Dynamic Mean Variance Asset Allocation: Numerics and Backtests

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Outline

1 Dynamic mean variance
   - Embedding result ⇒ quadratic target
   - Removal of spurious points

2 HJB PDE
   - Intuitive discretization
   - Semi-Lagrangian timestepping and explicit control
   - Unconditionally stable, monotone and consistent

3 Calibrate to historical market data (1926-2015)
   - Synthetic market: M-V optimal beats constant proportion
   - Backtests using real historical data: M-V optimal even better!
   - Constant proportion beats any deterministic glide path strategy\(^1\)
     \[\rightarrow \text{M-V optimal beats any glide path strategy}\]

\(^1\)Strategy used in Target Date funds (over $700 billion in US)
Dynamic Mean Variance: Abstract Formulation

Define:

\[ X = \text{Process} \]
\[ \frac{dX}{dt} = \text{SDE} \]
\[ x = (X(t) = x) = \text{State} \]
\[ W(X(t)) = \text{total wealth} \]

Control \( c(X(t), t) \) is applied to \( X(t) \)

Define admissible set \( \mathcal{Z} \), i.e.

\[ c(x, t) \in \mathcal{Z}(x, t) \]
Mean and Variance under control $c(X(t), t)$

Let:

\[
E_{t,x}^{c(\cdot)}[W(T)]
\]

\text{Reward}

= Expectation conditional on $(x, t)$ under control $c(\cdot)$

\[
\text{Var}_{t,x}^{c(\cdot)}[W(T)]
\]

\text{Risk}

= Variance conditional on $(x, t)$ under control $c(\cdot)$

Important:

- mean and variance of $W(T)$ are as observed at time $t$, initial state $x$. 
Basic Problem: Find Pareto Optimal Strategy

We desire to find the investment strategy $c^*(\cdot)$ such that, there exists no other other strategy $c(\cdot)$ such that

\[
\begin{align*}
\underbrace{E_{t,x}^c[W_T]}_{\text{Reward under strategy } c(\cdot)} & \geq \underbrace{E_{t,x}^{c^*(\cdot)}[W_T]}_{\text{Reward under strategy } c^*(\cdot)} \\
\underbrace{\text{Var}_{t,x}^c[W_T]}_{\text{Risk under strategy } c(\cdot)} & \leq \underbrace{\text{Var}_{t,x}^{c^*(\cdot)}[W_T]}_{\text{Risk under strategy } c^*(\cdot)}
\end{align*}
\]

and at least one of the inequalities is strict.

Scalarization: For $\lambda > 0$, find $c(\cdot)$ which solves

\[
\inf_{c(\cdot)} \left\{ \lambda \text{Var}_{t,x}^c[W_T] - E_{t,x}^c[W_T] \right\}
\]

Varying $\lambda$ traces out the efficient frontier.
Pareto optimal points

Let

\[ \mathcal{E} = E_{t,x}^{c(\cdot)} [W_T] \quad ; \quad \mathcal{V} = \text{Var}_{t,x}^{c(\cdot)} [W_T] \]

The achievable set \( \mathcal{Y} \) is

\[ \mathcal{Y} = \{ (\mathcal{V}, \mathcal{E}) : c(\cdot) \in Z \} , \]

Given \( \lambda > 0 \), define scalarization set \(^2\)

\[ S_{\lambda}(\mathcal{Y}) = \{ (\mathcal{V}, \mathcal{E}) \in \bar{\mathcal{Y}} : \lambda \mathcal{V} - \mathcal{E} = \inf_{(\mathcal{V}_*, \mathcal{E}_*) \in \mathcal{Y}} (\lambda \mathcal{V}_* - \mathcal{E}_*) \} \]

The efficient frontier \( \mathcal{Y}_P \) is

\[ \mathcal{Y}_P = \bigcup_{\lambda > 0} S_{\lambda}(\mathcal{Y}) \]

