | \, t_n = (n-1)\,\Delta t, \; n = 1, \ldots, m \}, \quad \Delta t = T/m. \tag{2.2}$$

Consider any two consecutive rebalancing times $t_n, t_{n+1} \in T_m$. In the case of discrete rebalancing, there is no intervention by the investor according to some control or investment strategy between rebalancing times, i.e. for $t \in \left(t_n^+, t_{n+1}^-\right)$. The amounts in the risky and risk-free asset are assumed to have the following dynamics in the absence of control,

$$\frac{dS(t)}{S(t^-)} = (\mu_t - \lambda\kappa)\,dt + \sigma_t dZ + d\left(\sum_{i=1}^{\pi(t)} (\xi_i - 1)\right), \qquad dB(t) = r_t B(t)\,dt, \qquad t \in \left(t_n^+, t_{n+1}^-\right). \tag{2.3}$$

Here, r_t denotes the continuously compounded risk-free rate, while μ_t and σ_t are the real world drift and volatility respectively, with r_t, μ_t and σ_t assumed to be deterministic, locally Lipschitz continuous functions[4] on $[0,T]$, and $\sigma_t^2 > 0, \forall t$. Z denotes a standard Brownian motion, $\pi(t)$ is a Poisson process with intensity $\lambda \geq 0$, and ξ_i are i.i.d. random variables with $\mathbb{E}[\xi_i - 1] = \kappa$. It is futhermore assumed that ξ_i, $\pi(t)$ and Z are mutually independent. Note that GBM dynamics for $S(t)$ can be recovered from (2.3) by setting the intensity parameter λ to zero.

Let ξ denote a random variable representing a generic jump multiplier with the same probability density function (pdf) $p(\xi)$ as the i.i.d. random variables ξ_i in (2.3). For concreteness, we consider two distributions of $\log \xi$, namely a normal distribution (Merton (1976) model) and an asymmetric double-exponential distribution (Kou (2002) model) - see Appendix B for more details. For subsequent reference, we also define $\kappa_2 = \mathbb{E}\left[(\xi - 1)^2\right]$.

Discrete portfolio rebalancing is modelled using the discrete impulse control formulation as discussed in for example Dang and Forsyth (2014); Van Staden et al. (2018, 2019a), which we now briefly summarize. Let u_n denote the impulse applied at rebalancing time $t_n \in \mathcal{T}_m$, which corresponds to the amount invested in the risky asset after rebalancing the portfolio at time t_n, and let $\mathcal{Z}$ denote the set of admissible impulse values. Suppose that the system is in state $x = (s,b) = (S(t_n^-), B(t_n^-))$ for some $t_n \in \mathcal{T}_m$. Letting $(S(t_n), B(t_n))$ denote the state of the system immediately after the application of the impulse u_n at time t_n, we define

$$S(t_n) \quad = \quad u_n, \qquad B(t_n) = (s+b) - u_n. \tag{2.4}$$

Let $\mathcal{A}$ denote the set of admissible impulse controls, defined as

$$\mathcal{A} = \left\{ \mathcal{U} = (\{t_n, u_n\})_{n=1,\ldots,m} : \; t_n \in \mathcal{T}_m \text{ and } u_n \in \mathcal{Z}, \text{ for } n = 1, \ldots, m \right\}. \tag{2.5}$$

For simplicity, the discrete admissible impulse control $\mathcal{U} \in \mathcal{A}$ associated with given fixed set of rebalancing times $\mathcal{T}_m$ will subsequently be written as only the set of impulses $\mathcal{U} \equiv \mathcal{U}_1 = \{u_n \in \mathcal{Z} : \; n = 1, \ldots, m\}$, while we define $\mathcal{U}_n \equiv \mathcal{U}_{t_n} = \{u_n, u_{n+1} \ldots, u_m\}$ to be the subset of impulses (and, implicitly, the corresponding rebalancing times) of $\mathcal{U}$ applicable to the time interval $[t_n, T]$.

2.2 Continuous portfolio rebalancing

In the case of continuous portfolio rebalancing, let $W^u(t)$ denote the controlled wealth process, and let $u : (W^u(t), t) \mapsto u(t) = u_t = u(W^u(t), t), \; t \in [0,T]$ be the adapted feedback control representing the amount invested in the risky asset at time t given wealth $W^u(t)$. In this case, we follow the example of Bjork et al. (2014); Zeng et al. (2013) in assuming that the dynamics of *unit* investments in the risky and risk-free assets

[4] The assumptions regarding r_t, μ_t and σ_t align with the assumptions of Bensoussan et al. (2014), so that the results reported in Bensoussan et al. (2014) can be extended to jump processes in this paper. Note that the volatility is assumed to be deterministic, which we argue is reasonable given that the results of Ma and Forsyth (2016) show that the effects of stochastic volatility, with realistic mean-reverting dynamics, are not important for long-term investors with time horizons greater than 10 years.

respectively (in the absence of control) are of the form (2.3), so that a single stochastic differential equation for the controlled wealth process[5] can be obtained. Specifically, the dynamics of $W^u(t)$ are given by (see for example Bjork (2009))

$$dW^u(t) = [r_t W^u(t) + \alpha_t u_t] dt + \sigma_t u_t dZ + u_t d\left(\sum_{i=1}^{\pi(t)} (\xi_i - 1)\right), \qquad t \in [0, T], \tag{2.6}$$

where $\alpha_t = \mu_t - \lambda\kappa - r_t$, with all the coefficients and sources of randomness having the same interpretation and properties as in (2.3). For proof that (2.6) is also the limiting case of the discrete impulse control formulation presented in Subsection 2.1 as $\Delta t \downarrow 0$, please refer to Van Staden et al. (2019a).

The set of admissible controls in the case of continuous rebalancing is defined as

$$\mathcal{A}^u = \left\{ u(t) | \, u(t) \in \mathbb{U}^{w,t}, \quad W^u(t) \text{ via (2.6) with } W^u(t) = w, \text{ and } t \in [0, T] \right\}, \tag{2.7}$$

where $\mathbb{U}^{w,t} \subseteq \mathbb{R}$ is the admissible control space applicable at time $t \in [0, T]$ given that the controlled wealth (2.6) is in state $W^u(t) = w$.

2.3 Investment constraints

We now describe the investment constraints considered in this paper, starting with the case of discrete rebalancing. Suppose that the system is in state $x = (s, b) = (S(t_n^-), B(t_n^-))$ for some $t_n \in \mathcal{T}_m$. We follow Dang and Forsyth (2014) in defining the bankruptcy (or insolvency) region $\mathcal{B}$ as

$$\mathcal{B} = \left\{ (s, b) \in \mathbb{R}^2 : \, W(s, b) \leq 0, \quad W \text{ given by (2.1)} \right\}. \tag{2.8}$$

In the case of discrete rebalancing, the following investment constraints will be considered sometimes individually and sometimes jointly, where $(S(t_n), B(t_n))$ is calculated according to (2.4):

$$S(t_n) \geq 0, \qquad n = 1, \ldots, m, \quad \text{(No short selling, risky asset)}, \tag{2.9}$$

$$B(t_n) \geq 0, \qquad n = 1, \ldots, m, \quad \text{(No short selling, risk-free asset)}, \tag{2.10}$$

$$\frac{S(t_n)}{W(S(t_n), B(t_n))} \leq q_{max}, \qquad n = 1, \ldots, m, \quad \text{(Leverage constraint)}, \tag{2.11}$$

as well as the solvency condition

$$\text{If } (s, b) \in \mathcal{B} \text{ at } t_n^- \quad \Rightarrow \begin{cases} \text{we require } (S(t_n) = 0, B(t_n) = W(s, b)) \\ \text{and remains so } \forall t \in [t_n, T]. \end{cases} \quad \text{(Solvency condition)} \tag{2.12}$$

The solvency condition (2.12) states that in the event of bankruptcy, defined to be the case when $(s, b) \in \mathcal{B}$, then the position in the risky asset has to be liquidated, total remaining wealth has to be placed in the risk-free asset, and all subsequent trading activities must cease. The maximum leverage constraint (2.11) ensures that the leverage ratio, defined here as the fraction of wealth invested in the risky asset after rebalancing, does not exceed some maximum value q_{max}, typically in the range $q_{max} \in [1.0, 2.0]$.

For theoretical purposes (see Section 3), we occasionally also consider a combination of (2.9) and (2.11) in constraints of the form

$$p_n \cdot W(S(t_n), B(t_n)) \leq \quad S(t_n) \quad \leq q_n \cdot W(S(t_n), B(t_n)), \; 0 \leq p_n \leq q_n \leq 1, \; n = 1, \ldots, m, \tag{2.13}$$

where we assume that p_n, q_n are specified by the investor for $n = 1, \ldots, m$.

Table 2.1 summarizes the combinations of constraints playing a key role in the subsequent results, as well as the associated naming conventions ("Description" column) and whether an analytical solution is available (see Section 3). Observe that Combination 1_{pq} refers to constraints of the form (2.13). In the case of discrete rebalancing, we will therefore consider the following concrete examples of the set of admissible impulse values

[5] In contrast, as observed in Dang et al. (2017), in the case of the discrete portfolio rebalancing presented in Subsection 2.1, it is conceptually simpler to model the dollar amounts invested in the risky and risk-free asset directly.

$\mathcal{Z}$,

$$
\begin{aligned}
\mathcal{Z}_0 &= \left\{u_n \in \mathbb{R}: \ (S,B) \ \text{ via } (2.4), \ \forall n\right\}, && \text{(No constraints)} \qquad (2.14)\\
\mathcal{Z}_{pq} &= \left\{u_n \in \mathbb{R}: \ (S,B) \ \text{ via } (2.4) \text{ s.t. } (2.13), \ \forall n\right\}, && \text{(Combination } 1_{pq}\text{)}\\
\mathcal{Z}_2 &= \left\{u_n \in \mathbb{R}: \ (S,B) \ \text{ via } (2.4) \text{ s.t. } (2.9),(2.12),(2.11) \text{ with } q_{max} = 1.5, \ \forall n\right\}, && \text{(Combination 2)}
\end{aligned}
$$

Note that Combination 1 in Table 2.1 is a special case of Combination 1_{pq} with $p_n = 0$ and $q_n = q_{max} = 1$ in (2.13) for all n.

Table 2.1: Combinations of constraints considered in this paper

Description	Short selling allowed?		Leverage constraint	If insolvent	Analytical solution available?	
	Risky asset	Risk-free asset			cMV	dMV
No constraints	Yes	Yes	None	Continue trading	Yes	Yes
Combination 1_{pq}	No	No	Lower bound $p \geq 0$, upper bound $q \leq 1$	Not applicable	No	Yes
Combination 1	No	No	$q_{max} = 1.0$	Not applicable	No	Yes
Combination 2	No	Yes	$q_{max} = 1.5$	Liquidate	No	No

In the case of continuous rebalancing, we do not consider Combination 2, while in this case Combination 1_{pq} imposes constraints of the form

$$
p_t W^u(t) \leq \quad u(t) \quad \leq q_t W^u(t), \quad 0 \leq p_t \leq q_t \leq 1, \quad \forall t \in [0,T], \tag{2.15}
$$

where p_t and q_t are locally Lipschitz continuous functions specified by the investor. As a result, the following concrete cases of the admissible control space $\mathbb{U}^{w,t}$ for continuous rebalancing will be considered,

$$
\begin{aligned}
\mathbb{U}_0^{w,t} &= \left\{u(t) \in \mathbb{R}: \ W^u \text{ via } (2.6), \ W^u(t) = w, \ t \in [0,T]\right\}, \qquad \text{(No constraints)} && (2.16)\\
\mathbb{U}_{pq}^{w,t} &= \left\{u(t) \in [p_t w, q_t w]: \ p_t, q_t \text{ as per } (2.15), \ W^u \text{ via } (2.6), \ W^u(t) = w, \ t \in [0,T]\right\}, && (2.17)
\end{aligned}
$$

In the case of continuous rebalancing, Combination 1 can be recovered from Combination 1_{pq} by setting $p_t \equiv 0$ and $q_t \equiv q_{max} = 1$ in (2.15) for all $t \in [0,T]$.

Remark 2.1. (Combinations of constraints) While some of the theoretical results in Section 3 are presented for Combination 1_{pq}, it is not necessarily a very practical set of constraints from an investor's perspective due to the requirement to specify the bounds in (2.13),(2.15). As a result, we instead follow Bensoussan et al. (2019) in highlighting an important special case of Combination 1_{pq}, namely Combination 1 (see Table 2.1) in our calculations and in the numerical results presented in Section 4 below. However, we observe that Combinations 1 and 1_{pq} present an extremely restrictive set of constraints, since even retail investors are typically able to leverage their investments to some extent. Combinations 1 and 1_{pq} effectively also rule out insolvency, since the initial wealth is positive and no borrowing in either asset is permitted. Note that in the case of Combination 2, a constant ρ together with the economically reasonable assumption that $\mu > r$ implies that a short position in the risky asset is never cMV-optimal, so the short-selling restriction in this particular case would not be active; however, as discussed in Section 4 below, this constraint might be active in the case of the dMV problem. Finally, if we were to rank the constraint combinations in terms of the extent to which it restricts investment decisions, we observe that Combination 2 can be informally ranked somewhere between the extremes of "No constraints" and Combination 1, an observation of significance that will be revisited in the subsequent results (see Section 4).

3 Analytical results

Recall that the cMV and dMV problems refer to the TCMV optimization problems using a constant risk-aversion parameter ρ and a wealth-dependent ρ of the form (1.2)-(1.3), respectively, in the objective (1.1).

In this section, we present the formulation and analytical solutions of the cMV and dMV problems, and extend the results of Bensoussan et al. (2014) to the case where the risky asset follows a jump-diffusion process. We also derive a number of additional analytical results that play an important role in the subsequent discussion.

In the case of discrete rebalancing, we fix a set of discrete rebalancing times $\mathcal{T}_m$ as in (2.2). Let $E^{x,t_n}_{\mathcal{U}_n}\left[W\left(T\right)\right]$ and $Var^{x,t_n}_{\mathcal{U}_n}\left[W\left(T\right)\right]$ denote the mean and variance of the terminal wealth $W\left(T\right)$, respectively, given that we are in state $x=(s,b)=\left(S\left(t_n^-\right),B\left(t_n^-\right)\right)$ for some $t_n\in\mathcal{T}_m$ and using discrete impulse control $\mathcal{U}_n\in\mathcal{A}$ over $[t_n,T]$. For subsequent reference, we also define the following constants for $n=1,\ldots,m$,

$$\hat{r}_n = \exp\left\{\int_{t_n}^{t_{n+1}} r_\tau d\tau\right\}, \qquad \hat{\alpha}_n = \exp\left\{\int_{t_n}^{t_{n+1}} \mu_\tau d\tau\right\} - \exp\left\{\int_{t_n}^{t_{n+1}} r_\tau d\tau\right\}, \tag{3.1}$$

$$\hat{\sigma}_n^2 = \exp\left\{\int_{t_n}^{t_{n+1}} \left(2\mu_\tau+\sigma_\tau^2+\lambda\kappa_2\right) d\tau\right\} - \exp\left\{\int_{t_n}^{t_{n+1}} 2\mu_\tau d\tau\right\}. \tag{3.2}$$

In the case of continuous rebalancing, the notation $E^{w,t}_{u}\left[W^u\left(T\right)\right]$ and $Var^{w,t}_{u}\left[W^u\left(T\right)\right]$ denote the mean and variance of terminal wealth, respectively, given wealth $W^u\left(t\right)=w$ at time t and the use of admissible control $u\in\mathcal{A}^u$ over the time period $[t,T]$.

3.1 Constant scalarization parameter

We now formally define problems $cMV_{\Delta t}\left(\rho\right)$ and $cMV\left(\rho\right)$ as the cMV problems (using a constant risk-aversion parameter $\rho>0$) in the cases of discrete and continuous rebalancing, respectively.

Given the state $x=(s,b)=\left(S\left(t_n^-\right),B\left(t_n^-\right)\right)$ for some $t_n\in\mathcal{T}_m$, the cMV problem in the case of discrete rebalancing is defined by (see for example Van Staden et al. (2018))

$$\left(cMV_{\Delta t}\left(\rho\right)\right):\ V^c_{\Delta t}\left(s,b,t_n\right) := \sup_{\mathcal{U}_n\in\mathcal{A}} \left(E^{x,t_n}_{\mathcal{U}_n}\left[W\left(T\right)\right]-\rho\cdot Var^{x,t_n}_{\mathcal{U}_n}\left[W\left(T\right)\right]\right), \quad \rho>0, \tag{3.3}$$

$$\text{s.t. } \mathcal{U}_n = \left\{u_n,\mathcal{U}^{c*}_{n+1}\right\} := \left\{u_n,u^{c*}_{n+1},\ldots,u^{c*}_m\right\}, \tag{3.4}$$

where $\mathcal{U}^{c*}_n=\{u^{c*}_n,\ldots,u^{c*}_m\}$ denotes the optimal control[6] for problem $cMV_{\Delta t}\left(\rho\right)$. We also define the following auxiliary function using $\mathcal{U}^{c*}_n$,

$$g^c_{\Delta t}\left(x,t_n\right) = E^{x,t_n}_{\mathcal{U}^{c*}_n}\left[W\left(T\right)\right]. \tag{3.5}$$

Lemma 3.1 gives the analytical solution to (3.3)-(3.15) in the case of no investment constraints.

