An HJB Equation Approach to Optimal Trade Execution

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Introduction

The Basic Problem

Broker buys/sells large block of shares on behalf of client

- Large orders will incur costs, due to price impact (liquidity) effects
- Slow trading minimizes price impact, but leaves exposure to stochastic price changes
- Fast trading will minimize risk due to random stock price movements, but price impact will be large
- What is the optimal strategy?
Introduction

**Previous Approaches**

**Almgren, Chriss** Mean-variance trade-off, discrete time, assume optimal asset positions are path-independent

**He, Mamaysky; Vath, Mnif, Pham** Maximize utility function, continuous time, dynamic programming, HJB equation.

**Almgren, Lorenz** Recognize that path-independent solution is not optimal. Suggest HJB equation, continuous time, mean variance tradeoff. No results.

**Lorenz (2008)** Mean variance tradeoff: analytic solution for simple cases
Introduction

Formulation

\[ P = B + \alpha S \]

= Trading portfolio

\[ B = \text{Bank account: keeps track of gains/losses} \]

\[ S = \text{Price of risky asset} \]

\[ \alpha = \text{Number of units of } S \]

\[ T = \text{Trading horizon} \]
For Simplicity: Sell Case Only

Sell

\[
\begin{align*}
t = 0 & \rightarrow B = 0, S = S_0, \alpha = \alpha_{sell} \\
t = T & \rightarrow B = B_L, S = S_T, \alpha = 0
\end{align*}
\]

- \( B_L \) is the cash generated by trading in \([0, T)\)
- Plus a final sale at \( t = T \) to ensure that zero shares owned.
- Success is measured by \( B_L \) (proceeds from sale).
- Maximize \( E[B_L] \), minimize \( Var[B_L] \)
Price Impact Modelling

Assume trades occur instantaneously in discrete amounts, leads to impulse control formulation.
← Problem: the price impact of two discrete trades independent of time interval between trades (unrealistic).

Alternate approach: assume trades occur continuously, at trading rate $v$.
← In this case, price impact can be a function of trade rate.
← Problem: real trading takes place discretely.

Neither model is perfect. We use the continuous trade model in the following.
Basic Problem

Trading rate \( v (\alpha = \text{number of shares}) \)

\[
\frac{d\alpha}{dt} = v .
\]

Suppose that \( S \) follows geometric Brownian Motion (GBM)

\[
dS = (\eta + g(v))S \ dt + \sigma S \ dZ
\]

- \( \eta \) is the drift rate of \( S \)
- \( g(v) \) is the permanent price impact
- \( \sigma \) is the volatility
- \( dZ \) is the increment of a Wiener process.
Basic Problem II

To avoid round-trip arbitrage (Huberman, Stanzl (2004))

\[ g(v) = \kappa_p v \]

\( \kappa_p \) permanent price impact factor (const.)

The bank account \( B \) is assumed to follow

\[ \frac{dB}{dt} = rB \ dt - vSf(v) \]

\( r \) is the risk-free return

\( f(v) \) is the temporary price impact

\(-vSf(v)\) represents the rate of cash generated when selling shares at price \( Sf(v) \) at rate \( v \).
**Temporary Price Impact**

The temporary price impact and transaction cost function $f(v)$ is assumed to be

$$f(v) = [1 + \kappa_s \text{sgn}(v)] \exp[\kappa_t \text{sgn}(v)|v|^\beta]$$

- $\kappa_s$ is the bid-ask spread parameter
- $\kappa_t$ is the temporary price impact factor
- $\beta$ is the price impact exponent
Control Problem

Select the control $v(t)$ (i.e. the selling strategy) so as to maximize

$$\max_{v(t)} \left( E^{t=0}[B_L] - \lambda V a r^{t=0}[B_L] \right)$$

$$E^{t=0}[\cdot] = \text{Expectation as seen at } t = 0$$

$$V a r^{t=0}[\cdot] = \text{Variance as seen at } t = 0$$  \hspace{1cm} (1)$$

$B_L$ is the total cash received from the selling strategy.

Varying $\lambda$ generates pairs $(E^{t=0}[B_L], V a r^{t=0}[B_L])$ along the efficient frontier. More intuitive result than usual power-law/exponential utility function approach. (What is the utility function of a bank?)
Control

The Liquidation Value

• If \((S, B, \alpha)\) are the state variables the instant before the end of trading \(t = T^-\), \(B_L\) is given by

\[
B_L = B - v_T(\Delta t)_T S f(v_T)
\]

\[
v_T = \frac{0 - \alpha}{(\Delta t)_T}
\]

• Choosing \((\Delta t)_T\) small, penalizes trader for not hitting target \(\alpha = 0\).