The efficient frontier is a collection of Pareto points

\(^2\bar{\mathcal{Y}} \) is the closure of \( \mathcal{Y} \).
Scalarization: intuition

Recall scalarization set:

\[ S_\lambda(\mathcal{Y}) = \{(\mathcal{V}, \mathcal{E}) \in \tilde{\mathcal{Y}} : \lambda \mathcal{V} - \mathcal{E} = \inf_{(\mathcal{V}_*, \mathcal{E}_*) \in \mathcal{Y}} (\lambda \mathcal{V}_* - \mathcal{E}_*)\} \tag{1} \]

Geometric interpretation:

- Consider the straight line (for fixed \( \lambda \))

\[ \lambda \mathcal{V} - \mathcal{E} = C_1 \tag{2} \]

Points in (1)

- Choose \( C_1 \) as small as possible, such that:
  \[ \rightarrow \] Intersection of \( \mathcal{Y} \) and straight line (2) has at least one point

\(^3\)We may not get all the Pareto points here if \( \mathcal{Y} \) is not convex
Move dotted lines line $\lambda \mathcal{V} - \mathcal{E} = C_1$ to the left as much as possible (decrease $C_1$)

Line will touch $\mathcal{V}$ at Pareto point
Problem

Pareto point

\[ \lambda V - \mathcal{E} = \inf_{(V^*, \mathcal{E}^*) \in \mathcal{Y}} (\lambda V^* - \mathcal{E}^*) \]  

(3)

Problem arises from variance

\[ V = E^c[W(T)^2] - (E^c[W(T)])^2 \]

\[ (E^c[W(T)])^2 \rightarrow \text{problem for dynamic programming} \]

Consider the optimization problem (for fixed \( \gamma \))

\[ \inf_{(V, \mathcal{E}) \in \mathcal{Y}} V + \mathcal{E}^2 - \gamma \mathcal{E} \]  

(4)

Note that

\[ V + \mathcal{E}^2 = E^c[W(T)^2] \]

Minimizing (4) can be done using dynamic programming
Embedded Objective Function Intuition

Examine points \((V, E) \in \mathcal{Y}\) such that (for fixed \(\gamma\))

\[
V + E^2 - \gamma E = \inf_{(V^*, E^*) \in \mathcal{Y}} V^* + E^2 - \gamma E^*
\]  

(5)

Geometric interpretation:

- Consider the parabola

\[
V + E^2 - \gamma E = C_2
\]

(6)

Points in (5)

- Choose \(C_2\) as small as possible, such that
  - Intersection of parabola and \(\mathcal{Y}\) has at least one point

Rewriting equation (6)

\[
V = - (E^2 - \gamma E) + C_2 = - (E - \gamma/2)^2 + \gamma^2/4 + C_2
\]

\[
= - (E - \gamma/2)^2 + C_3.
\]

Parabola faces left, symmetric about line \(E = \gamma/2\)
**Embedded Pareto Points**

Suppose \((\mathcal{V}_*, \mathcal{E}_*) \in \mathcal{Y}_P \rightarrow \exists \lambda > 0, C_1, \text{ s.t.} \)

\[
\lambda \mathcal{V}_* - \mathcal{E}_* = C_1
\]

Pick \(\gamma/2\), move parabola to left as much as possible, and intersect line \(\lambda \mathcal{V}_* - \mathcal{E}_* = C_1\) at a single point.
Tangency Condition

Parabola $\mathcal{V} = - (\mathcal{E} - \gamma/2)^2 + C_3$ tangent to line $\lambda \mathcal{V} - \mathcal{E} = C_1$ at $(\mathcal{V}_*, \mathcal{E}_*)$

\[
\left( \frac{\partial \mathcal{E}}{\partial \mathcal{V}} \right)_{\text{parabola}} = \lambda \quad ; \quad \lambda = \text{slope of dotted lines}
\]

\[
\Rightarrow \quad \gamma/2 = 1/(2\lambda) + \mathcal{E}_*
\]
Theorem 1 ((Li and Ng (2000); Zhou and Li (2000))

