Hedging Under Jump Diffusions with Transaction Costs

Peter Forsyth, Shannon Kennedy, Ken Vetzal
University of Waterloo

Computational Finance Workshop, Shanghai, July 4, 2008
Overview

- Single factor diffusion models for equities not adequate for risk management

- Alternatives:
  
  **Stochastic Volatility/Regime Switching**: can hedge with underlying plus small number of options (sometimes one)

  **Jump processes**: hedge with underlying plus infinite number of options!

- Obviously, hedging jumps is hard
Why Do We Need Jump Models?

- Equity return data suggests jumps.
- Typical local volatility surfaces
  - Heavy skew for short dated options
  - Consistent with jumps
- Large asset price changes more frequent than suggested by Geometric Brownian Motion
- Risk management: if we don’t hedge the jumps
  - We are exposed to sudden, large losses
Why Jumps?

Example: A Drug Company

- This is not Geometric Brownian Motion!
- 80% and 50% drops in one day!
Why Jumps?

S&P 500 monthly log returns since 1982

- Scaled to zero mean and unit standard deviation
- Standard normal distribution also shown
  - Extreme events more likely than simple GBM
  - Higher peak, fatter tail than normal distribution
Hedging

**Hedging the Jumps**

- If we believe that the underlying process has jumps, hedging portfolio must contain underlying plus options
- Hedging the jumps: previous work (Carr, He *et al*), good results for semi-static hedging (European options)
- We need a dynamic strategy for path dependent options
- Questions:
  - How many options do we need to reduce jump risk?
  - Will the bid-ask spread of the options in our hedging portfolio make a dynamic strategy too expensive?
Overview

- Assume price process is a jump diffusion
- Force delta neutrality (diffusion risk hedged)
- Isolate jump risk and transaction cost (bid/ask spread) terms
  - Model bid-ask spread as a function of moneyness
- At each hedge rebalance time
  - Minimize jump risk and transaction costs
- Test strategy by Monte Carlo simulation
Assumption: Stochastic Process for Underlying Asset $S$

\[
\frac{dS}{S} = \mu dt + \sigma dZ + (J - 1)dq
\]

$\mu$ = drift rate,
$\sigma$ = volatility,
$dZ =$ increment of a Wiener process

\[
dq = \begin{cases} 
0 & \text{with probability } 1 - \lambda dt \\
1 & \text{with probability } \lambda dt,
\end{cases}
\]

$\lambda$ = mean arrival rate of Poisson jumps; $S \rightarrow JS$. 

Computational Finance Workshop, Shanghai, July 4, 2008
**Option Price** \( V = V(S,t) \) **Given by PIDE/LCP**

\[
\min(V_\tau - \mathcal{L}V - \lambda \mathcal{I}V, V - V^*) = 0 \quad \text{American}
\]

\[
V_\tau = \mathcal{L}V - \lambda \mathcal{I}V \quad \text{European}
\]

\[
\mathcal{L}V \equiv \frac{\sigma^2}{2} S^2 V_{SS} + (r - \lambda \kappa) S V_S - (r + \lambda) V
\]

\[
\mathcal{I}V \equiv \int_0^\infty V(SJ) g^Q(J) \, dJ
\]

\[
T = \text{maturity date}, \quad \kappa = E^Q[J - 1], \quad V^* = \text{payoff},
\]

\[
r = \text{risk free rate}, \quad \tau = T - t,
\]

\[
g^Q(J) = \text{probability density function of the jump amplitude } J
\]
Hedging Strategy

Hedging Portfolio $\Pi$

\[ \Pi = -V + eS + \vec{\phi} \cdot \vec{I} + B \]

- Short option worth $V$
- Long $e$ units underlying worth $S$
- Long $N$ additional instruments worth $\vec{I} = [I_1, I_2, \ldots, I_N]^T$, with weights $\vec{\phi} = [\phi_1, \phi_2, \ldots, \phi_N]^T$
- Cash worth $B$
Jump Risk

- In $t \to t + dt$, $\Pi \to \Pi + d\Pi$.
- Use Ito’s formula for finite activity jump diffusions, force delta neutrality
- Assume mid-point option prices given by linear pricing PIDE
- Recall: $Q = \text{pricing measure}; \mathbb{P} = \text{real world measure}$
  - In practice, $Q$ measure parameters obtained by calibration
- $\mathbb{P}$ measure parameters unknown to hedger
Jump Risk

**Change in Delta Neutral Portfolio**

\[ d\Pi = \text{Jump Risk} = \lambda^Q dt \mathbb{E}^Q \left[ \Delta V - (\vec{\phi} \cdot \Delta \vec{I} + e\Delta S) \right] \]

\[ + dq^P \left[ -\Delta V + (\vec{\phi} \cdot \Delta \vec{I} + e\Delta S) \right] \]

\[ \Delta S = JS - S ; \quad \Delta V = V(JS) - V(S) \]

\[ \Delta \vec{I} = \vec{I}(JS) - \vec{I}(S) \]