Lemma 3.1. *(Analytical solution: Problem $cMV_{\Delta t}\left(\rho\right)$ - discrete rebalancing, no constraints) Fix a set of rebalancing times $\mathcal{T}_m$ and a state $x=(s,b)=\left(S\left(t_n^-\right),B\left(t_n^-\right)\right)$ with wealth wealth $w=s+b$ for some $t_n\in\mathcal{T}_m$. In the case of no constraints ($\mathcal{Z}=\mathcal{Z}_0$), the optimal amount invested in the risky asset at rebalancing time t_n for problem $cMV_{\Delta t}\left(\rho\right)$ in (3.3)-(3.4) is given by*

$$u^{c*}_n = \frac{1}{2\rho}\cdot\frac{\hat{\alpha}_n}{\hat{\sigma}_n^2}\cdot\left(\prod_{i=n+1}^{m}\hat{r}_i\right)^{-1}. \tag{3.6}$$

The auxiliary function $g^c_{\Delta t}$ and value function $V^c_{\Delta t}$ are respectively given by

$$g^c_{\Delta t}\left(x,t_n\right) = \left(\prod_{i=n}^{m}\hat{r}_i\right)\cdot w + \frac{1}{2\rho}\cdot\sum_{i=n}^{m}\frac{\hat{\alpha}_i^2}{\hat{\sigma}_i^2}, \qquad V^c_{\Delta t}\left(x,t_n\right) = g^c_{\Delta t}\left(x,t_n\right) - \frac{1}{4\rho}\cdot\sum_{i=n}^{m}\frac{\hat{\alpha}_i^2}{\hat{\sigma}_i^2}. \tag{3.7}$$

Proof. The proof relies on backward induction - see for example Van Staden et al. (2019a). □

In the case of continuous rebalancing, the cMV problem given wealth $W^u\left(t\right)=w$ at time t, is defined as (see for example Wang and Forsyth (2011))

$$\left(cMV\left(\rho\right)\right):\ V^c\left(w,t\right) := \sup_{u\in\mathcal{A}^u} \left(E^{w,t}_{u}\left[W^u\left(T\right)\right]-\rho\cdot Var^{w,t}_{u}\left[W^u\left(T\right)\right]\right), \quad \rho>0, \tag{3.8}$$

$$\text{s.t. } u^{c*}\left(t;y,v\right) = u^{c*}\left(t';y,v\right), \quad \text{for } v\geq t', t'\in\left[t,T\right], \tag{3.9}$$

where $u^{c*}\left(t;y,v\right)$ denotes the optimal control for problem $cMV\left(\rho\right)$ calculated at time t to be applied at some future time $v\geq t'\geq t$ given future state $W^u\left(v\right)=y$, while $u^{c*}\left(t';v,y\right)$ denotes the optimal control calculated at

[6] The resulting optimal control $\mathcal{U}^{c*}_n$ satisfies the conditions of a subgame perfect Nash equilibrium control, justifying the terminology "equilibrium" control often preferred (see e.g. Bensoussan et al. (2014); Bjork et al. (2014)). However, we will follow the example of Basak and Chabakauri (2010); Cong and Oosterlee (2016); Li and Li (2013); Wang and Forsyth (2011) and retain the terminology "optimal" control for simplicity.

some future time $t' \in [t, T]$ for problem $cMV(\rho)$, also to be applied at the same later time $v \geq t'$ given the same future state $W^u(v) = y$. To lighten notation and emphasize dependence on the given wealth level $W^u(t) = w$ at time t (which remains implicit in (3.9) for purposes of clarity), we will use the notation $u^{c*}(w,t)$ to denote the optimal control for problem (3.8)-(3.9). Using control u^{c*}, we define the following auxiliary function,

$$g^c(w,t) = E^{x,t_n}_{u^{c*}}\left[W^u(T)\right]. \tag{3.10}$$

Lemma 3.2 gives the analytical solution to (3.8)-(3.9) in the case of no investment constraints.

Lemma 3.2. *(Analytical solution: Problem cMV (ρ) - continuous rebalancing, no constraints). Suppose we are given wealth $W^u(t) = w$ at time $t \in [0,T]$. In the case of no investment constraints ($\mathbb{U}^{w,t} = \mathbb{U}_0^{w,t}$), the optimal amount invested in the risky asset at time t for problem cMV (ρ) in (3.8)-(3.9) is given by*

$$u^{c*}(w,t) = \frac{(\mu_t - r_t)}{2\rho\left(\sigma_t^2 + \lambda\kappa_2\right)} e^{-\int_t^T r_\tau d\tau}. \tag{3.11}$$

The auxiliary function g^c and value function V^c are respectively given by

$$g^c(w,t) = w \cdot e^{\int_t^T r_\tau d\tau} + \frac{1}{2\rho}\int_t^T \frac{(\mu_\tau - r_\tau)^2}{(\sigma_\tau^2 + \lambda\kappa_2)} d\tau, \quad V^c(w,t) = g^c(w,t) - \frac{1}{4\rho}\int_t^T \frac{(\mu_\tau - r_\tau)^2}{(\sigma_\tau^2 + \lambda\kappa_2)} d\tau. \tag{3.12}$$

Proof. See Basak and Chabakauri (2010); Zeng et al. (2013). □

As highlighted in Basak and Chabakauri (2010); Bjork et al. (2014), the optimal controls in the case of a constant ρ (see (3.6) and (3.11)) do not depend on the investor's current wealth w. For subsequent use, we also introduce the following definition that is standard in the literature (see for example Wang and Forsyth (2010)).

Definition 3.3. (Efficient frontier - cMV problem) Suppose that the system is in state $x_0 = (s_0, b_0)$ with initial wealth $w_0 = s_0 + b_0$ at time $t_0 \equiv t_1 = 0 \in \mathcal{T}_m$. Define the following sets associated with problems $cMV_{\Delta t}(\rho)$ and $cMV(\rho)$, respectively,

$$\mathcal{Y}_{cMV_{\Delta t}(\rho)} = \left\{\left(\sqrt{Var^{x_0,t_0}_{\mathcal{U}^{c*}}\left[W(T)\right]}, E^{x_0,t_0}_{\mathcal{U}^{c*}}\left[W(T)\right]\right)\right\},$$

$$\mathcal{Y}_{cMV(\rho)} = \left\{\left(\sqrt{Var^{w_0,t_0}_{u^{c*}}\left[W^u(T)\right]}, E^{w_0,t_0}_{u^{c*}}\left[W^u(T)\right]\right)\right\}. \tag{3.13}$$

The efficient frontiers associated with problems $cMV_{\Delta t}(\rho)$ and $cMV(\rho)$ are defined as $\bigcup_{\rho>0}\mathcal{Y}_{cMV_{\Delta t}(\rho)}$ and $\bigcup_{\rho>0}\mathcal{Y}_{cMV(\rho)}$, respectively.

3.2 Wealth-dependent scalarization parameter

We formulate the dMV problem in terms of the wealth-dependent scalarization parameter of the form (1.3), with the formulation (1.2) being a special case used for illustrative purposes in the numerical results in Section 4.

In the case of discrete rebalancing, given the set $\{\gamma_n : n = 1, \ldots, m\}$, we define $\rho_n = \gamma_n/(2w)$ as the risk-aversion parameter applicable at time $t_n \in \mathcal{T}$ for the interval $[t_n, t_{n+1})$. Given the state $x = (s,b) = (S(t_n^-), B(t_n^-))$ for some $t_n \in \mathcal{T}_m$, let $W(s,b) = s + b = w > 0$. Problem $dMV_{\Delta t}(\gamma_n)$ is then defined as (see for example Bensoussan et al. (2014))

$$(dMV_{\Delta t}(\gamma_n)): \; V^d_{\Delta t}(s,b,t_n) := \sup_{\mathcal{U}_n\in\mathcal{A}}\left(E^{x,t_n}_{\mathcal{U}_n}\left[W(T)\right] - \frac{\gamma_n}{2w}\cdot Var^{x,t_n}_{\mathcal{U}_n}\left[W(T)\right]\right), \quad \gamma_n > 0, \tag{3.14}$$

$$\text{s.t. } \mathcal{U}_n = \left\{u_n, \mathcal{U}^{d*}_{n+1}\right\} := \left\{u_n, u^{d*}_{n+1}, \ldots, u^{d*}_m\right\}, \tag{3.15}$$

where $\mathcal{U}^{d*}_n = \left\{u^{d*}_n, \ldots, u^{d*}_m\right\}$ is the optimal control for problem $dMV_{\Delta t}(\gamma_n)$, also used to define the following auxiliary functions:

$$g^d_{\Delta t}(x,t_n) = E^{x,t_n}_{\mathcal{U}^{d*}_n}\left[W(T)\right], \qquad h^d_{\Delta t}(x,t_n) = E^{x,t_n}_{\mathcal{U}^{d*}_n}\left[W^2(T)\right]. \tag{3.16}$$

The available analytical solutions to problem $dMV_{\Delta t}(\gamma_n)$ are presented in Lemma 3.4.

Lemma 3.4. *(Analytical solution: Problem $dMV_{\Delta t}(\gamma_n)$ - discrete rebalancing) Fix a set of rebalancing times $\mathcal{T}_m$ and a state $x = (s,b) = (S(t_n^-), B(t_n^-))$ with wealth $w = s + b > 0$ for some $t_n \in \mathcal{T}_m$. In the cases of (i) no* *constraints ($\mathcal{Z} = \mathcal{Z}_0$) and (ii) Combination 1_{pq} ($\mathcal{Z} = \mathcal{Z}_{pq}$), the optimal amount invested in the risky asset at rebalancing time t_n for problem $dMV_{\Delta t}(\gamma_n)$ in (3.14)-(3.15) is given by*

$$u_n^{d*} = C_n w, \quad \text{where} \quad C_n = F_n\left(\frac{\hat{\alpha}_n}{\gamma_n} \cdot \frac{A_{n+1} - \gamma_n \hat{r}_n\left(D_{n+1} - A_{n+1}^2\right)}{\hat{\alpha}_n^2\left(D_{n+1} - A_{n+1}^2\right) + \hat{\sigma}_n^2 D_{n+1}}\right), \tag{3.17}$$

while the auxiliary functions $g_{\Delta t}^d$ and $h_{\Delta t}^d$, defined in (3.16) are given by

$$g_{\Delta t}^d(x, t_n) = A_n w, \qquad h_{\Delta t}^d(x, t_n) = D_n w^2. \tag{3.18}$$

Here, A_n and D_n solve the following difference equations,

$$A_n = \left(\hat{r}_n + \hat{\alpha}_n C_n\right) A_{n+1}, \qquad n = 1, \ldots, m, \tag{3.19}$$

$$D_n = \left[\left(\hat{r}_n + \hat{\alpha}_n C_n\right)^2 + \hat{\sigma}_n^2 C_n^2\right] D_{n+1}, \qquad n = 1, \ldots, m, \tag{3.20}$$

with terminal conditions $A_{m+1} = 1$ and $D_{m+1} = 1$, respectively, while the function F_n depends on the combination of constraints,

$$F_n(y) = \begin{cases} y & \text{if } \mathcal{Z} = \mathcal{Z}_0, \quad (\text{No constraints}) \\ F_n^{pq}(y) & \text{if } \mathcal{Z} = \mathcal{Z}_{pq}, \quad (\text{Combination } 1_{pq}) \end{cases}, \quad \text{where} \quad F_n^{pq}(y) = \begin{cases} p_n & \text{if } y < p_n \\ y & \text{if } y \in [p_n, q_n] \\ q_n & \text{if } y > q_n. \end{cases}. \tag{3.21}$$

Finally, for all $n = 1, \ldots, m$, we have $D_n > 0$ and $\left(D_n - A_n^2\right) \geq 0$.

Proof. See Bensoussan et al. (2014). □

We introduce the following assumption, which is occasionally used for convenience to illustrate some practical implications of the analytical results.

Assumption 3.1. *(Constant process parameters) In the dynamics (2.3) and (2.6), we (occasionally) assume that the parameters are constants, i.e. let $r_t \equiv r > 0$, $\mu_t \equiv \mu > r$ and $\sigma_t \equiv \sigma > 0$ for all $t \in [0, T]$. Under this assumption, the constants (3.1)-(3.2) simplify to $\hat{r}_n \equiv \hat{r}$, $\hat{\alpha}_n \equiv \hat{\alpha}$ and $\hat{\sigma}_n^2 \equiv \hat{\sigma}^2$ for all $n = 1, \ldots, m$, where we define*

$$\hat{r} = e^{r\Delta t}, \qquad \hat{\alpha} = \left(e^{\mu\Delta t} - e^{r\Delta t}\right), \qquad \hat{\sigma}^2 = \left(e^{\left(2\mu + \sigma^2 + \lambda\kappa_2\right)\Delta t} - e^{2\mu\Delta t}\right). \tag{3.22}$$

The solution of the difference equations (3.19)-(3.20) in Lemma 3.4 becomes analytically intractable fairly quickly as $n \leq m - 2$. In Lemma 3.5 and Lemma 3.6 below, we present the explicit analytical solutions in the case of the penultimate rebalancing time $t_{m-1} = T - 2\Delta t$, which also corresponds to the case of an investor rebalancing twice in $[0, T]$. These results play an important role in the discussion in Section 4.

Lemma 3.5. *($dMV_{\Delta t}(\gamma)$-optimal fraction of wealth in risky asset at time t_{m-1}: No constraints) Assume that the system is in the state $x = (s, b) = \left(S\left(t_{m-1}^-\right), B\left(t_{m-1}^-\right)\right)$ with wealth $w = s + b > 0$ and that Assumption 3.1 is applicable. Furthermore, set $\gamma_n \equiv \gamma > 0$ for all n. In the case of no investment constraints, the $dMV_{\Delta t}(\gamma)$-optimal fraction of wealth C_{m-1} invested in the risky asset at time $t_{m-1} = T - 2\Delta t$ is given by*

$$C_{m-1} = \frac{\hat{r}\gamma - (\hat{r} - 1)\frac{\hat{\alpha}^2}{\hat{\sigma}^2}}{\gamma^2 \hat{r}^2 \frac{\hat{\sigma}^2}{\hat{\alpha}} + 2\gamma\hat{r}\hat{\alpha} + \hat{\alpha} + 2\frac{\hat{\alpha}^3}{\hat{\sigma}^2}}, \qquad \gamma > 0. \tag{3.23}$$

The function $\gamma \to C_{m-1}(\gamma)$ attains a unique, global maximum at $\gamma = \gamma_{m-1}^{max} > 0$, where

$$\gamma_{m-1}^{max} = \frac{\hat{\alpha}}{\hat{\sigma}^2} \cdot \frac{\hat{\alpha}(\hat{r} - 1) + \sqrt{\hat{\alpha}^2\left(1 + \hat{r}^2\right) + \hat{\sigma}^2}}{\hat{r}}. \tag{3.24}$$

Furthermore, for sufficiently small $\gamma > 0$, we have

$$C_{m-1}(\gamma) = -\hat{k}_0 + \hat{k}_1 \cdot \gamma - \hat{k}_2 \cdot \gamma^2 + \mathcal{O}\left(\gamma^3\right), \quad \text{where} \tag{3.25}$$

$$\hat{k}_0 = \frac{(\hat{r} - 1)\hat{\alpha}}{2\hat{\alpha}^2 + \hat{\sigma}^2}, \quad \hat{k}_1 = \frac{\hat{\sigma}^2 \hat{r}\left(2\hat{r}\hat{\alpha}^2 + \hat{\sigma}^2\right)}{\hat{\alpha}\left(2\hat{\alpha}^2 + \hat{\sigma}^2\right)^2}, \quad \hat{k}_2 = \frac{\hat{r}^2\hat{\sigma}^4}{\hat{\alpha}\left(2\hat{\alpha}^2 + \hat{\sigma}^2\right)^2}\left(\frac{(\hat{r} - 1)\left(2\hat{\alpha}^2 - \hat{\sigma}^2\right)}{\left(2\hat{\alpha}^2 + \hat{\sigma}^2\right)} + 2\right). \tag{3.26}$$

If $r\Delta t < 1$, which is a sufficient but not necessary condition, easily satisfied if economically reasonable parameters are used, we have $\hat{k}_0 > 0$, $\hat{k}_1 > 0$ and $\hat{k}_2 > 0$.