If

\[ \lambda V_0 - E_0 = \inf_{(V,E) \in Y} (\lambda V - E), \]  

(7)

then

\[ V_0 + E_0^2 - \gamma E_0 = \inf_{(V,E) \in Y} (V + E^2 - \gamma E), \]  

(8)

\[ \gamma = \frac{1}{\lambda} + 2E_0 \]

Implication

- We can determine all the Pareto points from (7) by solving problem (8)
Value function

Note:

\[ V + \mathcal{E}^2 - \gamma \mathcal{E} = E_{t,x}^c[(W(T) - \frac{\gamma}{2})^2] + \frac{\gamma^2}{4}, \]

Define value function\(^4\) (ignore \(\gamma^2/4\) term when minimizing)

\[ V(x, t) = \inf_{c(\cdot) \in \mathcal{Z}} E_{t,x}^c[(W(T) - \gamma/2)^2] \tag{9} \]

**Key Result:** Given point \((V^*, \mathcal{E}^*)\) on the efficient frontier, generated by control \(c^*(\cdot)\), then \(\exists \gamma\) s.t.

\[ \rightarrow c^*(\cdot)\text{ is an optimal control for (9)} \]

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\(^4\)Precommitment MV optimal \(\equiv\) quadratic target optimal. Precommitment
\[ \rightarrow \text{choose target wealth } \gamma/2 \text{ at time zero} \]
Spurious points

But, converse not necessarily true: i.e. there may be some \( \gamma \in (-\infty, +\infty) \) s.t. \( c^*(\cdot) \) which solves

\[
V(x, t) = \inf_{c(\cdot) \in \mathbb{Z}} E_{t,x}^c[(W(T) - \gamma/2)^2] \tag{10}
\]

does not correspond to a point on the efficient frontier
Basic Algorithm

Discretize the parameter $\gamma$

$$\gamma \in \Gamma^k = [-|\gamma_{\text{max}}^k|, -|\gamma_{\text{max}}^k| + h_k, \ldots, |\gamma_{\text{max}}^k|]$$  \hspace{1cm} (11)

$$h_k \to 0 \; ; \; \gamma_{\text{max}}^k \to \infty \; ; \; k \to \infty$$  \hspace{1cm} (12)

For each $\gamma_i$,

- Determine optimal control $c_{\gamma_i}^*(\cdot)$ by solving the embedded problem (solve HJB equation, store control)
- Using this control, compute $E_{t,x}^{c_{\gamma_i}^*(\cdot)}[(W_T)]$, $\text{Var}_{t,x}^{c_{\gamma_i}^*(\cdot)}[(W_T)]$ via Monte Carlo (one point on the frontier)

Does this converge to true efficient frontier as $k \to \infty$?
Problems

1. Controls which minimize $E_{t,x}^{c(\cdot)}[(W(T) - \gamma/2)^2]$ (from numerical solve)
   - May generate spurious points (e.g. non-convex $\mathcal{Y}$)

2. The control which minimizes

\[ E_{t,x}^{c(\cdot)}[(W(T) - \gamma/2)^2] \] (13)

may not be unique.
   - Numerical HJB solve for fixed $\gamma/2$
     - picks out only one control $c^*(\cdot)$
   - Does the control we compute correspond to a point in $\mathcal{Y}_P$?
Convergent Algorithm\textsuperscript{5}

For $k = 0, 1, \ldots$

- Solve value function $\forall \gamma_i \in \Gamma^k$
- Generate set of candidate points on the efficient frontier $\mathcal{A}^k$
- Determine upper left convex hull $S(\mathcal{A}^k)$
- Approximate points on efficient frontier: $\mathcal{A}^k \cap S(\mathcal{A}^k)$