Note: if \( \mathbb{E}^Q = \mathbb{E}^P \), deterministic drift term exactly compensates random term. But in general \( \mathbb{E}^Q \neq \mathbb{E}^P \), i.e. usually \( Q \) is more pessimistic than \( P \)
Minimizing Jump Risk

When a jump occurs $dq^P \neq 0$, the random change in $\Pi$ is

$$\Delta H(J) = -\Delta V + \vec{\phi} \cdot \Delta \vec{I} + e\Delta S$$

Let $W(J)$ be any positive weighting function. Consider:

$$F(\vec{\phi}, e)_{\text{jump}} = \int_0^\infty [\Delta H(J)]^2 W(J) dJ$$
Minimizing Jump Risk

If \( F(\vec{\phi}, e)_{jump} = 0 \), then both the deterministic and random component of jump risk is zero.

Objective: make \( F(\vec{\phi}, e)_{jump} \) (weighted jump risk) as small as possible

- Problem: What weighting function to use?
- Ideally, \( W(J) = \mathbb{P} \) measure jump distribution, but this is unobservable
- If you guess wrong, results can be be very bad
Weighting Function

Practical Solution: set $W(J)$ to be nonzero for likely jump sizes $S \rightarrow JS$ (triangular tails avoid numerical problems)

\[
\int_0^\infty [\Delta H(J)]^2 W(J) dJ
\]
Bid-Ask Spreads

- Assume that hedger buys/sells at PIDE midpoint price ± one half spread
- This represents a lost transaction cost at each hedge rebalance time

\[ F(\vec{\phi}, e)_{spread} = \sum_{portfolio} \left( \text{Money lost due to spreads} \right)^2 \]
Objective

**Objective Function**

At each hedge rebalance time, choose \((e, \vec{\phi})\) (weights in underlying and hedging options), so that

- Portfolio is Delta neutral
- Minimize

\[
\text{Objective Function} = \xi F(\vec{\phi}, e)_{jump} + (1 - \xi) F(\vec{\phi}, e)_{spread}
\]

\(\xi = 1 \rightarrow\) Minimize jump risk only

\(\xi = 0 \rightarrow\) Minimize trans. cost only
Review Assumptions: Synthetic Market

- Price process is Merton type jump diffusion
- All options in market can be bought/sold for the fair price plus/minus one half spread
- Mid-point option prices determined by linear pricing PIDE
- $\mathbb{Q}$ measure parameters: Andersen and Andreasen (2000)
- $\mathbb{P}$ measure market parameters: utility equilibrium model
- Hedger knows the $\mathbb{Q}$ measure market parameters
- Hedger does not know $\mathbb{P}$ measure market parameters
Basic Testing Method

- Choose target option, set of hedging instruments, hedging horizon
- Carry out MC simulations of hedging strategy, assume underlying follows a jump diffusion, with specified $\mathbb{P}$ measure parameters, option prices given by solution of PIDE
- Record discounted relative $P&L$ at end of hedging horizon (or exercise) $t = T^*$ for each MC simulation

$$
\text{Relative P & L} = \exp\{-rT^*\} \Pi(T^*) \cdot \frac{V(S_0, 0)}{V(S_0, 0)} = \text{Initial Target Option Price}
$$
Base Case

Base Case Example

• Target option: one year European straddle
• Hedging horizon: 1.0 years, rebalance 40 times
• Initial $S_0 = 100$
• Hedging portfolio: underlying plus five .25 year puts/calls with strikes near $S_0$ (liquidate portfolio at $t = .25, .50, .75$, buy new .25 year options)
• Case 1: no bid-ask spreads
• Case 2: flat relative bid-ask spreads

Relative Spread: underlying $= .002$
Relative Spread: options $= .10$
Optimization Weights

Recall that, at each rebalance date, we minimize:

\[
\text{Objective Function} = \xi \times (\text{Jump Risk}) + (1 - \xi) \times (\text{Transaction Cost})
\]

How to pick \(\xi\)?

- No *right* answer
- Tradeoff between risk and cost
- We simply compute the density of the \(P\&L\) for a range of \(\xi\) values, report results which give smallest standard deviation.
Base Case Results

Dotted - no transaction costs
Solid - transaction cost in market; not in objective function
Dashed - transaction cost in market; transaction cost in objective function
**Base Case Summary**

- Delta hedging alone not very good
- If there are bid-ask spreads, and you don’t take them into account when determining portfolio weights
  \[\rightarrow\text{Hedging with options worse then delta hedging!}\]
- Minimizing both jump risk and transaction costs
  \[\rightarrow\text{Small standard deviation}\]
  \[\rightarrow\text{Cumulative transaction cost comparable with relative spreads assumed for hedging options.}\]
A More Realistic Example

• Use better model for bid-ask spreads
• Allow a larger number of possible options for use in hedge portfolio (10 – 14 possible hedging options)
  – Consider .25 year puts/calls with strikes at $10 intervals, centered near $S = 100$
• Realistic bid-ask spread model should make deep out of the money options too expensive to use

Model relative spread as a function of moneyness ($K/S$)

Flat top data to avoid unrealistically large relative spread.