Proof. Result (3.23) follows from Lemma 3.4, with the first order optimality condition giving (3.24), where $\mu > r > 0$ ensures that $\hat{\alpha} > 0$ and $\hat{r} > 1$, so that $\gamma_{m-1}^{max} > 0$. Expanding $\gamma \to C_{m-1}(\gamma)$ up to second order gives (3.25)-(3.26). Since $\mu > r > 0$, then $\hat{k}_0 > 0$, $\hat{k}_1 > 0$, and additionally requiring $r\Delta t < 1$ is sufficient to ensure that $(\hat{r}-1)\left(2\hat{\alpha}^2 - \hat{\sigma}^2\right) + 2\left(2\hat{\alpha}^2 + \hat{\sigma}^2\right) > 0$, so that $\hat{k}_2 > 0$. ☐

Lemma 3.6 extends the results of Lemma 3.5 to the case of Combination 1 of investment constraints

Lemma 3.6. *($dMV_{\Delta t}(\gamma)$-optimal fraction of wealth in risky asset at time t_{m-1}: Combination 1) Assume that the system is in the state $x = (s,b) = \left(S\left(t_{m-1}^-\right), B\left(t_{m-1}^-\right)\right)$ with wealth $w = s + b > 0$ and that Assumption 3.1 is applicable. Furthermore, set $\gamma_n \equiv \gamma > 0$ for all n. In the case of Combination 1 of constraints, the $dMV_{\Delta t}(\gamma)$-optimal fraction of wealth C_{m-1} invested in the risky asset at time $t_{m-1} = T - 2\Delta t$ is given by*

$$C_{m-1} = \begin{cases} 1 & \text{if } 0 < \gamma < \gamma_{m-1}^{crit} \\ \left(\frac{\hat{\alpha}}{\hat{\sigma}^2} \cdot \frac{(\hat{r}+\hat{\alpha})}{2\hat{\alpha}(\hat{r}+\hat{\alpha})+\hat{r}^2+\hat{\sigma}^2}\right)\frac{1}{\gamma} - \left(\frac{\hat{\alpha}\hat{r}}{2\hat{\alpha}(\hat{r}+\hat{\alpha})+\hat{r}^2+\hat{\sigma}^2}\right) & \text{if } \gamma_{m-1}^{crit} \leq \gamma < \frac{\hat{\alpha}}{\hat{\sigma}^2} \\ \frac{\hat{r}\gamma - (\hat{r}-1)\frac{\hat{\alpha}^2}{\hat{\sigma}^2}}{\gamma^2\hat{r}^2\frac{\hat{\sigma}^2}{\hat{\alpha}} + 2\gamma\hat{r}\hat{\alpha} + \hat{\alpha} + 2\frac{\hat{\alpha}^3}{\hat{\sigma}^2}} & \text{if } \gamma \geq \frac{\hat{\alpha}}{\hat{\sigma}^2}, \end{cases} \tag{3.27}$$

where

$$\gamma_{m-1}^{crit} = \frac{\hat{\alpha}}{\hat{\sigma}^2} \cdot \frac{(\hat{r}+\hat{\alpha})}{3\hat{\alpha}\hat{r} + 2\hat{\alpha}^2 + \hat{r}^2 + \hat{\sigma}^2}. \tag{3.28}$$

Proof. This result follows from Lemma 3.4. If $\mu > r > 0$, then $\hat{\alpha} > 0$ and $\hat{r} > 1$, so $0 < \gamma_{m-1}^{crit} < \frac{\hat{\alpha}}{\hat{\sigma}^2}$. ☐

In the case of continuous rebalancing, the dMV problem given wealth $W^u(t) = w > 0$ at time t is defined as

$$\begin{aligned} (dMV(\gamma_t)):\ V^d(w,t) &:= \sup_{u \in \mathcal{A}^u} \left(E_u^{w,t}\left[W^u(T)\right] - \frac{\gamma_t}{2w} \cdot Var_u^{w,t}\left[W^u(T)\right]\right), && (3.29) \\ \text{s.t. } u^{d*}(t;y,v) &= u^{d*}(t';y,v), \quad \text{for } v \geq t', t' \in [t,T], && (3.30) \end{aligned}$$

where u^{d*} denotes the optimal control for problem $dMV(\gamma_t)$, and the interpretation of the time-consistency constraint (3.30) is the same as in the case of (3.9).

Using the techniques of Bjork et al. (2017), we have the following verification theorem and corresponding extended HJB equation associated with problem $dMV(\gamma_t)$ in (3.29)-(3.30) subject to Combination 1_{pq} of constraints.

Theorem 3.7. *(Verification theorem) Suppose that, for all $(w,t), (y,\tau) \in \mathbb{R}^+ \times [0,T]$, there exist real-valued functions $V^d(w,t)$, $g^d(w,t)$, $u^{d*}(w,t)$ and $f(w,t,y,\tau)$ with the following properties: 1) V^d, g^d and f are sufficiently smooth and solves the extended HJB system of equations (3.31)-(3.34), and 2) the function $u^{d*}(w,t)$ is an admissible control ($u^{d*} \in \mathcal{A}^u$) that attains the pointwise supremum in equation (3.31).*

$$\begin{aligned} &\frac{\partial V^d}{\partial t}(w,t) - \frac{\partial f}{\partial \tau}(w,t,w,t) - \left(\frac{\gamma_t'}{2w} + \lambda\frac{\gamma_t}{2w}\right)\left(g^d(w,t)\right)^2 - \lambda V^d(w,t) \\ &+ \sup_{u \in [p_t w, q_t w]} \Bigg\{ (r_t w + \alpha_t u)\left[\frac{\partial V^d}{\partial w}(w,t) - \frac{\partial f}{\partial y}(w,t,w,t) + \frac{\gamma_t}{2w^2}\left(g^d(w,t)\right)^2\right] \\ &\quad + \frac{1}{2}\sigma_t^2 u^2 \left[\frac{\partial^2 V^d}{\partial w^2}(w,t) - \frac{\gamma_t}{w^3}\left(g^d(w,t)\right)^2 + 2g^d(w,t)\frac{\gamma_t}{w^2}\frac{\partial g^d}{\partial w}(w,t)\right. \\ &\quad \left. - \frac{\gamma_t}{w}\left(\frac{\partial g^d}{\partial w}(w,t)\right)^2 - 2\frac{\partial^2 f}{\partial w \partial y}(w,t,w,t) - \frac{\partial^2 f}{\partial y^2}(w,t,w,t)\right] \\ &\quad + \lambda \int_0^\infty \left[f(w+u(\xi-1),t,w,t) - f(w+u(\xi-1),t,w+u(\xi-1),t)\right] p(\xi)\,d\xi \\ &\quad + \lambda \int_0^\infty \left[\frac{\gamma_t}{w} g^d(t,w) \cdot g^d(w+u(\xi-1),t) + V^d(w+u(\xi-1),t)\right] p(\xi)\,d\xi \\ &\quad - \lambda\gamma_t \int_0^\infty \frac{1}{2(w+u(\xi-1))}\left(g^d(w+u(\xi-1),t)\right)^2 p(\xi)\,d\xi \Bigg\} = 0, \end{aligned} \tag{3.31}$$

$$\frac{\partial g^d}{\partial t}(w,t) + \left(r_t w + \alpha_t u^{d*}\right)\frac{\partial g^d}{\partial w}(w,t) + \frac{1}{2}\sigma_t^2\left(u^{d*}\right)^2\frac{\partial^2 g^d}{\partial w^2}(w,t)$$
$$-\lambda g^d(w,t) + \lambda\int_0^\infty g^d\left(w + u^{d*}(\xi-1), t\right)p(\xi)\,d\xi \;=\; 0, \tag{3.32}$$
$$\frac{\partial f}{\partial t}(w,t,y,\tau) + \left(r_t w + \alpha_t u^{d*}\right)\frac{\partial f}{\partial w}(w,t,y,\tau) + \frac{1}{2}\sigma_t^2\left(u^{d*}\right)^2\frac{\partial^2 f}{\partial w^2}(w,t,y,\tau)$$
$$-\lambda f(w,t,y,\tau) + \lambda\int_0^\infty f\left(w + u^{d*}(\xi-1), t, y, \tau\right)p(\xi)\,d\xi \;=\; 0, \tag{3.33}$$
$$V^d(w,T) = w, \qquad g^d(w,T) = w, \qquad f(w,T,y,\tau) = w - \frac{\gamma(\tau)}{2y}w^2. \tag{3.34}$$

Then u^{d*} is the optimal control and V^d is the value function for problem dMV (γ_t) in (3.29)-(3.30) subject to Combination 1_{pq} of investment constraints. In addition, the functions g and f have the probabilistic representations

$$g^d(w,t) = E_{u^{d*}}^{w,t}\left[W^u(T)\right], \qquad f(w,t,y,\tau) = E_{u^{d*}}^{w,t}\left[W^u(T) - \frac{\gamma_\tau}{2y}\left(W^u(T)\right)^2\right], \tag{3.35}$$

where W^u denotes the controlled wealth process using $u^{d*}(w,t)$ in dynamics (2.6).

Proof. See Appendix A. □

We observe that by setting $\lambda \equiv 0$ in Theorem 3.7, we recover the extended HJB equation presented in Bensoussan et al. (2014), as expected. The next theorem gives a solution to the extended HJB equation presented in Theorem 3.7, as well as the solution in the case of no investment constraints.

Theorem 3.8. *(Analytical solution: Problem dMV (γ_t) - continuous rebalancing, with constraints and jumps, $\rho(t,w) = \gamma_t/(2w)$). A solution to the optimal amount invested in the risky asset u^{d*} for problem dMV (γ_t) satisfying the extended HJB equation of Theorem 3.7, subject to either (i) no investment constraints ($\mathbb{U}^{w,t} = \mathbb{U}_0^{w,t}$) or (ii) Combination 1_{pq} of constraints ($\mathbb{U}^{w,t} = \mathbb{U}_{pq}^{w,t}$), is given by*

$$u^{d*}(w,t) \;=\; c(t)\,w, \quad \text{where } c(t) = F_t\left(\frac{\mu_t - r_t}{\gamma_t\left(\sigma_t^2 + \lambda\kappa_2\right)}\left\{e^{-I_1(t;c)-I_2(t;c)} + \gamma_t e^{-I_2(t;c)} - \gamma_t\right\}\right). \tag{3.36}$$

Here, $I_1(t;c)$ and $I_2(t;c)$ are defined as

$$I_1(t;c) = \int_t^T \left(r_\tau + (\mu_\tau - r_\tau)\,c(\tau)\right)d\tau, \qquad I_2(t;c) = \int_t^T \left(\sigma_\tau^2 + \lambda\kappa_2\right)c^2(\tau)\,d\tau, \tag{3.37}$$

while F_t depends on the combination of constraints,

$$F_t(y) = \begin{cases} y & \text{if } \mathbb{U}^{w,t} = \mathbb{U}_0^{w,t} \qquad (\text{No constraints}) \\ F_t^{pq}(y) & \text{if } \mathbb{U}^{w,t} = \mathbb{U}_{pq}^{w,t}, \qquad (\text{Combination } 1_{pq}) \end{cases}, \quad \text{where } F_t^{pq}(y) = \begin{cases} p_t & \text{if } y < p_t \\ y & \text{if } y \in [p_t, q_t] \\ q_t & \text{if } y > q_t \end{cases}. \tag{3.38}$$

Furthermore, the value function V^d of problem dMV$_t$ (γ_t) is given by

$$V^d(w,t) \;=\; \left[e^{I_1(t;c)} - \frac{\gamma_t}{2}\cdot e^{2I_1(t;c)}\left(e^{I_2(t;c)} - 1\right)\right]w, \tag{3.39}$$

while the functions f and g^d, with probabilistic representations as in (3.35), are given by

$$g^d(w,t) = e^{I_1(t;c)}w, \qquad f(w,t,y,\tau) = g^d(w,t) - \left[\frac{\gamma_\tau}{2y}\cdot e^{2I_1(t;c)+I_2(t;c)}\right]w^2. \tag{3.40}$$

Proof. For the case of no investment constraints, see Bjork et al. (2014); Sun et al. (2016). For the case of Combination 1_{pq} of constraints, see Appendix A. □

As expected, setting $\lambda \equiv 0$ in the case of Combination 1_{pq} of constraints in Theorem 3.8 recovers the results presented in Bensoussan et al. (2014) for the case where the risky asset follows GBM dynamics. The existence of a unique solution to the integral equation (3.36) is established by the following lemma.

Lemma 3.9. *(Uniqueness of integral equation for c) The integral equation for $c(t)$ in (3.36) admits a unique solution in $C[0,T]$, the space of continuous functions on $[0,T]$ endowed with the supremum norm.*

Proof. Since σ_t is assumed to be locally Lipschitz continuous and therefore uniformly bounded on $[0,T]$, so is $\sigma_t^2 + \lambda\kappa_2$, therefore the same arguments as in Bensoussan et al. (2014) can be used to conclude the result of the lemma. $\square$

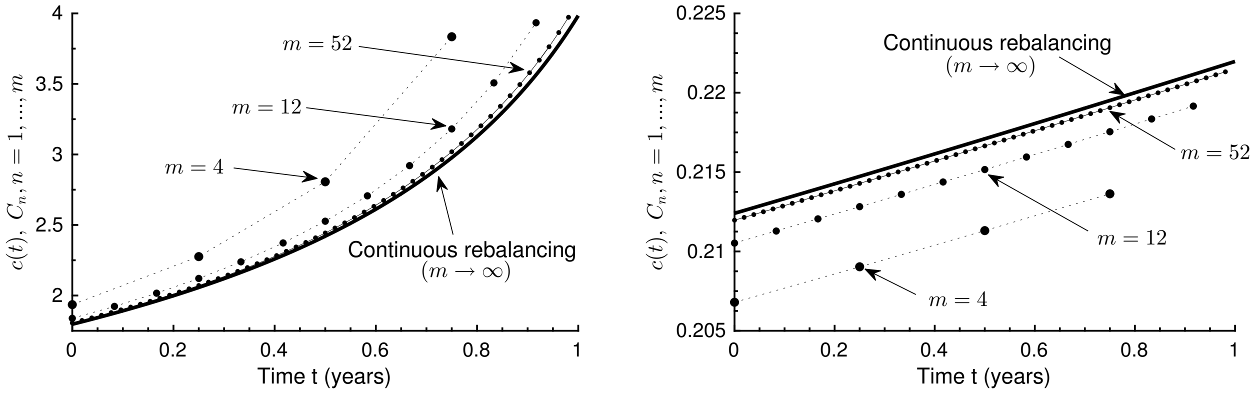
Lemma 3.10 gives the expected convergence $C_n \to c(t_n)$ as $\Delta t \downarrow 0$ (or $m \to \infty$) for the case of jumps in the risky asset process, which is illustrated in Figure 3.1.

Lemma 3.10. *(Convergence) Given $\gamma_t > 0$, $t \in [0,T]$, consider the continuous rebalancing problem $dMV(\gamma_t)$ subject to either (i) no constraints, or (ii) Combination 1_{pq} of constraints, in which case we are also given p_t, q_t with $0 \le p_t \le q_t \le 1$ for all $t \in [0,T]$. For a given set of rebalancing times $\mathcal{T}_m$, define the discrete rebalancing approximation to problem $dMV(\gamma_t)$ as the problem $dMV_{\Delta t}(\gamma_n)$ obtained by choosing $\gamma_n := \gamma_{t_n}$, $n = 1, \ldots, m$, and in the case of Combination 1_{pq}, setting*

$$p_n := p_{t_n}, \quad q_n := q_{t_n}, \quad n = 1, \ldots, m. \tag{3.41}$$

Then for all $\epsilon > 0$, there exists $K_\epsilon > 0$ independent of n such that $|C_n - c(t_n)| < K_\epsilon \epsilon$ for all $n = 1, \ldots, m$, where C_n and $c(t_n)$ is given by (3.17) and (3.36), respectively.

Proof. Since $\sigma_t^2 + \lambda\kappa_2$ is uniformly bounded on $[0,T]$, the result can be proven using similar arguments as in Bensoussan et al. (2014). $\square$



(a) No constraints ($\gamma = 0.4035$)

(b) Combination 1 ($\gamma = 7.2423$)

Figure 3.1: Illustration of the convergence of $C_n \to c(t_n)$, where $t_n = (n-1)\cdot(T/m)$, as $m \to \infty$. Initial wealth $w_0 = 100$, $T = 1$ year, $\gamma_t = \gamma_n = \gamma > 0, \forall t, n$. Kou model, parameters as in Appendix B.

To define the efficient frontier in the case of the dMV problem, we limit our attention to the case where $\gamma_n = \gamma_t \equiv \gamma > 0$, for all $n = 1, \ldots, m$ and all $t \in [0,T]$, since (as discussed in Section 4), this turns out to be not too restrictive.

Definition 3.11. (Efficient frontier - dMV problem) Suppose that the system is in state $x_0 = (s_0, b_0)$ with initial wealth $w_0 = s_0 + b_0 > 0$ at $t_0 \equiv t_1 = 0 \in \mathcal{T}_m$, and that the risk-aversion parameter is of the form $\rho(w) = \gamma/(2w)$ for some constant $\gamma > 0$. Define the following sets associated with problems $dMV_{\Delta t}(\gamma)$ and $dMV(\gamma)$, respectively:

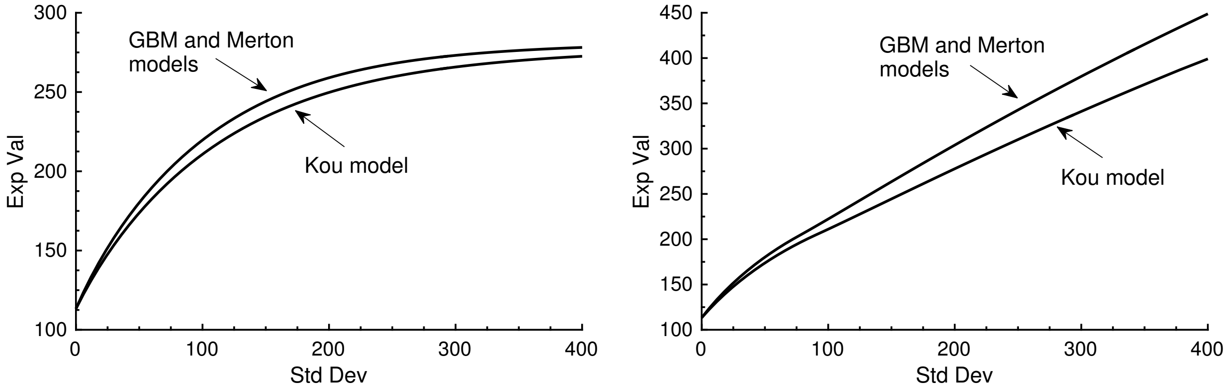
$$\begin{aligned} \mathcal{Y}_{dMV_{\Delta t}(\gamma)} &= \left\{ \left(\sqrt{Var^{x_0,t_0}_{\mathcal{U}^{d*}}[W(T)]}, E^{x_0,t_0}_{\mathcal{U}^{d*}}[W(T)] \right) \right\}, \\ \mathcal{Y}_{dMV(\gamma)} &= \left\{ \left(\sqrt{Var^{w_0,t_0}_{u^{d*}}[W^u(T)]}, E^{w_0,t_0}_{u^{d*}}[W^u(T)] \right) \right\}. \end{aligned} \tag{3.42}$$

The efficient frontiers associated with problems $dMV_{\Delta t}(\gamma)$ and $dMV(\gamma)$ are then defined as $\bigcup_{\gamma>0} \mathcal{Y}_{dMV_{\Delta t}(\gamma)}$ and $\bigcup_{\gamma>0} \mathcal{Y}_{dMV(\gamma)}$, respectively.