• Optimal strategy will avoid the state \(\alpha \neq 0\)

→ Numerical solution insensitive to \((\Delta t)_T\) if sufficiently small
Pre-commitment vs. Time-consistent

We are maximizing (as seen at $t = 0$)

$$\max_{v(t)} \left( E^{t=0}[B_L] - \lambda Var^{t=0}[B_L] \right)$$

This is the pre-commitment policy, i.e. the strategy as a function of $(S, B, \alpha, t)$ is computed at $t = 0$.

- The trader follows this strategy even if (2) computed at $t > 0$ would yield a different strategy.
- The pre-commitment policy is not time-consistent in this sense.
- It is also possible to determine a time consistent mean-variance optimal strategy (Basak and Chabakauri, 2008)
How Do We Measure Success?

Suppose we lived in a world where our model of the process for the risky asset, and the price impact functions was perfect
• Suppose we followed the pre-commitment policy for thousands of trades, of same stock

We then compute the mean and standard deviation of the trades
• Any other strategy (including time consistent) must result in a smaller mean gain for the same standard deviation, compared to the pre-commitment strategy
↔ Time-consistent = pre-commitment + constraints
Dynamic Programming and Efficient Frontier

We would like to use Dynamic Programming and derive an HJB equation for the optimal strategy \( v^*(t) \).

\[
\max_{v(t)} \left( E^{t=0}[B_L] - \lambda Var^{t=0}[B_L] \right)
\]

\( E^{t=0}[\cdot] = \text{Expectation} \)

\( Var^{t=0}[\cdot] = \text{Variance} \)  \( (3) \)

But the variance term in the objective function causes difficulty. Solution (Li, Ng(2000); Zhou, Li (2000); theoretical analysis, not numerical)
Theorem 1 (Equivalent LQ problem). If $v^*(t)$ is the optimal control of Mean-Variance problem (3) then $v^*(t)$ is also the optimal control of problem

$$\max_{v(t)} E^{t=0} [\mu B_L - \lambda B_L^2]$$

(4)

where

$$\mu = 1 + 2\lambda E^{t=0}_{v^*} [B_L]$$

(5)

where $v^*$ is the optimal control of problem (4).
LQ Problem II

At first glance, this does not seem to be very useful
• \( \mu \) is a function of the optimal control \( u^* \)
\( \rightarrow \) Not known until the problem is solved

Since \( \lambda > 0 \), we can rewrite the LQ problem as

\[
\min_{v(t)} \left( E^{t=0} \left[ (B_L - \frac{\gamma}{2})^2 \right] - \frac{\gamma^2}{4} \right)
\]

\[
\gamma = \frac{\mu}{\lambda}
\]
LQ Problem III

For fixed $\gamma$, an optimal control of the original problem is an optimal control of

$$\min_{v(t)} E^{t=0}[(B_L - \frac{\gamma}{2})^2].$$

(6)

Possible solution method: pick a value of $\gamma$, solve (6) for optimal strategy $v^*(t)$. Then, with known $v^*(t)$, compute

$$E_{v^*}^{t=0}[B_L] ; \quad \lambda = \frac{1}{\gamma - 2E_{v^*}[B_L]}$$

Note: effectively parameter $\lambda$ replaced by parameter $\gamma$. 

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Choosing different values of $\gamma$ corresponds to different choices of $\lambda$.

$\lambda$ determines the optimal strategy $v^*_\lambda(t)$.

We generate pairs of points (for each $\lambda$) given by:

$$
\left( E^{t=0}_{v^*_\lambda}[B_{L}^2], E^{t=0}_{v^*_\lambda}[B_{L}] \right)
$$

This can then be converted to points on the efficient frontier.

Varying $\gamma$ traces out efficient frontier.

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A Better Method

Define a pseudo-bank account $\mathcal{B}(t)$

$$\mathcal{B}(t) = B(t) - \frac{\gamma e^{-r(T-t)}}{2}.$$  (8)

so that the control problem becomes ($\gamma$ disappears)

$$\min_{\nu(t)} E^{t=0}[\mathcal{B}_L^2].$$  (9)

Assume the state of the strategy is fully specified by the variables $(S, B, \alpha, t)$. 