\textsuperscript{5}Tse, Forsyth, Li (2014, SIAM Cont. Opt.); Dang, Forsyth, Li (2016, Num. Math.)
Convergence result

Recall def’n of scalarization set:

\[ S_\lambda(X) = \{ (V_*, E_*) \in \bar{X} : \lambda V_* - E_* = \inf_{(V, E) \in X} \lambda V - E \} , \] (14)

Suppose \( S_\lambda(Y) \neq \emptyset, \lambda > 0 \) (i.e. \( S_\lambda(Y) \) are points on the efficient frontier for fixed \( \lambda \))

**Theorem 2**

Suppose \( \Gamma^k \) is systematically refined \(^6\) as \( k \to \infty \), and let \((V_k, E_k) \in S_\lambda(A^k)\). Let \((V_*, E_*)\) be a limit point of \( \{(V_k, E_k)\} \). Then \((V_*, E_*)\) is on the original efficient frontier.

**Remark 1**

All points on the approximate efficient frontier \( A^k \cap S(A^k) \) are valid points on the true efficient frontier as \( k \to \infty \). \(^7\)

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\(^6\)Any reasonable refinement satisfies this condition

\(^7\)There may some gaps in the approximate frontier if there are 3 or more points on a straight line segment.
Asset allocation: risk free bond, stock index

Risk free bond $B$

$$ dB = rB \, dt $$

$r = \text{risk-free rate}$

Amount in risky stock index $S$ (jump diffusion)

$$ dS = (\mu - \rho \kappa) S \, dt + \sigma S \, dZ + (J - 1) S \, dq $$

$\mu = \mathbb{P} \text{ measure drift}$ ; $\sigma = \text{volatility}$

$dZ = \text{increment of a Wiener process}$

$$ dq = \begin{cases} 
0 & \text{with probability } 1 - \rho \, dt \\
1 & \text{with probability } \rho dt,
\end{cases} $$

$log J \sim \text{double exponential}.$ ; $\kappa = E[J - 1]$
Optimal Control

Define:

\[ X = (S(t), B(t)) = \text{Process} \]
\[ x = (S(t) = s, B(t) = b) = (s, b) = \text{State} \]
\[ (s + b) = \text{total wealth} \]

Let \((s, b) = (S(t^-), B(t^-))\) be the state of the portfolio the instant before applying a control

The control \(c(s, b) = (d, B^+)\) generates a new state

\[ b \rightarrow B^+ \]
\[ s \rightarrow S^+ \]
\[ S^+ = (s + b) - B^+ - d \] 

Note: we allow cash withdrawals of an amount \(d \geq 0\) at a rebalancing time
Optimal de-risking (free cash flow)

Let

\[ F(t) = \frac{\gamma}{2} e^{-r(T-t)} \]

= discounted target wealth

Proposition 1 (Dang and Forsyth (2016))

If \( W_t > F(t), \ t \in [0, T] \), an optimal MV strategy is

- Withdraw cash \( d = W_t - F(t) \) from the portfolio
- Invest the remaining amount \( F(t) \) in the risk-free asset.

We will refer to the amount withdrawn as a free cash flow.  

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\[ \text{See also: Ehrbar, J. Econ. Theory (1990); Cui, Li, Wang, Zhu Mathematical Finance (2012); Bauerle, Grether Mathematical Methods of Operations Research (2015).} \]
Constraints on the strategy

The investor can continue trading only if solvent

\[ W(s, b) = s + b > 0 . \]  
\[ \text{Solvency condition} \] (15)

In the event of bankruptcy, the investor must liquidate

\[ S^+ = 0 \quad ; \quad B^+ = W(s, b) \quad ; \quad \text{if } W(s, b) \leq 0 . \]  
\[ \text{bankruptcy} \]