Figure 3.2 illustrates the efficient frontiers (Definition 3.11) constructed using the results of Theorem 3.8. It is clear that using a jump-diffusion model for the risky asset can potentially have a material effect[7] on the

[7] The fact that the frontiers for the GBM and Merton models is not entirely unexpected - see Van Staden et al. (2019b).

investment outcomes, illustrating the importance of the extension of the results of Bensoussan et al. (2014) to jump processes as presented in this section.



(a) No constraints (b) Combination 1

Figure 3.2: Efficient frontiers for the dMV problem with continuous rebalancing, where $\rho(w) = \gamma/(2w)$, $\gamma > 0$. Initial wealth $w_0 = 100$, $T = 20$. All model parameters as in Appendix B.

3.3 Comparison of objective functionals

In order to explain the consequences of using different risk aversion parameter formulations in conjunction with the time-consistency constraint in dynamic MV optimization, the objective functionals presented in Lemma 3.12 plays a key role in the subsequent discussion.

Lemma 3.12. *(Objective functionals - discrete rebalancing). Assume that the system is state* $x = (s, b) = (S(t_n^-), B(t_n^-))$ *with wealth* $w = s + b > 0$ *for some* $t_n \in \mathcal{T}_m$. *Let* $E_{u_n}^{x,t_n}[\cdot]$ *and* $Var_{u_n}^{x,t_n}[\cdot]$ *denote the expectation and variance, respectively, using impulse* $u_n \in \mathcal{Z}$ *at time* t_n, *and define* $X_{n+1} \coloneqq \left(S\left(t_{n+1}^-\right), B\left(t_{n+1}^-\right)\right)$.

Problem $cMV_{\Delta t}(\rho)$ *in (3.3)-(3.4) can be solved using the following backward recursion,*

$$V_{\Delta t}^c(x, t_n) = \sup_{u_n \in \mathcal{Z}} J_{\Delta t}^c(u_n; x, t_n), \quad n = m, \ldots, 1, \quad \text{where} \tag{3.43}$$

$$J_{\Delta t}^c(u_n; x, t_n) = E_{u_n}^{x,t_n}\left[V_{\Delta t}^c(X_{n+1}, t_{n+1})\right] - \rho \cdot Var_{u_n}^{x,t_n}\left[g_{\Delta t}^c(X_{n+1}, t_{n+1})\right], \tag{3.44}$$

with terminal conditions $V_{\Delta t}^c(s, b, t_{m+1}) = g_{\Delta t}^c(s, b, t_{m+1}) = s + b$.

Problem $dMV_{\Delta t}(\gamma_n)$ *in (3.14)-(3.15) can be solved using the following backward recursion,*

$$V_{\Delta t}^d(x, t_n) = \sup_{u_n \in \mathcal{Z}} J_{\Delta t}^d(u_n; x, t_n), \quad n = m, \ldots, 1, \quad \text{where} \tag{3.45}$$

$$\begin{aligned} J_{\Delta t}^d(u_n; x, t_n) &= E_{u_n}^{x,t_n}\left[V_{\Delta t}^d(X_{n+1}, t_{n+1})\right] - \frac{\gamma_n}{2w} \cdot Var_{u_n}^{x,t_n}\left[g_{\Delta t}^d(X_{n+1}, t_{n+1})\right] \\ &\quad + H_{\Delta t}^d(u_n; x, t_n), \end{aligned} \tag{3.46}$$

with terminal conditions $V_{\Delta t}^d(s, b, t_{m+1}) = g_{\Delta t}^d(s, b, t_{m+1}) = s + b$, *and with the functional* $H_{\Delta t}^d$ *given by*

$$H_{\Delta t}^d(u_n; x, t_n) = \frac{\gamma_n}{2w} \cdot E_{u_n}^{x,t_n}\left[\left(\frac{\gamma_{n+1}}{\gamma_n} \cdot \frac{w}{W\left(t_{n+1}^-\right)} - 1\right) \cdot Var_{\mathcal{U}_{n+1}^{d*}}^{X_{n+1},t_{n+1}}\left[W(T)\right]\right], \tag{3.47}$$

where we use the convention $\gamma_{m+1} \equiv \gamma_m$ *in (3.47) for the case when* $n = m$.

Proof. Follows from the problem definitions in conjunction with the time-consistency constraints. □

For subsequent use, we note that in the special case where $\gamma_n \equiv \gamma > 0$ for all n, the functional $H_{\Delta t}^d$ in (3.47) reduces to

$$H_{\Delta t}^d(u_n; x, t_n) = \frac{\gamma}{2w} \cdot E_{u_n}^{x,t_n}\left[\left(\frac{w}{W\left(t_{n+1}^-\right)} - 1\right) \cdot Var_{\mathcal{U}_{n+1}^{d*}}^{X_{n+1},t_{n+1}}\left[W(T)\right]\right]. \tag{3.48}$$

Lemma 3.12 shows how the time-consistency constraint enables us to reduce the cMV and dMV problems to a series of single-period objective functions, which is consistent with the game-theoretic formulation of Bjork and Murgoci (2014) where the TCMV optimization problem is viewed as a multi-period game played by the investor against their own future incarnations. Specifically, we make the following observations:

- In the case of the cMV problem, Basak and Chabakauri (2010) observes that the two components of the objective functional $J^c_{\Delta t}$ in (3.44) has a simple intuitive interpretation: (i) $E^{x,t_n}_{u_n}\left[V^c_{\Delta t}\left(X_{n+1}, t_{n+1}\right)\right]$ gives the expected future value of the choice $u_n \in \mathcal{Z}$, while (ii) $Var^{x,t_n}_{u_n}\left[g^c_{\Delta t}\left(X_{n+1}, t_{n+1}\right)\right]$ can be interpreted as an adjustment, weighted by the investor's risk-aversion parameter ρ, quantifying the incentive of the investor at time t_n to deviate from the choice that maximizes the expected future value (see Basak and Chabakauri (2010)).

- In the case of the dMV problem, the first two components of the objective functional $J^d_{\Delta t}$ in (3.46) has a very similar intuitive interpretation as in the case of the cMV problem. However, the addition of the functional $H^d_{\Delta t}$ in (3.47) complicates matters significantly, so that the dMV problem no longer admits this straightforward interpretation. Observe that the functional $H^d_{\Delta t}$ vanishes if $n = m$, i.e. at the last rebalancing time $t_m = T - \Delta t$, or equivalently if the investor rebalances only once[8] at the start of $[0, T]$. This observations turns out to be critical in understanding the impact of rebalancing frequency on the MV outcomes discussed below, since rebalancing once presents one extreme end of the spectrum of rebalancing frequency possibilities, with continuous rebalancing at the other extreme end.

To analyze the implications of the functional $H^d_{\Delta t}$ in (3.46), we present the following theorem examining the behavior of $H^d_{\Delta t}$ in the case where a fixed parameter $\gamma > 0$ (see (3.48)) in $\rho(w) = \gamma/(2w)$ takes on extreme values.

Theorem 3.13. *(Problem $dMV_{\Delta t}(\gamma)$: γ-dependence of functional $H^d_{\Delta t}$) Let $\gamma_n \equiv \gamma > 0$ for all n. Assume that the system is in state $x = (s, b) = \left(S\left(t_n^-\right), B\left(t_n^-\right)\right)$ with wealth $w = s + b > 0$ at $t_n \in \mathcal{T}_m$, where $n \in \{1, \ldots, m-1\}$, and that $\mu_t > r_t, \forall t \in [0, T]$. Furthermore, assume that the values of $\hat{r}_n$, $\hat{\alpha}_n$ and $\hat{\sigma}^2_n$ in (3.1)-(3.2) do not depend on γ. In the case of no investment constraints, the functional $H^d_{\Delta t}$ (3.47) satisfies*

$$\left|H^d_{\Delta t}\left(u_n; x, t_n\right)\right| \quad \rightarrow \quad \begin{cases} 0, & \text{as } \gamma \rightarrow \infty, \\ \infty, & \text{as } \gamma \downarrow 0. \end{cases} \qquad (\textit{No constraints}) \tag{3.49}$$

In the case of Combination 1 of constraints, the functional $H^d_{\Delta t}$ satisfies

$$\left|H^d_{\Delta t}\left(u_n; x, t_n\right)\right| \quad \rightarrow \quad \begin{cases} 0, & \text{as } \gamma \rightarrow \infty, \\ 0, & \text{as } \gamma \downarrow 0. \end{cases} \qquad (\textit{Combination } 1) \tag{3.50}$$

Proof. Note that in both the cases of no constraints and Combination 1, the analytical solution of Lemma 3.4 gives the following expression for $H^d_{\Delta t}$ at arbitrary rebalancing time $t_n \in \mathcal{T}_m$,

$$H^d_{\Delta t}\left(u_n; x, t_n\right) \quad = \quad \gamma \cdot \frac{1}{2w} \cdot \left(D_{n+1} - A^2_{n+1}\right) \cdot E^{x,t_n}_{u_n}\left[W\left(t^-_{n+1}\right) \cdot \left(w - W\left(t^-_{n+1}\right)\right)\right], \tag{3.51}$$

so that the γ-dependence of $H^d_{\Delta t}$ is limited to the term $\gamma \cdot \left(D_{n+1} - A^2_{n+1}\right)$. We give an outline of the proof of (3.49), since the proof of (3.50) proceeds along similar lines. First, we observe that as a result of (3.51), proving (3.49) requires us to show that in the case of no investment constraints, we have

$$\gamma \cdot \left(D_{n+1} - A^2_{n+1}\right) \rightarrow \begin{cases} 0, & \text{as } \gamma \rightarrow \infty \\ \infty, & \text{as } \gamma \downarrow 0 \end{cases}, \text{ for all } n = 1, \ldots, m-1. \tag{3.52}$$

We prove (3.52) using backward induction. To establish that (3.52) holds for the base case of $n = m - 1$, we recall that the results of Lemma 3.4 imply that in the case of no investment constraints, we have

$$\gamma \cdot \left(D_m - A^2_m\right) = \frac{1}{\gamma} \cdot \frac{\hat{\alpha}^2_m}{\hat{\sigma}^2_m}, \quad C_m = \frac{1}{\gamma} \cdot \frac{\hat{\alpha}_m}{\hat{\sigma}^2_m}, \quad A_m = \hat{r}_m + \frac{1}{\gamma} \cdot \frac{\hat{\alpha}^2_m}{\hat{\sigma}^2_m}, \quad D_m = A^2_m + \left(\frac{1}{\gamma} \cdot \frac{\hat{\alpha}_m}{\hat{\sigma}_m}\right)^2. \tag{3.53}$$

It is clear from (3.53) that $\gamma \cdot \left(D_{n+1} - A^2_{n+1}\right)$ satisfies (3.52) for $n = m - 1$. Furthermore, A_m and D_m are bounded as $\gamma \rightarrow \infty$, and we observe that $A_m > 0$. For the induction step, fix an arbitrary $n \in \{1, \ldots, m-1\}$,

[8] If the investor rebalances only once in $[0, T]$, the cMV and dMV formulations can be viewed as trivially equivalent, in the sense that $\forall \gamma_m > 0$, $\exists \rho \equiv \gamma_m/(2w) > 0$ such that $u^{d*}_m = u^{c*}_m \in \mathcal{Z}$.

and assume that $\gamma \cdot \left(D_{n+1} - A_{n+1}^2\right)$ satisfies (3.52). To treat the case of $\gamma \to \infty$, assume that A_{n+1} and D_{n+1} are bounded as $\gamma \to \infty$. Recalling that $\hat{r}_n$, $\hat{\alpha}_n$ and $\hat{\sigma}_n^2$ do not depend on γ, the expression for C_n (3.17) in the case of no constraints together with the stated assumptions guarantee that $C_n \sim \mathcal{O}(1/\gamma) \to 0$ as $\gamma \to \infty$. This implies that $(\hat{r}_n + \hat{\alpha}_n C_n)$ and $\hat{\sigma}_n^2 C_n^2$ are bounded as $\gamma \to \infty$. Since A_{n+1} and D_{n+1} are assumed to be bounded as $\gamma \to \infty$, A_n and D_n obtained by solving the difference equations (3.19)-(3.20) are also bounded as $\gamma \to \infty$. Furthermore, $\gamma \cdot C_n^2 \sim \mathcal{O}(1/\gamma)$ as $\gamma \to \infty$, so $\gamma \cdot C_n^2 \cdot \hat{\sigma}_n^2 D_{n+1} \to 0$ as $\gamma \to \infty$. Since we can rearrange the results of Lemma 3.4 to obtain

$$\gamma \cdot \left(D_n - A_n^2\right) = \left(\hat{r}_n + \hat{\alpha}_n C_n\right)^2 \gamma \cdot \left(D_{n+1} - A_{n+1}^2\right) + \gamma \cdot C_n^2 \cdot \hat{\sigma}_n^2 D_{n+1}, \tag{3.54}$$

we have therefore established that $\gamma \cdot \left(D_n - A_n^2\right) \to 0$ as $\gamma \to \infty$. To treat the case where $\gamma \downarrow 0$, assume that $A_{n+1} > 0$, and recall from Lemma 3.4 that $D_{n+1} > 0$ and $D_{n+1} - A_{n+1}^2 \geq 0$ for all n. Since $\hat{\sigma}_n > 0$, and the assumption $\mu_t > r_t, \forall t \in [0, T]$ also implies that $\hat{\alpha}_n > 0$, we therefore have

$$0 < \left[1 - \frac{\hat{\alpha}_n^2 \left(D_{n+1} - A_{n+1}^2\right)}{\hat{\alpha}_n^2 \left(D_{n+1} - A_{n+1}^2\right) + \hat{\sigma}_n^2 D_{n+1}}\right] \leq 1, \tag{3.55}$$

which implies that $\left(\hat{r}_n + \hat{\alpha}_n C_n\right)^2 > 0$. Using the fact that $D_{n+1} > 0$ and $\gamma > 0$, we also have $\gamma \cdot C_n^2 \cdot \hat{\sigma}_n^2 D_{n+1} \geq 0$. Since (3.52) by assumption, the expression (3.54) therefore implies that $\gamma \cdot \left(D_n - A_n^2\right) \to \infty$ as $\gamma \downarrow 0$. Finally, since $A_n = \left(\hat{r}_n + \hat{\alpha}_n C_n\right) A_{n+1}$, we have $A_n > 0$. Therefore, we conclude by backward induction that (3.52) and therefore (3.49) hold for all $n \in \{1, \ldots, m-1\}$. □

Theorem 3.13 is particularly valuable in that it describes the dependence of the functional $H_{\Delta t}^d$ on γ in the limiting cases without solving the difference equations (3.19)-(3.20) explicitly (as noted above, the analytical solution of these equations become intractable for $n \leq m-2$). To illustrate the conclusions of Theorem 3.13, the following lemma gives concrete examples of functional $H_{\Delta t}^d$ for the simplest non-trivial case where the difference equations can be solved analytically, namely at the penultimate rebalancing time $t_{m-1} = T - 2\Delta t$.

Lemma 3.14. *(Problem $dMV_{\Delta t}(\gamma)$ - Examples of the functional $H_{\Delta t}^d$ at $t_{m-1} \in \mathcal{T}_m$) Let $\gamma_n \equiv \gamma > 0$ for all n. Assume that the system is in state $x = (s, b) = \left(S\left(t_{m-1}^-\right), B\left(t_{m-1}^-\right)\right)$ with wealth $w = s + b > 0$ at $t_{m-1} \in \mathcal{T}_m$, and that Assumption 3.1 is applicable. In the case of no investment constraints, the functional $H_{\Delta t}^d$ in (3.47) at time t_{m-1} is given by*

$$H_{\Delta t}^d\left(u_{m-1}; x, t_{m-1}\right) \;=\; \frac{1}{\gamma} \cdot \frac{1}{2w} \cdot \frac{\hat{\alpha}^2}{\hat{\sigma}^2} \cdot E_{u_{m-1}}^{x, t_{m-1}}\left[W\left(t_m^-\right) \cdot \left(w - W\left(t_m^-\right)\right)\right], \tag{3.56}$$

while in the case of Combination 1 of constraints, $H_{\Delta t}^d$ is given by

$$H_{\Delta t}^d\left(u_{m-1}; x, t_{m-1}\right) \;=\; \begin{cases} \gamma \cdot \frac{1}{2w} \cdot \hat{\sigma}^2 \cdot E_{u_{m-1}}^{x, t_{m-1}}\left[W\left(t_m^-\right) \cdot \left(w - W\left(t_m^-\right)\right)\right] & \textit{if } 0 < \gamma < \frac{\hat{\alpha}}{\hat{\sigma}^2} \\ \frac{1}{\gamma} \cdot \frac{1}{2w} \cdot \frac{\hat{\alpha}^2}{\hat{\sigma}^2} \cdot E_{u_{m-1}}^{x, t_{m-1}}\left[W\left(t_m^-\right) \cdot \left(w - W\left(t_m^-\right)\right)\right] & \textit{if } \gamma \geq \frac{\hat{\alpha}}{\hat{\sigma}^2}. \end{cases} \tag{3.57}$$

Proof. At rebalancing time t_{m-1}, we can solve the difference equations (3.19)-(3.20) explicitly (see for example (3.53)) to obtain $\left(D_m - A_m^2\right)$, and substitute the result into (3.51) to obtain (3.56) and (3.57), respectively. □

4 The economic challenges of using a wealth-dependent ρ

In this section, we present a detailed overview of the economic challenges that may be experienced by the MV investor using a wealth-dependent risk aversion parameter (1.2)-(1.3) in conjunction with the time-consistency constraint. These observations are compared and contrasted with the results obtained using a constant risk aversion parameter ρ in the same setting. We use the analytical solutions of Section 3 wherever possible, and where analytical solutions are not available (see Table 2.1), we solve the cMV and dMV problems numerically using the algorithm of Van Staden et al. (2018).