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A Better Method II

Let

\[ V(S, B, \alpha, t) = \mathbb{E}_{t=0}^t [B_L^2] \]

At \( t = 0 \) note that (assuming real bank account = 0)

\[ \gamma = -2B_0e^{rT} \]

This means that if we examine \( V(S_0, B_0, \alpha_0, t = 0) \) for various \( B_0 \rightarrow \) we can determine \( v^{\star}(t), \mathbb{E}_{t=0}^t [B_L^2] \) for any \( \lambda \).
Solution of the Optimal Control Problem

Recall $V = V(S, B, \alpha, \tau = T - t) = E_{u^*}^{t=T-\tau}[B_L^2]$. Let

$$\mathcal{L}V \equiv \frac{\sigma^2 S^2}{2} V_{SS} + \eta SV_S + r BV_B .$$

Then, using usual arguments, $V(S, B, \alpha, \tau)$ is determined by

$$V_\tau = \mathcal{L}V + r BV_B + \min_{v \in Z} \left[ -vSf(v)V_B + vV_{\alpha} + g(v)SV_S \right]$$

$$Z = [v_{min}, v_{max}]$$

with the payoff $V(S, B, \alpha, \tau = 0) = B_L^2$.
**Numerical Method**

**Step 1** Solve HJB equation once with initial condition $V(S, B, \alpha, \tau = 0) = B^2_L$.

→ This determines optimal control $v^*(t), E_{v^*}^{t=0}[B^2_L]$.

**Step 2** Solve PDE problem again, using known control from Step 1, with initial condition $U(S, B, \alpha, \tau = 0) = B_L$, this gives $U = E_{v^*}^{t=0}[B_L]$.

**Step 3** Solution of these two PDEs allows us to generate points along the entire efficient frontier.
Nonlinear HJB equation solved using finite difference with semi-Lagrangian timesteping

- Optimal trade rate at each node determined by discretizing $[v_{\min}, v_{\max}]$, and using linear search (expensive but bullet proof)

- Consistent, stable, monotone $\rightarrow$ converges to viscosity solution of HJB equation.
Further Simplification

Since $V = V(S, B, \alpha, \tau)$, we need to solve a $3 - d$ HJB equation.

For many reasonable price impact functions, we have the property ($V = E[B^2_L], U = E[B_L]$)

\[
\begin{align*}
V(\xi S, \xi B, \alpha, \tau) &= \xi^2 V(S, B, \alpha, \tau) = \xi^2 E_{\alpha^*}^{t=0}[B^2_L] \\
U(\xi S, \xi B, \alpha, \tau) &= \xi U(S, B, \alpha, \tau) = \xi E_{\alpha^*}^{t=0}[B_L] \\
\xi &= \text{constant}
\end{align*}
\]

This similarity reduction can be used to reduce the $3 - d$ PDEs to $2 - d$ PDEs.
### Optimal Liquidation Example

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Volatility $\sigma$</td>
<td>0.40</td>
</tr>
<tr>
<td>Trading Horizon (years) $T$</td>
<td>1/12</td>
</tr>
<tr>
<td>Drift Rate $\eta$</td>
<td>0.10</td>
</tr>
<tr>
<td>Risk Free Rate $r$</td>
<td>0.0</td>
</tr>
<tr>
<td>Pretrade Price $S_0$</td>
<td>100</td>
</tr>
<tr>
<td>Initial Shares $\alpha_{sell}$</td>
<td>1.0</td>
</tr>
<tr>
<td>Permanent Impact Factor $\kappa_p$</td>
<td>0.0</td>
</tr>
<tr>
<td>Temporary Impact Factor $\kappa_t$</td>
<td>0.002</td>
</tr>
<tr>
<td>Relative Bid-Ask Spread $\kappa_s$</td>
<td>0.0</td>
</tr>
<tr>
<td>Temporary Impact Exponent $\beta$</td>
<td>1.0</td>
</tr>
<tr>
<td>Minimum Trading Rate $v_{min}$</td>
<td>$-25/T$</td>
</tr>
<tr>
<td>Maximum Trading Rate $v_{max}$</td>
<td>0.0</td>
</tr>
</tbody>
</table>
• $n_S \times n_\alpha = (S \text{ nodes}) \times (\alpha \text{ nodes})$, $\mathcal{B}$ eliminated by similarity reduction
• Average price obtained during one month period, pre-trade price 100
Sanity Check: Two Simple Cases

Trade at constant rate \( v = 1/T = 12 \). Minimizes price impact, (approximately) maximizes expected gain.

\[
E[e^{-\kappa t/T} \frac{1}{T} \int_0^T S(t) \, dt] \simeq 98.04
\]

Standard Deviation \[e^{-\kappa t/T} \frac{1}{T} \int_0^T S(t) \, dt] \simeq 6.55 \]