Leverage is also constrained

\[ \frac{S^+}{W^+} \leq q_{\text{max}} \]

\[ W^+ = S^+ + B^+ = \text{Total Wealth} \]
HJB PIDE

Find optimal control $c(\cdot)$ ⇒ solve for value function

$$V(x, t) = \inf_{c \in \mathbb{Z}} \left\{ E_{t,x}^c \left[ (W(T) - \gamma/2)^2 \right] \right\},$$

Define:

$$\mathcal{L} V \equiv \frac{\sigma^2 s^2}{2} V_{ss} + (\mu - \rho \kappa) s V_s - \rho V,$$

$$\mathcal{J} V \equiv \int_0^\infty p(\xi)V(\xi s, b, \tau) \, d\xi$$

$$p(\xi) = \text{jump size density} ; \rho = \text{jump intensity}$$

and the intervention operator $\mathcal{M}(c) \; V(s, b, t)$

$$\mathcal{M}(c) \; V(s, b, t) = V(S^+(s, b, c), B^+(s, b, c), t)$$
HJB PIDE II

Value function, control \( c(\cdot) \Rightarrow \) solve impulse control HJB equation

\[
\max \left[ V_t + \mathcal{L} V + rbV_b + \mathcal{J} V, \; V - \inf_{c \in \mathcal{Z}} (\mathcal{M}(c) \; V) \right] = 0
\]

Discretize computational domain \((s, b) \in [0, \infty) \times (-\infty, +\infty)\)

\[
\{ s_1, s_2, \ldots, s_{i_{\text{max}}} \} \; ; \; \{ b_1, \ldots, b_{j_{\text{max}}} \}
\]

Constant timesteps, discretize control

\[
\Delta \tau = \tau^{n+1} - \tau^n \; ; \; B^+ \in \{ b_1, \ldots, b_{j_{\text{max}}} \}
\]

Discretization parameter \( h \)

\[
\max_i (s_{i+1} - s_i) = \max_j (b_{j+1} - b_j) = \max_n (\tau^{n+1} - \tau^n) = O(h)
\]
Computational Domain

(S,B) ∈ [0, ∞) x [−∞, +∞]

9 If µ > r it is never optimal to short S
Intuitive Derivation of Discretization

Consider a set of discrete rebalancing times \( \{t_1, t_2, \ldots \} \)

Define

\[
t_m^+ = t_m + \epsilon \quad ; \quad t_m^- = t_m - \epsilon \quad ; \quad \epsilon \to 0^+ \quad \tag{16}
\]

At \( t = t_m^+ \), \( s = S(t) \) and \( b = B(t) \)

Step \( [t_m^+, t_{m+1}^-] \) (bond amount constant)

- The value function \( V(s, b, t) \) evolves according to the PIDE

\[
V_t + \hat{\mathcal{L}} V + \hat{\mathcal{J}} V = 0,
\]
Evolution over \([t_{m+1}^-, t_{m+1}^+]\)

Step \([t_{m+1}^-, t_{m+1}^+]\) (Stock amount constant)

- Pay interest earned in \([t_m^-, t_{m+1}^-]\)

\[
V(s, b, t_{m+1}^-) = V(s, be^{r\Delta t}, t_{m+1}) \quad \text{; by no-arbitrage}
\]

\[
\Delta t = t_{m+1} - t_m
\]

Step \([t_{m+1}^-, t_{m+1}^+]\)

- Optimal rebalance

\[
V(s, b, t_{m+1}) = \underbrace{\min_c V(S^+(s, b, c), B^+(s, b, c), t_{m+1}^+)}_{\text{rebalance}}
\]
Now, we write these steps down in backwards time $\tau = T - t$