For convenience, the numerical results in this section are based on the assumption of constant process parameters, i.e. Assumption 3.1, which can be relaxed without fundamentally affecting our conclusions. We also set $\gamma_t = \gamma_n \equiv \gamma > 0$ for all n and t, so that $\rho(w) = \gamma / (2w)$ in all numerical results for the dMV problem. As discussed below, this assumption is also not too limiting. Furthermore, the parameter values used throughout (see Appendix B) are calibrated to inflation-adjusted, long-term US market data (89 years), which ensures that realistic conclusions can be drawn from the numerical results.

4.1 Arguments in the literature in favor of using a wealth-dependent ρ

Before discussing the economic challenges associated with using a wealth-dependent ρ, we discuss the arguments put forward in the literature in favor of the formulation (1.2)-(1.3).

As an introduction, it is useful to recall the context within which the formulation (1.2) was originally proposed. As observed in Section 3, the optimal controls (amount invested the risky asset) in the case of the cMV problem, (3.6) and (3.11), do not depend on the investor's current wealth w. Bjork et al. (2014) correctly points out that this is not an economically reasonable conclusion.

We make two observations regarding this issue. First, instead of arguing that the cMV formulation should be changed in some way by for example using a different formulation for ρ, note that (3.6) and (3.11) hold only under the assumption of no investment constraints, in particular including the assumptions of (i) infinite leverage being allowed, (ii) trading continuing in the event of bankruptcy, and (iii) zero transaction costs. If realistic investment constraints are applied to the cMV problem, as the results below as well as those of Cong and Oosterlee (2016); Van Staden et al. (2018); Wang and Forsyth (2011) show, the cMV-optimal controls do depend on the investor's wealth. Explained differently, it is not necessarily problematic if the optimal controls (3.6) and (3.11) turned out to be economically unreasonable, since they result from applying economically unreasonable constraints (in this case, no investment constraints). Secondly, whether (3.6) and (3.11) are actually economically unreasonable ultimately also depends on the investor's perspective: for example, in the case of no constraints the *fraction* of wealth invested in the risky asset for the cMV problem is inversely proportional to current wealth, in contrast to the case of the dMV problem where this fraction is independent of wealth (see optimal controls (3.17) and (3.36)). Therefore the cMV investor would, even in the case of no investment constraints, de-risk the portfolio as wealth grows over time and T is approached, which is arguably a more desirable result from the perspective of e.g. pension funds than the dMV result. We return to this perspective in more detail in Subsection 4.2.

As to why the wealth-dependent ρ should, in particular, be inversely proportional to wealth as in (1.2), Bjork et al. (2014) offers two motivating arguments which we briefly discuss. We emphasize that we do *not* argue that these motivating arguments are invalid in some sense; instead, we simply wish to present an alternative perspective on each argument that may be valid in some circumstances.

- First, Bjork et al. (2014) notes that since $Var\left[W\left(T\right)\right]$ is typically quadratic in current wealth w in the available analytical solutions, multiplying by $\gamma/\left(2w\right)$ would standardize the variance term so that the MV objective functional is measured in units of currency. We note that this might be a desirable attribute for some investors, but it is not clear that this should be desirable more generally, since in portfolio optimization settings it is quite typical to have an objective functional that is not measured in units of currency, for example utility maximization problems (e.g.Joshi and Paterson (2013); Merton (1969); Rogers (2013)). Furthermore, if an investor desired an objective functional in units of currency, this might also be achieved more directly through for example mean-standard deviation or mean-CVaR optimization (e.g. Kronborg and Steffensen (2014); Miller and Yang (2017)), instead of adjusting the risk-aversion parameter formulation in MV optimization.

- Second, Bjork et al. (2014) points out that in the single-period setting of Markowitz (1952), MV optimization is performed using the rate of return as the random variable, not the terminal wealth. With initial wealth $w > 0$, this would appear to suggest an objective of the form $\mathbb{E}\left[W\left(T\right)/w\right] - \rho \cdot Var\left[W\left(T\right)/w\right]$, so that factoring out $1/w$ leads to an objective equivalent to that of the dMV problem in this single-period setting. However, we note that the cMV and dMV problems are actually equivalent in a single-period setting, in the sense that both result in the same achievable set of MV outcomes regardless of the investment constraints applied - see Lemma 3.12, where the objective functional $H^d_{\Delta t}$ in (3.47) vanishes for $n = m$, so that $\forall \gamma \equiv \gamma_m > 0$, $\exists \rho \equiv \rho\left(w\right) = \gamma/\left(2w\right)$ such that $u^{d*}_m = u^{c*}_m \in \mathcal{Z}$. By contrast, in multi-period (dynamic) portfolio optimization settings, intermediate investment decisions play a critical role, and the dMV formulation (and in particular, the presence of the functional $H^d_{\Delta t}$) effectively means that we are no longer performing MV optimization in its intuitive sense, as will be discussed in detail in Subsection 4.2 below.

More recently, Landriault et al. (2018) offers a third argument in favor of requiring ρ to be inversely proportional to wealth, relevant in the particular case where the investor has a logarithmic or power utility function. To follow the underlying reasoning, observe that in utility theory (see Joshi and Paterson (2013)), given an investor utility function $w \rightarrow h\left(w\right)$ of wealth w, the coefficient of absolute risk aversion is defined as $-h''\left(w\right)/h'\left(w\right)$. Landriault et al. (2018), inspired by Pratt (1964), argues that it is natural to set $\rho\left(w\right)$ to be proportional to this quantity, which implies that if h is a logarithmic or power utility function, the risk-aversion parameter should indeed be inversely proportional to wealth.

We make a number of observations regarding this issue. MV and expected utility maximization are often viewed as competing alternatives in portfolio optimization settings - see for example Johnstone and Lindley (2013); Loistl (2015); Markowitz (2014). Somewhat controversially, the MV objective is sometimes viewed as a type of (expected) utility function in its own right (Grinold and Kahn (2000); Johnstone and Lindley (2013); Li et al. (2016); Nakamura (2015)), and the approximate relationship between MV optimization and expected utility maximization can be established for a wide variety of reasonable choices of h (Levy and Markowitz (1979); Loistl (2015); Markowitz (2014); Sargent (1987)). In particular, setting $\rho(w) \equiv -h''(w) / [2 \cdot h'(w)]$ achieves only approximate equivalence between MV and expected utility maximization, and then only in very specific circumstances, for example if both T and volatility σ is very small[9]. However, even if this approximation turns out to be accurate, it raises a much more fundamental issue. Specifically, if the investor has a given utility function h encoding their risk-reward preferences, the appropriate investment objective is perhaps more accurately given by $\mathbb{E}[h(W(T))]$ as is standard (see Rogers (2013)) in the utility maximization framework. Performing MV optimization with $\rho(w)$ proportional to $-h''(w)/h'(w)$ implies that the investor obtains an investment strategy that is inconsistent with both MV optimization (see discussion below) and with their risk-reward preferences.

In summary, we note that for each of the arguments accepted in the literature in favor of using the formulation $\rho(w) = \gamma/(2w)$, another perspective is possible. As noted above, we do not necessarily view the perspectives presented on these issues as more valid than the original arguments, but merely that not all investors might share the original perspective on these issues.

4.2 Challenges of using a wealth-dependent ρ

We now identify some significant economic challenges an investor might face in the practical application the formulation (1.2)-(1.3) in conjunction with the time-consistency constraint. For ease of reference, the various economic challenges arising as a consequence of using dMV formulation are identified below as Challenge 1 through Challenge 9.

Remark 4.1. (Economic challenges) We emphasize that the challenges presented in this section (with the possible exception of Challenge 1 below) are not mathematical in nature, but economic. By this, we mean that while the dMV formulation is mathematically sound (at least for positive wealth levels), it is associated with a number of attributes which an investor is likely to find particularly problematic in a practical application. We present no rank-ordering of these economic challenges, since their relative importance depends on the investor's point of view and on the particular application of the dMV problem, as discussed below. Furthermore, we view these challenges not in terms of some causal hierarchy (i.e. one challenge causing another), but as being interconnected, with each challenge highlighting a different aspect of the consequences of the formulation (1.2)-(1.3) in conjunction with the time-consistency constraint.

We start with the most obvious challenge, unsurprisingly also the most frequently mentioned in the literature.

Challenge 1. (dMV value function is unbounded for $w < 0$) *The dMV problem is economically unsound if $w < 0$, since this implies an unbounded value function due to the simultaneous maximization of both the expected value and variance of terminal wealth.*

We note that Challenge 1 does not arise in the original proposal[10] of Bjork et al. (2014), and thus might not be problematic under some specific circumstances. However, in more general settings, this challenge becomes very relevant, and difficult to address analytically. Despite the attention it has received in literature, whether it is just noted (e.g. Wu (2013)) or whether a concrete solution is proposed (e.g. Bensoussan et al. (2014); Cui et al. (2017, 2015)), we observe that it is not too difficult to address in any practical/numerical implementation of the dMV problem, since it is simultaneously (i) easy to identify and (ii) easy to address in any numerical algorithm (see Cong and Oosterlee (2016); Van Staden et al. (2018); Wang and Forsyth (2011)).

The next challenge presents a very practical problem that might arise when an investor attempts to explain the results from the dMV problem.

Challenge 2. (MV intuition does not apply to dMV optimization) *An investor using a wealth-dependent ρ in conjunction with the time-consistency constraint does not actually perform dynamic MV portfolio optimization in the intuitive sense in which it is usually understood, with one exception: in the case of discrete rebalancing, the usual intuition applies only at the final rebalancing time $t_m = T - \Delta t$.*

[9]Informally, this follows since we require $[W(T) - w]$ to be very small almost surely.

[10]The dMV-optimal controlled wealth process is simply GBM in the specific formulation of the problem considered in Bjork et al. (2014), and thus always positive.

To explain Challenge 2, we observe that it is standard in literature to define MV optimization as the maximization of the vector $\{E[W(T)], -Var[W(T)]\}$, subject to control admissibility requirements and constraints - see for example Hojgaard and Vigna (2007); Zhou and Li (2000). This definition also aligns with an intuitive understanding of what dynamic MV optimization should entail. Using the standard linear scalarization method for solving multi-criteria optimization problems (Yu (1971)), the MV objective (1.1) with *constant* $\rho > 0$ (i.e. the cMV formulation) is thus obtained, so that varying $\rho \in (0, \infty)$ enables us to solve the original multi-criteria MV problem (see e.g. Hojgaard and Vigna (2007)).

If ρ is no longer a scalar but instead inversely proportional to wealth, the resulting dMV objective is no longer consistent with maximizing the vector $\{E[W(T)], -Var[W(T)]\}$, and therefore does not align with either the intuitive understanding or usual definition of MV optimization.

To clarify this observation, consider Lemma 3.12, and in particular the economic consequences of the implicit incentive encoded by the functional $H^d_{\Delta t}$, faced by the dMV investor but not by the cMV investor. At time $t_n \in \mathcal{T}_m$, the investor is given $\mathcal{U}^{d*}_{n+1}$ (since the problem is solved backwards in time) and wishes to maximize $J^d_{\Delta t}$ in (3.46). All else being equal, a choice $u_n \in \mathcal{Z}$ achieving a relatively larger value of $H^d_{\Delta t}$ is to be preferred. Making a small investment u_n in the risky asset (possibly even short-selling the risky asset) at time t_n would achieve a larger value of $H^d_{\Delta t}$, again all else being equal. It also implies that very risky "future" strategies $\mathcal{U}^{d*}_{n+1}$ over $[t_{n+1}, T]$ is likely to be counter-balanced by a very low-risk strategy at time t_n. Note how this runs completely counter to the intuition underlying the MV optimization framework. In particular, $H^d_{\Delta t}$ contributes an incentive for the investor to invest in such a way that the end-of-period wealth $W\left(t^-_{n+1}\right)$ is *small* compared to the "current" wealth w at time t_n, an observation which is discussed more rigorously below. Here we simply highlight that the analytical results presented in Lemma 3.14 confirm this perspective explicitly, while the more general results of Theorem 3.13 (discussed in more detail below) can be used to show that if the impact of $H^d_{\Delta t}$ can be limited in some way, superior MV outcomes are easily obtained. Therefore, we conclude that the presence of the functional $H^d_{\Delta t}$ in the dMV objective (3.46) significantly complicates the intuitively expected behavior of the dMV problem. Finally, the exception noted in Challege 2 arises since $H^d_{\Delta t}$ vanishes when $n = m$.

The next challenge focuses only on the MV outcomes of terminal wealth.

Challenge 3. (dMV-optimal strategy not as MV-efficient as cMV-optimal strategy) *The efficient frontiers obtained using a wealth-dependent ρ show a substantially worse MV trade-off for terminal wealth than those obtained using a constant ρ, regardless of the combination of investment constraints, rebalancing frequency, or risky asset model under consideration.*

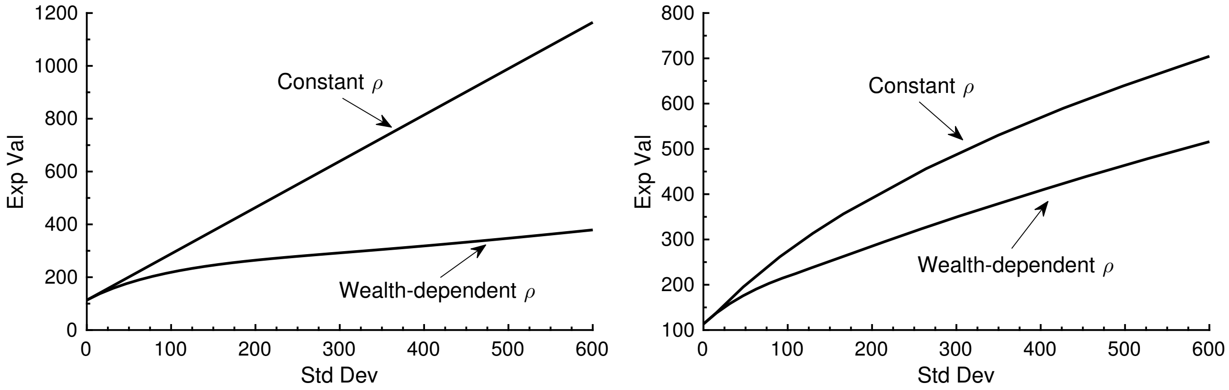
Challenge 3 is based on observation, illustrated in Figure 4.1, that the dMV efficient frontier (Definition 3.11) always appears to show a worse MV trade-off than the corresponding cMV efficient frontier (Definition 3.3). First observed in Wang and Forsyth (2011), this observation has been confirmed subsequently without exception using many different model assumptions and investment constraint combinations (Cong and Oosterlee (2016); Van Staden et al. (2018)). As observed in Figure 4.1, the gap between the cMV and dMV frontiers are narrower in two cases: (i) for extremely risk-averse investors, all wealth is simply invested in the risk-free asset regardless of the exact form of the risk-aversion parameter, and (ii) the application of constraints appear to narrow the gap between the cMV and dMV efficient frontiers. The latter case is discussed in more detail below (see Challenge 5).

Challenge 3 is to be expected given the results of Lemma 3.12. Informally, as noted in the discussion of Challenge 1, the cMV formulation is actually consistent with maximizing the MV trade-off of terminal wealth in the usual sense of performing multi-criteria optimization, which is *not* the case for the dMV formulation. It is therefore only natural that the dMV strategy would underperform the cMV strategy in terms of the resulting efficient frontiers.

The next challenge describes a very significant practical problem associated with the dMV formulation.