- Same target option (one year straddle)
- Forty rebalances
- Optimization method should pick out cheapest options to minimize jump risk
## Realistic Spread Model: Results

<table>
<thead>
<tr>
<th>Hedging Strategy</th>
<th>Mean</th>
<th>Standard Deviation</th>
<th>Percentiles 0.02%</th>
<th>Percentiles 0.2%</th>
</tr>
</thead>
<tbody>
<tr>
<td>Delta Hedge</td>
<td>0.0565</td>
<td>1.0395</td>
<td>-11.55</td>
<td>-9.01</td>
</tr>
<tr>
<td>Ten Hedging Options</td>
<td>-0.0639</td>
<td>0.0230</td>
<td>-0.1493</td>
<td>-0.1250</td>
</tr>
<tr>
<td>Fourteen Hedging Options</td>
<td>-0.0667</td>
<td>0.0206</td>
<td>-0.1251</td>
<td>-0.1152</td>
</tr>
</tbody>
</table>

- Note that ten hedging options $\rightarrow$ 99.98% of the time we can lose no more than 15% of the initial option premium
- Note positive mean for simple delta hedging
Some Analysis: No Transaction Costs

Suppose that the weighting function $W(J)$ is such that for any function $f(J)$

$$
\int_0^\infty f^2(J) g^p(J) \, dJ \leq \int_0^\infty f^2(J) W(J) \, dJ < \infty
$$

Recall notation:

\begin{align*}
T &= \text{Expiry time of option} \\
\Delta H(J) &= \text{Jump Risk} \\
\Delta t &= \text{Hedge rebalance interval}
\end{align*}
Global Bound: Hedging Error

Theorem 1. In the limit as $\Delta t \to 0$, and if at each hedge rebalance time

- The hedge portfolio $\Pi$ is delta neutral
- $\int_0^\infty [\Delta H(J)]^2 W(J) \, dJ \leq \epsilon$

Then

$$E^\mathcal{P}[(\text{Total Hedging Error})^2_T] \leq C_1 \epsilon$$

where $C_1$ is a constant.
Analysis

**Adding in Transaction Costs**

**Theorem 2.** In the limit as $\Delta t \to 0$, assuming $\Pi$ is delta neutral and, at each rebalance time

\[
\xi \left\{ \int_0^\infty \left[ \Delta H_J(S_t, t) \right]^2 W(J) \, dJ \right\} + (1 - \xi) \left\{ \text{Transaction Costs} \right\}^2 < \epsilon \Delta t^2
\]

holds for some $\xi \in (0, 1)$. Then

\[
E^\mathbb{P}[\left( \text{Total Hedging Error} \right)_T^2] \leq C_2 \epsilon
\]
Analysis

Adding in Transaction Costs II

• It is always possible to minimize transaction costs by not trading
• It may not be possible to minimize both transaction costs and jump risk as required by the Theorem
• As a practical solution, we attempt to make the objective function as small as possible at each rebalance time

\[
\text{Objective Function} = \xi \left\{ \int_0^\infty \left[ \Delta H_J(S_t, t) \right]^2 W(J) \, dJ \right\} + (1 - \xi) \left\{ \text{Transaction Costs} \right\}^2
\]
Analysis

Adding in Transaction Costs III

It follows from this result that if $\xi$ is fixed for given $\Delta t$, and we minimize the local objective function at each rebalance time, then the best choice for $\xi$ is

$$\xi = C_3(\Delta t)^2$$

This is observed in the numerical experiments.

Note that this means that more weight is put on the transaction cost term in the objective function as $\Delta t \to 0$.

This is required to avoid infinite transaction costs.
Conclusions

- In market with jumps → delta hedging is bad
- Need to use additional options in the hedging portfolio
- If hedging portfolio is determined only on basis of minimizing jump risk → bid-ask spreads cause poor results when hedging with options
- If both jump risk and transaction costs minimized
  - Standard deviation much reduced compared to delta hedge
  - Relative cumulative transaction costs $\approx 6 - 7\%$
- Similar results for American options
Let’s Start a Hedge Fund 😊

- Recall that hedge fund managers typically receive 20% of the gain in an investment portfolio, but *no penalty if a loss*.
- Hedge fund strategy
  - Select asset which has large, infrequent jumps
  - Sell contingent claims (on this asset) with positive gamma, delta hedge
  - In a market with jumps, recall that this strategy has a positive mean
- This means that we, as hedge fund managers make money most of the time (and collect large bonuses)
- When a jump occurs, the investors are left with large, unhedged losses, hedge fund is bankrupt, but we retire rich!
- Sound familiar?