- Define $V_{i,j}^n \equiv$ discrete solution $V_h(s_i, b_j, \tau^n)$

\[
\tilde{V}_{i,j}^n = \min_{c \in \mathbb{Z}_h} V_h(S^+(s_i, b_j e^{r\Delta\tau}, c), B^+(s_i, b_j e^{r\Delta\tau}, c), \tau^n)
\]

\[
\frac{V_{i,j}^{n+1}}{\Delta\tau} - L_h V_{i,j}^{n+1} - J_h V_{i,j}^{n+1} = \frac{\tilde{V}_{i,j}^n}{\Delta\tau}
\]

Formally: Semi-Lagrangian timestepping and explicit impulse control
Discretization Properties

1. Positive coefficient method used to discretize $P$,
2. Jump term: fixed point iteration + FFT for dense matrix-vector product
3. Linear interpolation used to approximate $V_h$ at off grid points (needed for optimal control)

Assume strong comparison property holds:
- Consistent, $\ell_\infty$ stable, monotone
  - $\rightarrow$ Convergence to viscosity solution
Example Asset Allocation: Constant Proportions

According to Benjamin Graham\textsuperscript{10}, most investors should

- Pick a fraction $p$ of wealth to invest in a diversified equity fund (e.g. $p = 1/2$).
- Invest $(1 - p)$ in bonds
- Rebalance to maintain this asset mix
  $\rightarrow$ i.e. a constant proportion strategy

How does this strategy compare with standard target date funds, which follow a glide path over time $T$?

Typical glide path strategy\textsuperscript{11}

$$p(t) = (110 - \text{your age})$$

\textsuperscript{10}Benjamin Graham, \textit{The Intelligent Investor}

\textsuperscript{11}This used to be $(100 - \text{your age})$ but people are living longer
Constant Proportion Beats Glide Path

Consider any glide path strategy \( p(t) \)

\[
p(t) = \text{fraction of wealth invested in equities}
\]

Define a constant weight strategy \( p^* \) where

\[
p^* = \frac{1}{T} \int_0^T p(s) \, ds
\]

= time average fraction in equities

Let \( W \) denote total wealth. We can prove (GBM + jumps) \(^{12}\)

\[
\begin{array}{c}
\text{constant weight} & \text{glide path} \\
\overline{E[W(T)]} & \overline{E[W(T)]} \\
\text{constant weight} & \text{glide path} \\
\overline{\text{Var}[W(T)]} & \text{Var}[W(T)] \leq \text{Var}[W(T)]
\end{array}
\] (17)

Backtests on historical data and MC simulations\(^{13}\) indicates (17) holds in general \( \rightarrow \) constant proportion beats glide path

\(^{12}\)Graf (2013), Forsyth and Vetzal (2016)

\(^{13}\)Esch and Michaud (2014)
Monte Carlo Simulation Results

- Inflation-adjusted equity: jump diffusion\textsuperscript{14} model estimated using CRSP\textsuperscript{15} total return index and CPI data (1926 to 2015)
- Inflation-adjusted bonds: average real 3M T-bills (1926 to 2015)

<table>
<thead>
<tr>
<th>Strategy</th>
<th>Expected Value</th>
<th>Standard Deviation</th>
<th>Prob(W(T)) &lt; 300</th>
<th>Prob(W(T)) &lt; 400</th>
</tr>
</thead>
<tbody>
<tr>
<td>Constant Proportion $p = 0.5$</td>
<td>417</td>
<td>299</td>
<td>0.41</td>
<td>0.60</td>
</tr>
<tr>
<td>M-V Optimal Control</td>
<td>417</td>
<td>117</td>
<td>0.13</td>
<td>0.22</td>
</tr>
</tbody>
</table>

\textbf{Table:} Investment horizon $T = 30$ years. Initial investment $W(0) = 100$. Optimal de-risking; no trading if insolvent; maximum leverage $= 1.5$, rebalancing once/year.