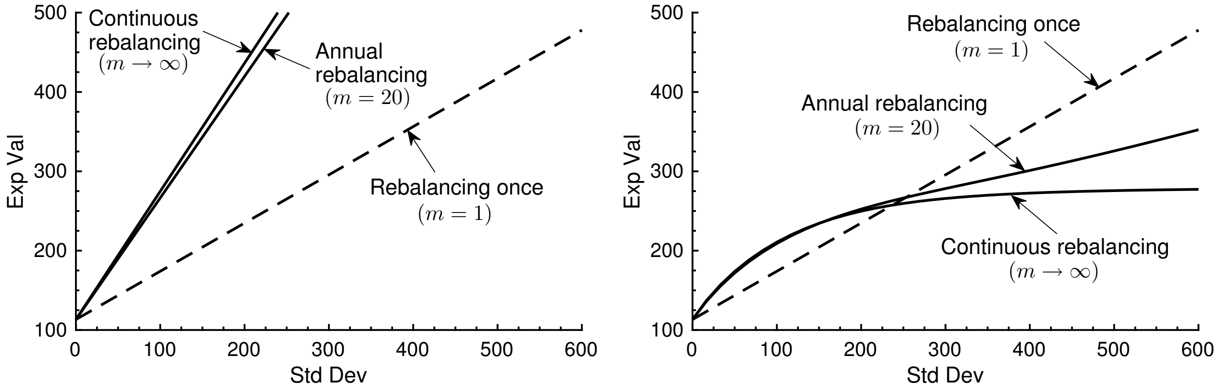
Challenge 4. (dMV mean-variance outcomes are adversely affected by increasing the portfolio rebalancing frequency) *The more frequently the investor using a wealth-dependent ρ rebalances the portfolio, the potentially worse the resulting MV outcomes of terminal wealth. In other words, increasing the rebalancing frequency can lower the dMV efficient frontier. There appears to be two groups of dMV-investors less affected by this phenomenon: (i) extremely risk-averse investors, and (ii) investors implementing Combination 1 of investment constraints.*

Intuition suggests that when transaction costs are zero, an investor rebalancing their portfolio more frequently should achieve a result no worse than the result obtained if the investor were to rebalance less frequently. However, as Figure 4.2 (no investment constraints) and Figure 4.3 (Combinations 1 and 2) illustrate, this intuition is accurate in the case of the cMV formulation, but does not hold in the case of the dMV formulation.



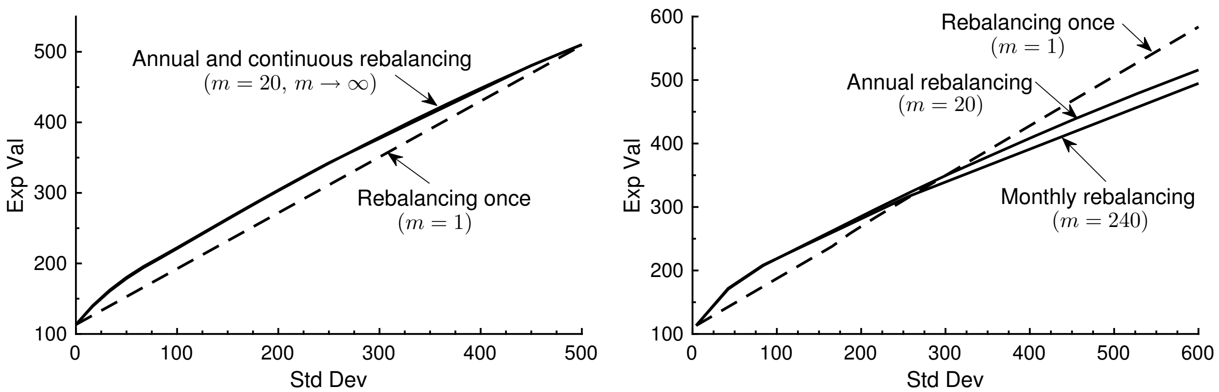
(a) No constraints (b) Combination 2

Figure 4.1: Efficient frontiers for a constant vs. wealth-dependent ρ. Discrete rebalancing (annually, $m = 20$), Merton model.



(a) Constant ρ, no constraints (b) Wealth-dependent ρ, no constraints

Figure 4.2: Effect of rebalancing frequency on the efficient frontiers for a constant and a wealth-dependent ρ. No investment constraints, Kou model. Note the same scale on the y-axis. The dotted lines in subfigures (a) and (b) are identical as a consequence of Lemma 3.12.



(a) Wealth-dependent ρ, Combination 1 (b) Wealth-dependent ρ, Combination 2

Figure 4.3: Effect of rebalancing frequency on the efficient frontiers for wealth-dependent ρ, Combinations 1 and 2 of constraints. Merton model.

We can explain this strange phenomenon informally, by noting that more frequent rebalancing increases the number of times the investor has to act consistently with the dMV objective functional (3.46) which includes the incentive encoded by the functional $H_{\Delta t}^{d}$ (see the discussion of Challenge (2) and Challenge (3)).

More rigorously, we can explain Challenge 4 as follows. Lemma 3.12 shows that rebalancing only once in $[0,T]$ will result in identical efficient frontiers for the dMV and cMV problems ($H_{\Delta t}^{d}$ vanishes when $n=m$), regardless of the set of investment constraints under consideration[11]. Suppose now that the investor rebalances twice in $[0,T]$. Considering the results of Lemma 3.14 for the cases of no constraints and Combination 1, we observe the following. First, observe that the form of $H_{\Delta t}^{d}$ for both these cases (3.51) implies that $H_{\Delta t}^{d}$ adds an incentive to the objective functional $J_{\Delta t}^{d}$ in (3.46) to choose u_n such that $W\left(t_{n+1}^{-}\right)\cdot\left(w-W\left(t_{n+1}^{-}\right)\right)$ is maximized. Since the function $y\rightarrow y\left(w-y\right)$ attains an unconstrained maximum at $y^{*}=w/2$, we see that at each rebalancing time t_n when the investor maximizes the functional $J_{\Delta t}^{d}$ in (3.46), component $H_{\Delta t}^{d}$ contributes an incentive to invest a relatively small fraction ($\ll 1$) of wealth in the risky asset. The relative role $H_{\Delta t}^{d}$ plays in the overall objective $J_{\Delta t}^{d}$ obviously depends on a number of factors. For example, as noted above, the more frequently the investor rebalances in $[0,T]$, the more often $J_{\Delta t}^{d}$ is maximized, and the more often the incentive implied by $H_{\Delta t}^{d}$ plays a role (however small) in the investment decision.

For a more general explanation when the investor rebalances m times in $[0,T]$, we can rely on the results of Theorem 3.13 to explain the two exceptions highlighted in Challenge 4. In particular, Theorem 3.13 shows that these two exception arise precisely because the suppression of $H_{\Delta t}^{d}$ benefits the MV outcomes. Explaining the first exception (extremely risk-averse investors), we note that for both no constraints and Combination 1, (3.49) and (3.50) show that $H_{\Delta t}^{d}\rightarrow 0$ as $\gamma\rightarrow\infty$, thus the dMV frontiers behave more like cMV frontiers in the case of extreme risk aversion. However, for investors that are less risk-averse, choosing smaller values of γ *magnifies* the effect of $H_{\Delta t}^{d}$ in the case of no constraints (3.56), since $H_{\Delta t}^{d}\rightarrow\infty$ as $\gamma\downarrow 0$. As a result, as we move along the standard deviation axis in Figure 4.2(b), the more pronounced the adverse impact on the MV outcomes. In contrast, in the case of Combination 1, (3.51) shows that $H_{\Delta t}^{d}\rightarrow 0$ as $\gamma\downarrow 0$, explaining the second exception noted in Challenge 4, which is illustrated by Figure 4.3(a). In other words, Combination 1 turns out to be one example of a very effective way to reduce the adverse impact of $H_{\Delta t}^{d}$ on MV outcomes, in that for this particular set of constraints (arguably very restrictive, as discussed in Remark 2.1), the dMV investor acts somewhat more like the cMV investor and thus improves the resulting MV outcomes.

Unfortunately, as Figure 4.3(b) shows for the case of Combination 2, the fundamental challenge that $H_{\Delta t}^{d}$ forms part of the objective functional $J_{\Delta t}^{d}$ (3.46) of the dMV problem, and thereby adversely impacts MV outcomes, simply cannot be managed by imposing some constraints on the problem. For example, the impact of the rebalancing frequency on MV outcomes in the case of Combination 2, for which no analytical solution is known, is qualitatively between the extremes of no constraints (Figure 4.2(b)) and Combination 1 (Figure 4.3(a)), as expected - see Remark 2.1.

The next challenge is also deeply problematic from a practical investment perspective.

Challenge 5. (The constrained dMV-optimal strategy outperforms the corresponding unconstrained strategy) *In the case of a wealth-dependent ρ, applying investment constraints improves the MV outcomes compared to those obtained in the case of no constraints. In other words, even though the unconstrained dMV investor should intuitively also be able to follow the investment strategies of a constrained dMV investor, the constrained investor achieves a higher efficient frontier. Similarly, more stringent investment constraints (e.g. Combination 1) improves the MV outcomes relative to those subject to less stringent investment constraints (e.g. Combination 2).*

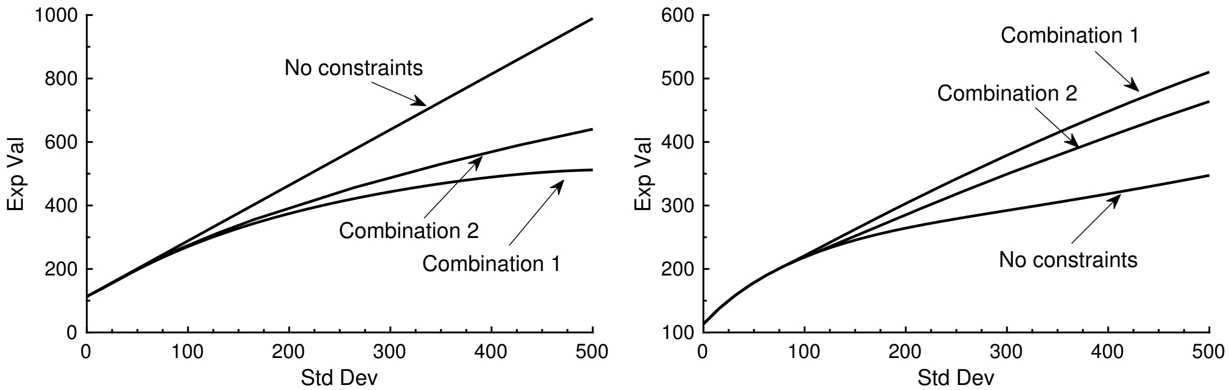
Challenge 5, first noted in the numerical experiments of Wang and Forsyth (2011), has subsequently been confirmed in experiments formulated using many different underlying models, sets of investment constraints and rebalancing frequencies - see for example Wong (2013), Bensoussan et al. (2014) and Van Staden et al. (2018). Figure 4.4(a) shows that Challenge 5 does not occur in the case of the cMV problem (see Van Staden et al. (2018); Wang and Forsyth (2011) for more examples), in contrast to the case of the dMV problem illustrated in Figure 4.4(b). Furthermore, since Combination 2 can be viewed as qualitatively between the extremes of no constraints and Combination 1 (Remark 2.1), Figure 4.4(b) illustrates the "hierarchy effect" mentioned in Challenge 5 that occurs in the case of the dMV problem, whereby relatively more strict constraints results in better MV outcomes.

Based on the assumption of GBM dynamics for the risky asset and the available analytical solutions (i.e. the cases of no constraints and Combination 1), Bensoussan et al. (2019) presents a rigorous and detailed study of the phenomenon described by Challenge 5. Bensoussan et al. (2019) accurately concludes that the time-consistency constraint is responsible for Challenge 5, which can be also be seen in our results. For example, the recursive relationship for the dMV problem presented in Lemma 3.12, and in particular the functional $H_{\Delta t}^{d}$, owe their

[11] If $n=m$, the objective functionals (3.44) and (3.46) are equivalent, in the sense that $\forall\gamma_m>0$ for the dMV problem, we can set $\rho=\gamma_m/\left(2w\right)$ for the cMV problem to obtain the identical objective ($H_{\Delta t}^{d}$ vanishes if $n=m$).

existence to the time-consistency constraint. Furthermore, other examples in literature (see for example Forsyth (2019)) show that in certain settings, the time-consistency constraint can indeed have undesirable consequences. However, for the purposes of this paper, we observe that cMV problem is *also* subject to the time-consistency constraint, and it is clear from comparing Figures 4.4(a) and 4.4(b) that Challenge 5 arises only in the case of the dMV formulation. We therefore agree with Bensoussan et al. (2019) that the time-consistency constraint plays a critical role, but also observe that this problem can apparently be avoided altogether in a dynamic MV setting if a constant ρ is used, without revisiting the notion of time-consistency.

Finally, the results of Theorem 3.13 suggests an explanation of Challenge 5 that is perhaps more intuitive than the explanation offered by Bensoussan et al. (2019), but by necessity also less rigorous, since it helps to explain the results from Combination 2 where no analytical solution is available. As noted above, Theorem 3.13 shows that Combination 1 of constraints acts to reduce the adverse impact of $H_{\Delta t}^{d}$ on MV outcomes, since in this case $H_{\Delta t}^{d} \rightarrow 0$ as $\gamma \downarrow 0$ and as $\gamma \rightarrow \infty$. Informally, we can argue that the the dMV investor acts more like the cMV investor, so that the dMV efficient frontier improves (see discussion of Challenge 3). Therefore, in the case of Combination 2, due to the informal ranking of constraints in terms of restrictiveness noted in Remark 2.1, we expect the dMV frontier to be closer to the cMV frontier than in the case of no constraints, but not as close as in the case of Combination 1. This explains the phenomenon illustrated in Figure 4.1, whereby the cMV and dMV frontiers are closer to each other for Combination 2 than for no constraints, a result that follows from the cMV (resp. dMV) frontier for Combination 2 being lower (resp. higher) than the corresponding frontiers in the case of no constraints.



(a) Constant ρ (b) Wealth-dependent ρ

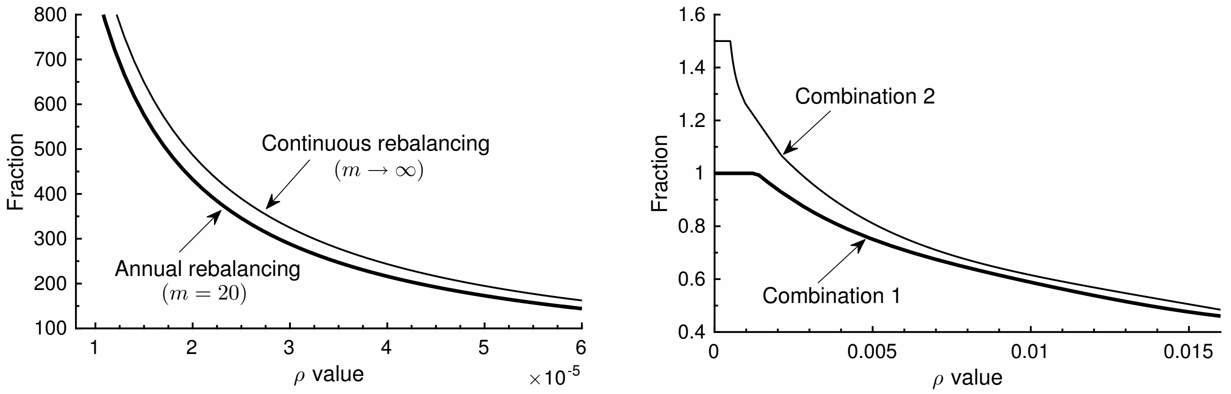
Figure 4.4: Effect of investment constraints on the efficient frontiers for a constant and a wealth-dependent ρ. Discrete rebalancing (annually, $m = 20$), Merton model.

The next challenge is especially problematic for interpreting the dMV formulation and associated results.

Challenge 6. (Role of γ in $\rho(w) = \gamma/(2w)$ is economically ambiguous) *Smaller values of γ in $\rho(w) = \gamma/(2w)$ do not necessarily imply more risk-seeking (or technically, less risk-averse) behavior on the part of the investor. In particular, except at the final rebalancing time $t_m = T - \Delta t$, the optimal fraction of wealth invested in the risky asset does not monotonically increase as γ decreases. This appears to hold regardless of the combination of investment constraints or the discrete rebalancing frequency under consideration.*

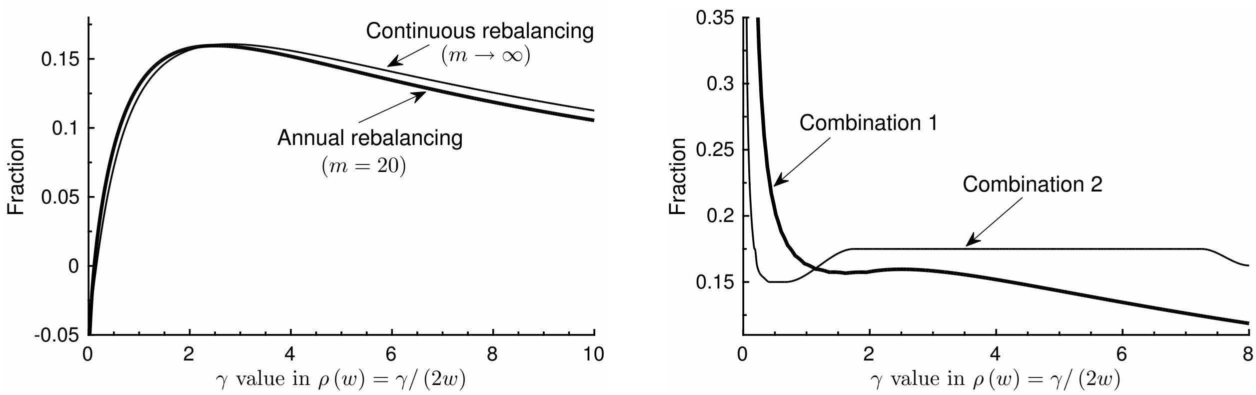
Challenge 6 is illustrated by Figure 4.5 and Figure 4.6. Before discussing the causes of Challenge 6 in more detail, we make a few observations.