Trade at maximum possible rate, minimizes standard deviation

- Expected gain = standard deviation = 0.
Example

Two Simple Cases II

- point \((6.55, 98.04)\), maximizes gain
- point \((0, 0)\) minimizes standard deviation
Optimal strategy

Consider the point on the efficient frontier

\[
\text{Expected Gain} = 96.17 \\
\text{Standard Deviation} = 3.76
\]

- Examine the value of \( v^*(S, B = 0, \alpha = 1, t = 0) \)
- Recall that we assume \( S = 100 \) at \( t = 0 \)
- This means that we can interpret this as examining how the optimal strategy would change if the asset price suddenly moved away from \( S = 100 \).
Optimal strategy: \( v^*(S, B = 0, \alpha = 1, t = 0) \)
Sawtooth Pattern?

Why do we get this sawtooth pattern in the optimal trade rate as a function of $S$?
How can the optimal strategy be a non-smooth function of $S$?

Semi-Lagrangian timestepping

Optimal trade at each point $(S, B, \alpha, t)$ determined by finding the minimum value of the objective function along a curve in $(S, B, \alpha, t)$ space.

- We plot the value of the objective function along this curve, for fixed initial point $(B, \alpha)$ at two nearby initial points in the $S$ direction.
- Optimal trade rate determined by global minimum

- Objective function has multiple local minima

- Objective function is very flat between local minima

- Location of global minimum is not a continuous function of $S$
Value Surface

\[ E_{v^*}^{t=0}[B_L^2] \ ( B = -100) \]

- Note \( V_S \approx V_\alpha \approx V \approx 0 \) for large region. Similarity reduction \( \rightarrow V_B \approx 0 \)
Uniqueness

Optimal Strategy: Uniqueness

Recall HJB equation

\[ V_\tau = \mathcal{L}V + \min_{v \in [v_{\text{min}}, v_{\text{max}}]} \left[ -v S f(v) V_B + v V_\alpha + g(v) S V_S \right] . \]

- If \( V_S = V_B = V_\alpha = 0 \)
  \( \iff \) Optimal control can be any value \( v \in [v_{\text{min}}, v_{\text{max}}] \).
Example: Non-uniqueness

Suppose we want a strategy which produces zero standard deviation
- Obvious method: sell immediately at infinite rate
  This gives zero standard deviation, but zero gain.

But this strategy is not unique. Another strategy:
- Do nothing until $T - \varepsilon$.
  Then sell at infinite rate.
- This strategy also produces zero standard deviation (and zero gain).

There are an infinite number of such strategies
Uniqueness

Optimal Strategy: Discrete Trade Rates

Instead of allowing a continuous set of trade rates in $[v_{\text{min}}, v_{\text{max}}]$

Allow only a set of twenty five possible trading rates in $[v_{\text{min}}, v_{\text{max}}]$

Solve HJB problem again.

Does this restriction to a fixed number of possible trading rates result in a large change in the efficient frontier?
Uniqueness

**Optimal Strategy: Discrete Trade Rates**

![Graph showing the comparison between Continuous Trade Rate and Discrete Trade Rate with respect to Expected Gain and Standard Deviation.]

![Graph showing the comparison between Continuous Rate and Discrete Rate with respect to Asset Price and Trade Rate.]

Clearly, many strategies which give almost the same efficient frontier!
Uniqueness

**Optimal Strategy: Discrete Trade Rates**

The efficient frontier is very stable
- Appears to converge rapidly as mesh/timesteps refined.

The optimal trading rate is ill-posed.
- Strategy is non-unique in many cases.

This is actually useful in practice
- We get virtually the same efficient frontier for nearby strategies.
- Precise choice of trading strategy at start of trading not crucial.
Conclusions 1

- One solve of the HJB equation, plus linear PDE, gives us the optimal (pre-commitment) strategy for all points on the efficient frontier.
- Semi-Lagrangian solution of HJB PDE is independent (within reason) of:
  - Form of price impact functions
  - Stochastic process for underlying asset (i.e. easy to add jumps, regime switching).
- We are finding the truly optimal mean-variance strategy
  - For liquidation, if stochastic price moves higher, sell faster; if price drops, sell slower
Conclusions II

• Solution for efficient frontier is well-posed
  ← Optimal strategy appears not well posed
  ← Many strategies which give same frontier
  ← Practical result: efficient frontier not sensitive to precise strategy!

• General selling rule:
  ← Cut your gains, ride your losses.

• Question: is the pre-commitment mean-variance objective function what we want to do?
Disclaimer

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