\textbf{Standard deviation reduced by 250\%, shortfall probability reduced by $3 \times$}

\textsuperscript{14}Jump size had double exponential distribution (Kou, 2002)
\textsuperscript{15}Capitalization weighted index of all stocks traded on major US exchanges.
Cumulative Distribution Function: \( IRR^{16} \)

\[ E[W(T)] = 417 \] same for both strategies

Optimal policy: Contrarian:
- when market goes down → increase stock allocation;
- when market goes up → decrease stock allocation

Optimal allocation gives up gains \( \gg \) target in order to reduce variance and probability of shortfall.

Investor must pre-commit to target wealth

\( MV \) optimal beats constant proportion, consequently it also beats any glide path!

\[ ^{16} \text{Internal rate of return (i.e. effective rate of return) } = \log(W(T)/W(0))/T \]
Strategy Heat Map

Fraction in Risky Asset

Real Wealth

Time (years)

$W_0 = 100$

Red: maximum leverage

Blue: 100% bond
Back Testing

M-V optimal performance on historical data

- Compute and store strategy based on estimated parameters for entire historical period (January 1, 1926 - December 31, 2014).
- $E[W(T)]$ same as for constant proportion strategy ($p = .5$), for this set of average parameters.
- Select starting date
- Compare:
  - Optimal MV strategy (based on average parameters, not tuned to this period)
  - Constant proportion strategy
Back Test, Real Returns: Jan 1, 1985 - Dec 31, 2014

Mean Variance Optimal

50% Stocks 50% Bonds

Back Test, Real Returns: Jan 1, 1930 - Dec 31, 1959

Note *Falling Knife* effect in 1932

Can we fix this: regime switching plus machine learning?

\[ W(1930) = 100. \] Maximum leverage 1.5. Optimal MV strategy computed using parameters for 1926-2015 period. Yearly rebalancing.
Bootstrap Resampling: 1926-2015

More Scientific Test: Resampling

Use real historical data, quarterly returns

- Randomly draw 30 years of returns (with replacement) from historical returns (blocksize 10 years)
- 10,000 simulations, each block starts at random quarter

<table>
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<tr>
<th>Strategy</th>
<th>Expected Value</th>
<th>Standard Deviation</th>
<th>$P_r(W(T)) &lt; 300$</th>
<th>Expected Free Cash</th>
</tr>
</thead>
<tbody>
<tr>
<td>Constant Proportion $p = 0.5$</td>
<td>385</td>
<td>183</td>
<td>0.38</td>
<td>0.0</td>
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<tr>
<td>M-V Optimal Control</td>
<td>431</td>
<td>84</td>
<td>0.07</td>
<td>40</td>
</tr>
</tbody>
</table>

Table: $T = 30$ years. $W(0) = 100$. Yearly rebalancing. Optimal de-risking; no trading if insolvent; maximum leverage = 1.5.

Performs even better on actual historical data than on synthetic market data!
Resampled Cumulative Distribution Function: IRR

Internal rate of return, (i.e. effective rate of return) = \( \log\left(\frac{W(T)}{W(0)}\right)/T \)

\[ \text{Prob( Internal Rate of Return < IRR)} \]

0 0.01 0.02 0.03 0.04 0.05 0.06 0.07 0.08 0.09 0.10
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

MV Optimal (no free cash)

MV Optimal (plus free cash)

Constant Proportion (p=0.5)
Conclusions

- M-V strategy is very robust
  - Insensitive to calibration ambiguity
  - MC tests: insensitive to random perturbations of synthetic market SDE parameters
  - Stochastic volatility: typical parameters, insignificant for long term investors
  - 10 year treasuries (instead of 3-M) similar results
  - Good results on historical backtests

- Similar results for accumulation, decumulation
- M-V beats constant proportion, i.e. probability of shortfall $2 - 3\times$ smaller
  - Constant proportion beats any deterministic glide path
- M-V optimal equivalent to minimizing quadratic loss w.r.t. wealth target
  - Optimal strategy is M-V optimal and quadratic loss optimal

- More sophisticated models
  - Regime switching? (machine learning approach being investigated)