First, Figure 4.5 shows that this problem appears not to arise at all in the case of the cMV formulation. Second, this challenge seems to be largely overlooked in the available literature concerned with the dMV problem. For example, Bensoussan et al. (2019, 2014) models $\gamma = \gamma_t$ by means of a logistic function which is justified on the basis that investors "become more risk-averse, relative to their current wealth, as time evolves", while Wang and Chen (2019) makes use of $\gamma = \gamma_t = c/t, c > 0$ in a pension fund setting, justifying this choice by noting that as "the retirement time approaches, the suggestion usually given to the investor in pension plans is to decrease the investment in the risky asset." While these observations regarding the evolution of risk preferences might be economically reasonable, the results of Figure 4.6 show that γ does not necessarily encode risk preferences in such a straightforward way. Complicating the definition of $\rho(w, t)$ even further using economic reasoning as in Cui et al. (2017, 2015) may be problematic if the underlying economic intuition regarding the role of γ in the simplest case $\rho(w) = \gamma/(2w)$ turns out to be ambiguous.



(a) Constant ρ, no constraints (b) Constant ρ, with constraints ($m = 20$)

Figure 4.5: cMV-optimal fraction of wealth invested in the risky asset at time $t = 0$ as a function of $\rho > 0$, for initial wealth $w = 100$. Merton model, maturity $T = 20$ years.



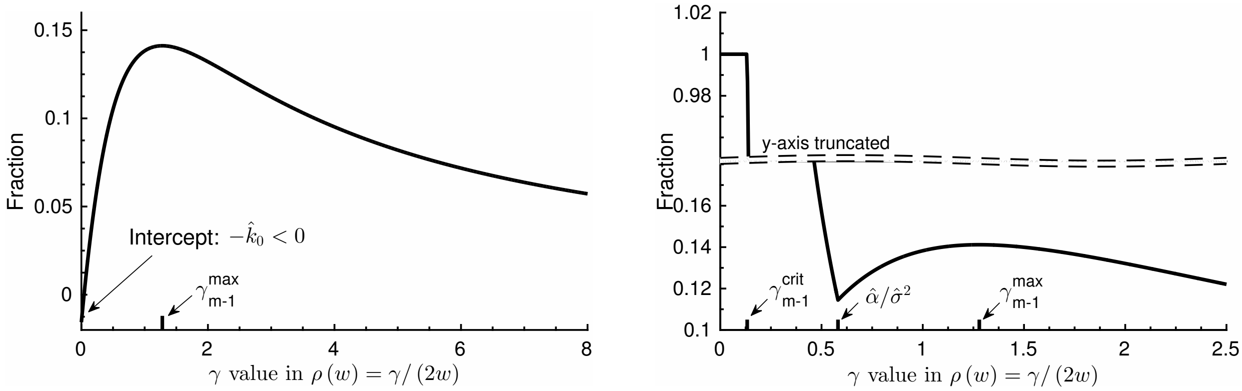
(a) Wealth-dependent ρ, no constraints (b) Wealth-dependent ρ, with constraints ($m = 20$)

Figure 4.6: dMV-optimal fraction of wealth invested in the risky asset at time $t = 0$ as a function of $\gamma > 0$, $C_0(\gamma)$, for initial wealth $w = 100$, where $\rho(w) = \gamma/(2w)$. Merton model, maturity $T = 20$ years.

Explaining the causes of Challenge 6 is not straightforward, since the dMV-optimal control's dependence on γ is very complex due to the integral equation (3.36) in the case of continuous rebalancing and the difference equations (3.19)-(3.20) in the case of discrete rebalancing. However, Lemma 3.5 and Lemma 3.6 rigorously show that the function $\gamma \to C_{m-1}(\gamma)$ (see Figure 4.7) exhibit all the key qualitative characteristics of the function $\gamma \to C_0(\gamma)$ (see Figure 4.6), and is therefore instructive for understanding the underlying causes of Challenge 6.

We note that the result of Lemma 3.5, illustrated in Figure 4.7(a), is not unexpected given the results of Theorem 3.13, and in particular the special case given in Lemma 3.14 applicable to rebalancing time t_{m-1}. Specifically, in the case of no constraints, we know that $H^d_{\Delta t} \to 0$ as $\gamma \to \infty$, so that the dMV problem has a structural similarity to the cMV problem as γ becomes large. This explains why the monotone decreasing behavior of $\gamma \to C_{m-1}(\gamma)$ for large γ in Figure 4.7(a) is comparable to that of Figure 4.5(a). In contrast, as $\gamma \downarrow 0$, in the case of no constraints $H^d_{\Delta t} \to \infty$. Lemma 3.5 shows that in the case of t_{m-1}, there is a value of γ, namely γ^{max}_{m-1}, where the contribution of $H^d_{\Delta t}$ effectively overwhelms the other terms of objective $J^d_{\Delta t}$ (3.46), so that its implied incentive to invest a relatively small fraction of wealth in the risky asset dominates. This explains the parabolic behavior in (3.25), which is illustrated in Figure 4.7(a).

Now consider Lemma 3.6, which extends the results of Lemma 3.5 to the case of Combination 1 of investment constraints. In this case, as $\gamma \downarrow 0$, the fact that $H^d_{\Delta t} \to 0$ (see Theorem 3.13 and Lemma 3.14) means that the dependence on γ for small γ illustrated in Figure 4.7(b) is more comparable to the dependence on ρ for small ρ illustrated in Figure 4.5(b).



(a) Wealth-dependent ρ, no constraints

(b) Wealth-dependent ρ, Combination 1

Figure 4.7: Function $\gamma \rightarrow C_{m-1}(\gamma)$: *dMV*-optimal fraction of wealth invested in the risky asset at time $t_{m-1} = T - 2\Delta t$ as a function of $\gamma > 0$, $C_{m-1}(\gamma)$, for wealth $w = 100$, where $\rho(w) = \gamma/(2w)$. Merton model, maturity $T = 20$ years, $m = 2$.

Unfortunately, the impact of $H^d_{\Delta t}$ cannot be ignored entirely, even in the case of Combination 1 of constraints. Specifically, considering the results of Lemma 3.6, we observe that if $\gamma \geq \frac{\hat{\alpha}}{\hat{\sigma}^2}$, the expression (3.27) is identical to the no constraints case in (3.23). Suppose for the moment that $\gamma^{max}_{m-1} > \frac{\hat{\alpha}}{\hat{\sigma}^2}$, where γ^{max}_{m-1} is defined in (3.24). Then even in the case of Combination 1, as γ increases, the dMV-optimal fraction of wealth in the risky asset $\gamma \rightarrow C_{m-1}(\gamma)$ in (3.27) is (i) constant if $\gamma \in \left(0, \gamma^{crit}_{m-1}\right)$, (ii) decreasing if $\gamma \in \left[\gamma^{crit}_{m-1}, \frac{\hat{\alpha}}{\hat{\sigma}^2}\right)$, (iii) increasing if $\gamma \in \left[\frac{\hat{\alpha}}{\hat{\sigma}^2}, \gamma^{max}_{m-1}\right]$, and finally (iv) decreasing if $\gamma \in \left(\gamma^{max}_{m-1}, \infty\right)$. This is illustrated in Figure 4.7(b). This is just one example of possible behavior however, since depending on the underlying parameters and rebalancing frequency, it might be the case that $\gamma^{max}_{m-1} < \frac{\hat{\alpha}}{\hat{\sigma}^2}$, with either $\gamma^{max}_{m-1} < \gamma^{crit}_{m-1}$ or $\gamma^{max}_{m-1} > \gamma^{crit}_{m-1}$ possible. Regardless of the exact behavior, the fact that γ has a non-monotonic or economically ambiguous influence on the dMV-optimal strategy is a very concerning aspect of the dMV formulation.

Given this interesting dependence of the dMV-optimal control on γ, the next challenge is perhaps not surprising.

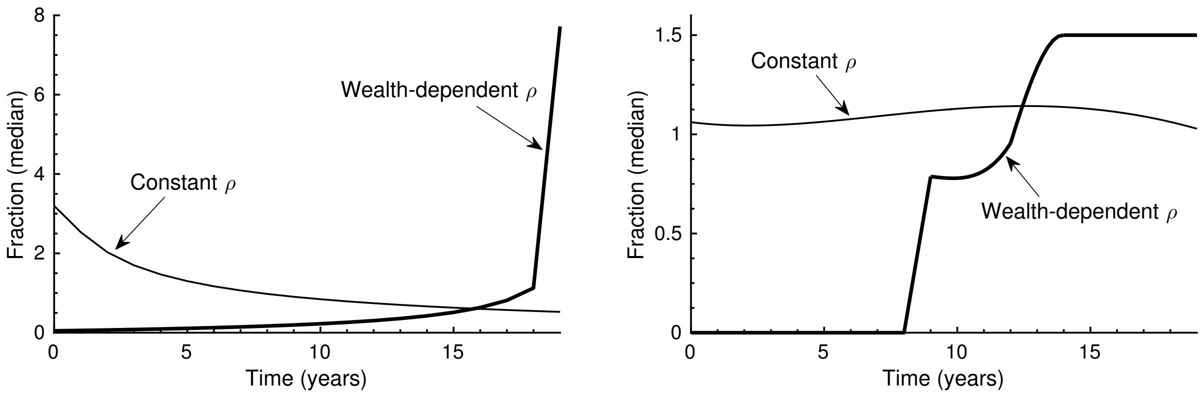
Challenge 7. (dMV-optimal strategy potentially calls for economically counterintuitive positions in underlying assets) *In the case of using a wealth-dependent ρ, it might be optimal to short the risky asset. Furthermore, even for a well-performing risky asset ($\mu \gg r$), it might be dMV-optimal, in both the constrained and unconstrained case, to invest all wealth in the risk-free asset for a substantial portion of the investment time horizon. Neither of these positions are intuitively expected in a dynamic MV optimization framework.*

Comparing results of Lemmas 3.14, 3.5 and 3.6, we observe that the shorting of the risky asset highlighted in Challenge 7 can also be explained as a consequence of the functional $H^d_{\Delta t}$ in the dMV objective becoming dominant for certain values of γ. Shorting the risky asset is not intuitively expected in the MV framework (and is indeed never cMV optimal) if there is a single risky asset and $\mu > r$, since an otherwise identical short and long position incurs the same risk as measured by the variance, but at the cost of negative expected returns in the case of a short position. The possibility that shorting the risky asset might be dMV-optimal is therefore deeply counterintuitive from a MV perspective.

As to the second part of Challenge 7, namely that it might be dMV-optimal to invest all wealth in the risk-free asset, see Bensoussan et al. (2019) for a rigorous discussion. Here we simply note that in the case of Combination 2, where no analytical solution is available, Figure 4.8(b) shows that even when $\mu \gg r$ (see parameters in Appendix B), the dMV-investor spends more than a third of the investment time horizon of $T = 20$ years, and in particular the critical early years, with zero investment in the risky asset (i.e. all wealth invested in the risk-free asset).

We explore this strange phenomenon in more detail as part of the explanation of the next economic challenge associated with the dMV formulation.

Challenge 8. (*dMV*-optimal strategy has an undesirable risk profile for the long-term investor) *Using a wealth-dependent ρ results in an optimal investment strategy with a very undesirable risk profile, especially from the perspective of long-term investors with a fixed investment time horizon, such as institutional investors like pension funds. This appears to remain true regardless of the combination of investment constraints under consideration.*



(a) No constraints, $Stdev\left[W\left(T\right)\right] = 400$

(b) Combination 2, $Stdev\left[W\left(T\right)\right] = 400$

Figure 4.8: Median fraction of wealth invested in the risky asset over time by rebalancing according to the optimal control achieving a standard deviation of terminal wealth equal to 400. Discrete rebalancing ($m = 20$), Kou model, 1 million Monte Carlo simulations.

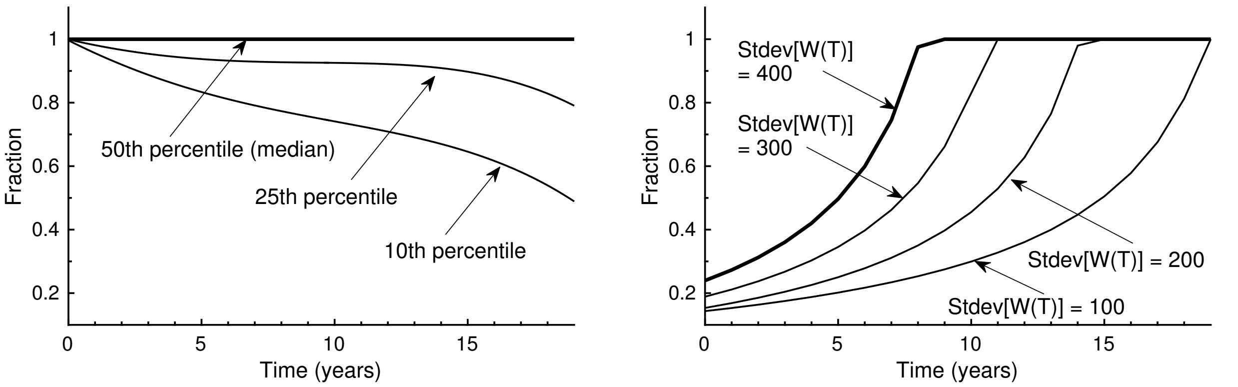
Figures 4.8 and 4.9 plots the fraction of wealth invested in the risky asset over time according to the cMV and dMV-optimal strategies, with the values of ρ and γ chosen to obtain the desired standard deviation of terminal wealth. Observe that in the case of the cMV formulation, this fraction depends on wealth even in the case of no constraints (as highlighted in Subsection 4.1). In the case of the dMV formulation, this fraction depends on wealth only in the case of Combination 2. In all cases where this fraction depends on wealth, the data for Figures 4.8 and 4.9 is obtained by solving the problems using the algorithm of Van Staden et al. (2018), outputting the optimal controls, and rebalancing the portfolio in a Monte Carlo simulation at each rebalancing time according to the saved controls (see Van Staden et al. (2018) for more details), so that we obtain a distribution of the fraction invested in the risky asset over time that enables the plotting of certain percentiles of this distribution over time.

Figure 4.8 and Figure 4.9(b) show that regardless of the investment constraints, the dMV-optimal fraction of wealth *increases* as $t \to T$. What's more, this increase in risk exposure over time is observed even if we impose additional downside risk constraints (Bi and Cai (2019)), allow for consumption (Kronborg and Steffensen (2014)), allow for T to be a random variable (Landriault et al. (2018)), impose a stochastic mortality process on investors (Liang et al. (2014)), include a model for reinsurance (Li and Li (2013)), allow for stochastic volatility (Li et al. (2016)), include a model of random wage income for the investor (Wang and Chen (2018)), or model the funding of a random liability over time from the portfolio (Zhang et al. (2017)). In other words, it appears that this increase is not a function of the constraints or modelling assumptions, but from the wealth-dependent ρ formulation itself, since the challenge is not observed in the case of a constant ρ.

Specifically, in the case of a constant ρ, Figure 4.8 and Figure 4.9(a) shows a much more desirable risk profile for a long-term investor with a fixed time horizon. As $t \to T$, provided previous returns were favorable, the investor de-risks the portfolio over time (see e.g. 25th percentile in Figure 4.9(a)), with no such adaptability present in the wealth-dependent ρ case.

We again observe that the presence of the functional $H_{\Delta t}^{d}$ in the dMV objective functional (3.46) is the source of this problem. Consider the final rebalancing time $t_m = T - \Delta t$. In this case, the cMV and dMV investors act similarly since $H_{\Delta t}^{d}$ vanishes, and we specifically note that the dMV-optimal strategy is inversely proportional to γ, see (3.53). Suppose now that the dMV investor chooses a small value of γ, then this implies a large dMV-optimal position in the risky asset at time $t_m = T - \Delta t$. However, Lemmas 3.5 and 3.6 shows that at time $t_{m-1} = T - 2\Delta t$, a small value of γ might *not* translate into a large position in the risky asset. In fact, due to the role of $H_{\Delta t}^{d}$ (see for example Lemma 3.14, or the general case in Lemma 3.12), there might be a significant incentive for the investor to make a very small investment in the risky asset at time t_{m-1}, with similar observations holding for t_n, $n < m-1$. As a result, if the dMV-investor sets a risk target for the standard deviation of terminal wealth, then the positions in the risky asset has to be very large at later rebalancing times compared to earlier rebalancing times if this target is to be achieved, resulting in the increasing risk exposure as $t \to T$ observed in Figures 4.8 and 4.9. These observations are also discussed rigorously in Bensoussan et al. (2019) for the case where analytical solutions are available.

Challenge 8 is closely connected to Challenge 7, since it might be dMV-optimal to invest zero wealth in the risky asset at earlier times (see Figure 4.8(b)). It is clearly also closely connected to Challenge 3, since the dMV investor might achieve the same overall risk as the cMV investor by taking large positions in the risky assets in later periods, resulting in the same or similar standard deviation of terminal wealth, but at a much lower level of expected wealth, since the low investment in the risky asset during early periods does not allow the wealth to grow sufficiently over time.



(a) Constant ρ: Combination 1, $Stdev\left[W\left(T\right)\right] = 400$ (b) Wealth-dependent ρ: Combination 1

Figure 4.9: Fraction of wealth invested in the risky asset over time by rebalancing according to the optimal control achieving the desired standard deviation of terminal wealth target. Discrete rebalancing ($m = 20$), Kou model. Note the same scale on the y-axis.

The final challenge that we discuss is closely connected to Challenge 7 and Challenge 8.

Challenge 9. (dMV-optimal strategy can exhibit undesirable discontinuities) *The optimal investment strategy using a wealth-dependent* ρ *can exhibit undesirable discontinuities or "cliff-effects" when economically reasonable constraints are applied. For example, as the investor's wealth crosses a certain threshold in the case of Combination 2 of constraints, either all wealth or no wealth is invested in the risky asset, with effectively no transition between these extremes. This makes the resulting investment strategy not just economically unreasonable, but also impractical to implement.*

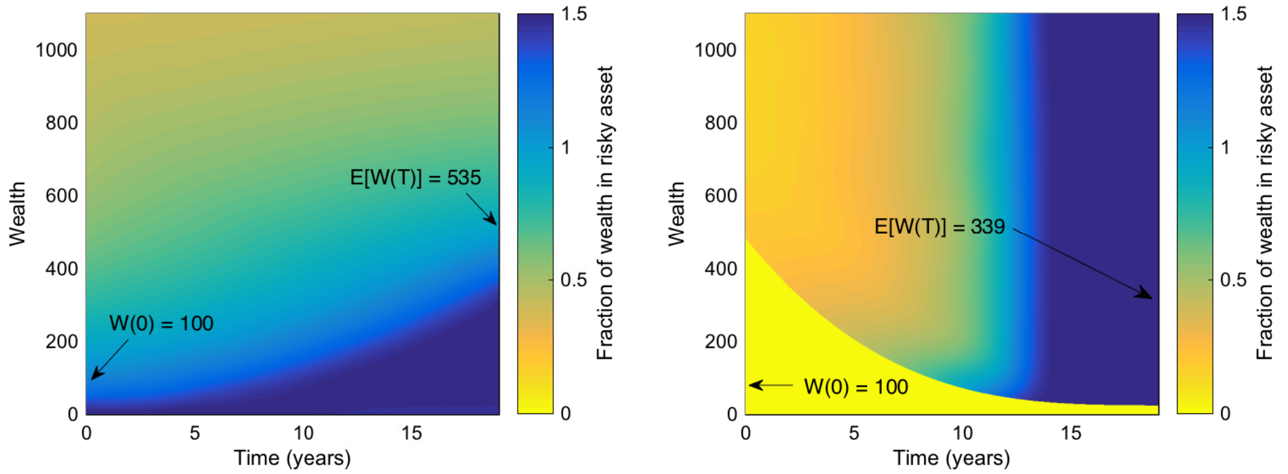
Challenge 9 is illustrated by Figure 4.10, which illustrates the cMV- and dMV-optimal controls for Combination 2 expressed as a fraction of wealth invested in the risky asset over time. We observe the very fast transition from a zero investment in the risky asset to investing all wealth in the risky asset as the wealth increases above a certain level, especially pronounced as $t \rightarrow T$. As observed in Challenge 9, this makes the dMV-optimal strategy very challenging to implement, especially if wealth fluctuates over this region of discontinuity.

The specific case of Combination 2 illustrated in Figure 4.10 is analyzed in detail in Van Staden et al. (2018). Here it is sufficient to give the following intuitive explanation of the discontinuity in Figure 4.10(b). As observed in discussing Challenge 8, the dMV investor takes the largest positions in the risky asset as $t \rightarrow T$. However, for the dMV formulation to be meaningful (see discussion of Challenge 1), any reasonable set of constraints should be such that the investment in the risky asset is zero if $w \equiv 0$, see for example (2.12). This implies that there should always be a "yellow strip" as at the bottom of Figure 4.10(b), the width of which is theoretically infinitesimal as $t \rightarrow T$. However, any numerical scheme solving this problem in practice can only approximate this strip by a finite size (which shrinks as the mesh is refined). Since the problem is solved recursively backwards, the transition from zero investment to non-zero investment in the risky asset is somewhat smoothed due to iterated conditioning, but remains unavoidable and economically undesirable.

5 Conclusion

In this paper, we extend the results of Bensoussan et al. (2014) to the case where the risky asset follows any of the jump-diffusion processes commonly encountered in finance. We summarize the arguments presented in literature in favor of using a wealth-dependent risk aversion parameter, and discuss alternative perspectives that an investor might hold regarding these arguments.

Using both numerical and analytical results, we present and explain a set of economic challenges resulting from using a wealth-dependent risk aversion parameter in combination with the time-consistency constraint. We show that these challenges appear not to arise when a constant risk aversion parameter is used. We conclude that when time-consistent MV optimization is performed, which in particular implies that a time-consistency



(a) Constant ρ (b) Wealth-dependent ρ

Figure 4.10: Combination 2: Optimal control achieving a standard deviation of terminal wealth equal to 400 as a fraction of wealth in the risky asset. Discrete rebalancing ($m = 20$), Kou model.

constraint is enforced, then a constant risk aversion parameter is preferable to a wealth dependent risk aversion parameter in a practical setting where realistic investment constraints are imposed.

Appendix A: Proofs of Theorems 3.7 and 3.8

Proof of Theorem 3.7

Let $\mathcal{L}^u$ and $\mathcal{H}^u$ be the following infinitesimal operators associated with the controlled wealth process (2.6),

$$\begin{aligned}\mathcal{L}^u \phi\left(w,t\right) &= \frac{\partial \phi}{\partial t}\left(w,t\right) + \left(r_t w + \alpha_t u\right)\frac{\partial \phi}{\partial w}\left(w,t\right) + \frac{1}{2}\sigma_t^2 u^2 \frac{\partial^2 \phi}{\partial w^2}\left(w,t\right) \\ &\quad -\lambda \phi\left(w,t\right) + \lambda \int_0^\infty \phi\left(w + u\left(\xi - 1\right), t\right) p\left(\xi\right) d\xi, \quad (A.1)\\ \mathcal{H}^u g^d\left(w,t\right) &= 2\rho\left(w,t\right)\cdot g^d\left(w,t\right)\cdot \mathcal{L}^u g^d\left(w,t\right), \quad (A.2)\end{aligned}$$

where $\phi : \mathbb{R}^+ \times [0,T] \to \mathbb{R}$ is a suitably smooth function. Define the following functions:

$$G\left(w,t,y\right) = \rho\left(w,t\right)y^2, \qquad \left(G \diamond g^d\right)\left(w,t\right) = G\left(w,t,g^d\left(w,t\right)\right), \qquad f^{y,\tau}\left(w,t\right) = f\left(w,t,y,\tau\right). \quad (A.3)$$

By the results derived in Bjork et al. (2017), if V^d, g^d, f and u^{d*} are sufficiently smooth functions that satisfy the following extended HJB system of equations,

$$\sup_{u \in \mathbb{U}^{w,t}} \left\{\mathcal{L}^u V^d\left(w,t\right) - \mathcal{L}^u\left(G \diamond g^d\right)\left(w,t\right) + \mathcal{H}^u g^d\left(w,t\right) - \mathcal{L}^u f\left(w,t,w,t\right) + \mathcal{L}^u f^{w,t}\left(w,t\right)\right\} = 0, \quad (A.4)$$

$$\mathcal{L}^{u^{d*}} g^d\left(w,t\right) = 0, \qquad \mathcal{L}^{u^{d*}} f^{y,\tau}\left(w,t\right) = 0, \quad (A.5)$$

$$V^d\left(w,T\right) = w, \qquad g^d\left(w,T\right) = w, \qquad f^{y,\tau}\left(w,T\right) + \frac{\gamma\left(\tau\right)}{2y}w^2 = w, \quad (A.6)$$

where $u^{d*} := u^{d*}\left(w,t\right)$ is the pointwise supremum attained for each $\left(w,t\right) \in \mathbb{U}^{w,t}$ in (A.4), then we can conclude the results of Theorem 3.7. Substituting the definitions (A.1)-(A.3) and $\rho\left(t,w\right) = \gamma\left(t\right)/\left(2w\right)$ into the extended HJB system (A.4)-(A.6) and simplifying the resulting expressions, we obtain the extended HJB system (3.31)-(3.34) in Theorem 3.7. The probabilistic representations (3.35) of g^d and f follows from the backward equations (A.5) (or equivalently (3.32)-(3.33)) and terminal conditions (A.6) together with standard results - see for example Applebaum (2004); Oksendal and Sulem (2005).

Proof of Theorem 3.8

Suppose that the optimal control is of the form $u^{d*}(w,t) = c(t)\,w$, for some non-random function of time $c \in C[0,T]$ that does not depend on w. At this stage, no other assumption is made regarding $c(t)$. Let W^{d*} denote the controlled wealth dynamics (2.6) using control u^{d*}. Define the auxiliary functions:

$$\mathcal{E}(\tau; w, t) = E_{u^{d*}}^{w,t}\left[W^{d*}(\tau)\right], \quad \mathcal{Q}(\tau; w, t) = E_{u^{d*}}^{w,t}\left[\left(W^{d*}(\tau)\right)^2\right], \quad \text{for } \tau \in [t, T]. \tag{A.7}$$

Using standard derivations (see for example Oksendal and Sulem (2005)), we obtain the following ODEs for $\mathcal{E}(\tau; w, t)$ and $\mathcal{Q}(\tau; w, t)$, respectively:

$$\frac{d\mathcal{E}}{d\tau}(\tau; w, t) = \left[r_\tau + (\mu_\tau - r_\tau)\, c(\tau)\right] \mathcal{E}(\tau; w, t), \quad \tau \in (t, T], \tag{A.8}$$

$$\mathcal{E}(t; w, t) = w, \quad \text{and} \tag{A.9}$$

$$\frac{d\mathcal{Q}}{d\tau}(\tau; w, t) = \left[2r_\tau + 2(\mu_\tau - r_\tau)\, c(\tau) + \left(\sigma_\tau^2 + \lambda\kappa_2\right) c^2(\tau)\right] \mathcal{Q}(\tau; w, t), \quad \tau \in (t, T], \tag{A.10}$$

$$\mathcal{Q}(t; w, t) = w^2. \tag{A.11}$$

Solving the ODEs (A.8)-(A.11), and evaluating the solution at $\tau = T$, we have

$$\mathcal{E}(T; w, t) = e^{I_1(t;c)} w, \qquad \mathcal{Q}(T; w, t) = w^2 \cdot e^{2I_1(t;c) + I_2(t;c)}, \tag{A.12}$$

where $I_1(t;c)$ and $I_2(t;c)$ are defined in (3.37). Using the probabilistic representations (3.35) of g^d and f, the ansatz $u^{d*}(w,t) = c(t)\,w$ therefore implies that

$$g^d(w,t) = \mathcal{E}(T; w, t), \qquad f(w, t, y, \tau) = g^d(w,t) - \frac{\gamma_\tau}{2y}\mathcal{Q}(T; w, t), \tag{A.13}$$

with g^d and f satisfying the backward equations (3.32) and (3.33) with terminal conditions (3.34), respectively, a fact which can be verified by direct calculation. Using (A.13), we obtain the value function as

$$V^d(w,t) = f(w, t, w, t) + \frac{\gamma_t}{2w}\left[g^d(w,t)\right]^2. \tag{A.14}$$

Consider now the HJB equation (3.31), which can be written more compactly as

$$\frac{\partial V^d}{\partial t}(w,t) - \frac{\partial f}{\partial \tau}(w,t,w,t) - \left(\frac{\gamma_t'}{2w} + \lambda\frac{\gamma_t}{2w}\right)\left(g^d(w,t)\right)^2 - \lambda V^d(w,t) + \sup_{u \in \mathbb{U}^{w,t}}\left\{\Phi^{w,t}(u)\right\} = 0, \tag{A.15}$$

where $\Phi^{w,t} : \mathbb{U}^{w,t} \to \mathbb{R}$ is the objective function of the embedded local optimization problem in equation (3.31). If g^d, f and V^d is as in (A.13)-(A.14), then $\Phi^{w,t}$ simplifies to the following concave and quadratic function in u,

$$\begin{aligned}
\Phi^{w,t}(u) = &-\left[\frac{\gamma_t}{2w}\left(\sigma_t^2 + \lambda\kappa_2\right) e^{2I_1(t;c)+I_2(t;c)}\right] \cdot u^2 \\
&+ (\mu_t - r_t)\left[e^{I_1(t;c)} - \gamma_t e^{2I_1(t;c)+I_2(t;c)} + \gamma_t e^{2I_1(t;c)}\right] \cdot u \\
&+ w(r_t + \lambda)\left[e^{I_1(t;c)} + \gamma_t e^{2I_1(t;c)}\right] - \gamma_t w\left(r_t + \frac{1}{2}\lambda\right) e^{2I_1(t;c)+I_2(t;c)}.
\end{aligned} \tag{A.16}$$

From the first order condition, the function $u \to \Phi^{w,t}(u)$ attains a maximum at u^*, where

$$u^* = F_t\left(\frac{\mu_t - r_t}{\gamma_t\left(\sigma_t^2 + \lambda\kappa_2\right)}\left\{e^{-I_1(t;c)-I_2(t;c)} + \gamma_t e^{-I_2(t;c)} - \gamma_t\right\}\right) \cdot w, \tag{A.17}$$

with F_t given by (3.38). Comparing (A.17) with the anzatz $u^{d*}(w,t) = c(t)\,w$, we see that $c(t)$ satisfies the integral equation (3.36).

It now only remains to verify that the HJB equation (A.15) is satisfied by $u^{d*}(w,t) = c(t)\,w$. Using (A.13), (A.14) and (A.16), together with the fact that g^d and f satisfy the backward equations (3.32) and (3.33), we obtain

$$\Phi^{w,t}\left(u^{d*}\left(w,t\right)\right) = -\frac{\partial f}{\partial t}\left(w,t,w,t\right) + \lambda f\left(w,t,w,t\right) + \frac{\gamma_t}{w} g^d\left(w,t\right)\left[-\frac{\partial g^d}{\partial t}\left(w,t\right) + \lambda g^d\left(w,t\right)\right]$$

$$= -\left[\frac{\partial V^d}{\partial t}\left(w,t\right) - \frac{\partial f}{\partial \tau}\left(w,t,w,t\right) - \left(\frac{\gamma_t'}{2w} + \lambda\frac{\gamma_t}{2w}\right)\left(g^d\left(w,t\right)\right)^2 - \lambda V^d\left(w,t\right)\right], \tag{A.18}$$

so that the first equation (3.31) in the extended HJB system (3.31)-(3.34) is therefore satisfied. This completes the proof of Theorem 3.8.

Appendix B: Parameters for numerical results

Throughout Appendix B, we assume that Assumption 3.1 is applicable. We therefore set $r_t \equiv r$, $\mu_t \equiv \mu$ and $\sigma_t \equiv \sigma$ for all $t \in [0,T]$ in the underlying asset dynamics (2.3).

In order to parameterize (2.3), the same calibration data and techniques are used as detailed in Dang and Forsyth (2016); Forsyth and Vetzal (2017). In terms of the empirical data sources, the risky asset data is based on inflation-adjusted daily total return data (including dividends and other distributions) for the period 1926-2014 from the CRSP's VWD index[12], which is a capitalization-weighted index of all domestic stocks on major US exchanges. A jump is only identified in the historical time series if the absolute value of the inflation-adjusted, detrended log return in that period exceeds 3 standard deviations of the "geometric Brownian motion change" (see Dang and Forsyth (2016)), which is a highly unlikely event. In the case of the Merton (1976) model, $p(\xi)$ is the log-normal pdf, so that we assume $\log\xi$ is normally distributed with mean $\widetilde{m}$ and variance $\widetilde{\gamma}^2$. In the case of the Kou (2002) model, $p(\xi)$ is of the form

$$p(\xi) = \quad \nu\zeta_1\xi^{-\zeta_1-1}\mathbb{I}_{[\xi\geq 1]}(\xi) + (1-\nu)\,\zeta_2\xi^{\zeta_2-1}\mathbb{I}_{[0\leq\xi<1]}(\xi), \quad \upsilon\in[0,1] \text{ and } \zeta_1>1, \zeta_2>0, \tag{B.19}$$

where ν denotes the probability of an upward jump (given that a jump occurs). The calibrated parameters for the risky asset dynamics are provided in Table B.1 for each of the models considered.

Table B.1: Calibrated risky asset parameters

Parameters	μ	σ	λ	$\widetilde{m}$	$\widetilde{\gamma}$	υ	ζ_1	ζ_2
GBM	0.0816	0.1863	n/a	n/a	n/a	n/a	n/a	n/a
Merton	0.0817	0.1453	0.3483	-0.0700	0.1924	n/a	n/a	n/a
Kou	0.0874	0.1452	0.3483	n/a	n/a	0.2903	4.7941	5.4349

The risk-free rate is based on 3-month US T-bill rates[13] over the period 1934-2014, and has been augmented with the NBER's short-term government bond yield data[14] for 1926-1933 to incorporate the impact of the 1929 stock market crash. Prior to calculations, all time series were inflation-adjusted using data from the US Bureau of Labor Statistics[15]. This results in a risk-free rate of $r = 0.00623$.

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[12] Calculations were based on data from the Historical Indexes 2015©, Center for Research in Security Prices (CRSP), The University of Chicago Booth School of Business. Wharton Research Data Services was used in preparing this article. This service and the data available thereon constitute valuable intellectual property and trade secrets of WRDS and/or its third party suppliers.

[13] Data has been obtained from `See http://research.stlouisfed.org/fred2/series/TB3MS`.

[14] Obtained from the National Bureau of Economic Research (NBER) website, `http://www.nber.org/databases/macrohistory/contents/chapte`

[15] The annual average CPI-U index, which is based on inflation data for urban consumers, were used - see `http://www.bls.gov.cpi`
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