On the distribution of terminal wealth under dynamic mean-variance optimal investment strategies

Pieter M. van Staden∗  Duy-Minh Dang†  Peter A. Forsyth‡

October 12, 2019

Abstract

We compare the distributions of terminal wealth obtained from implementing the optimal investment strategies associated with the different approaches to dynamic mean-variance (MV) optimization available in the literature. This includes the pre-commitment MV (PCMV) approach, the dynamically optimal MV (DOMV) approach, as well as the time-consistent MV approach with a constant risk aversion parameter (cTCMV) and wealth-dependent risk aversion parameter (dTCMV), respectively. For benchmarking purposes, a constant proportion (CP) investment strategy is also considered. To ensure that terminal wealth distributions are compared on a fair and practical basis, we assume that an investor, otherwise agnostic about the philosophical differences of the underlying approaches to dynamic MV optimization, requires that the same expected value of terminal wealth should be obtained regardless of the approach. We present first-order stochastic dominance results proving that for wealth outcomes below the chosen expected value target, the cTCMV strategy always outperforms the DOMV strategy, and an appropriately chosen CP strategy always outperforms the dTCMV strategy. We also show that the PCMV strategy results in a terminal wealth distribution with fundamentally different characteristics than any of the other strategies. Finally, our analytical results are very effective in explaining the numerical results currently available in the literature regarding the relative performance of the various investment strategies.

Keywords: Asset allocation, constrained optimal control, time-consistent, mean-variance

AMS Subject Classification: 91G, 65N06, 65N12, 35Q93

1 Introduction

Originating with Markowitz (1952), mean-variance (MV) portfolio optimization forms the foundation of modern portfolio theory (Elton et al. (2014)), in part due to its intuitive nature. In dynamic settings (see for example Zhou and Li (2000)), MV optimization aims to obtain an investment strategy that maximizes the expected value of the terminal wealth of the portfolio, for a given level of risk as measured by the associated variance of the terminal wealth.

It is well-known that variance does not satisfy the law of iterated expectations. As a result, the MV objective is not separable in the sense of dynamic programming, resulting in three main approaches to MV optimization that can be identified in the literature.

In the first approach, referred to as pre-commitment MV (PCMV) optimization, the resulting optimal investment strategy is typically time-inconsistent when viewed from the perspective of the original MV objective (Basak and Chabakauri (2010)). However, in practice the PCMV problem is solved using the embedding approach of Li and Ng (2000); Zhou and Li (2000), and the resulting PCMV-optimal investment strategy is time-consistent from the perspective of the induced quadratic objective function used in the corresponding embedding problem (Vigna (2014, 2016)). Therefore, the PCMV-optimal investment strategies considered in this paper are in fact feasible to implement as trading strategies (see Strub et al. (2019)).

The second approach, referred to as time-consistent MV (TCMV) optimization, is based on a game-theoretic approach to the MV problem (Basak and Chabakauri (2010); Bjork and Murgoci (2014)). The TCMV-optimal investment strategies are guaranteed to be time-consistent, since optimization is performed only over a subset of investment strategies which are time-consistent from the perspective of the original MV problem. Equivalently,

∗School of Mathematics and Physics, The University of Queensland, St Lucia, Brisbane 4072, Australia, email: pieter.vanstaden@uq.edu.au
†School of Mathematics and Physics, The University of Queensland, St Lucia, Brisbane 4072, Australia, email: duyminh.dang@uq.edu.au
‡Cheriton School of Computer Science, University of Waterloo, Waterloo ON, Canada, N2L 3G1, paforsyt@waterloo.ca
in the TCMV approach the MV problem is solved subject to a time-consistency constraint on the admissible investment strategies (Cong and Oosterlee (2016b); Wang and Forsyth (2011)). Two main variations of the TCMV approach can be found in the literature, depending on the treatment of the risk aversion parameter which encodes the investor’s risk preferences in an MV setting. Specifically, the risk-aversion parameter is either assumed to be a constant over the entire investment time horizon (see for example Basak and Chabakauri (2010)), or it is assumed to be “wealth-dependent”, in particular, inversely proportional to the investor’s wealth at any given point in time (Björk et al. (2014)). To distinguish between these two cases, we refer to the TCMV approach using a constant risk aversion parameter as the cTCMV approach, and to the case using wealth-dependent risk aversion parameter as the dTCMV approach.

The third approach, namely the dynamically-optimal MV (DOMV) optimization approach of Pedersen and Peskir (2017), entails solving an infinite number of problems with the MV objective dynamically forward in time. In particular, starting from an initial wealth and initial time, each new wealth level attained over time results in a new MV problem that has to be solved, resulting in a new optimal strategy to be implemented only at that time instant and for that particular wealth level. The resulting DOMV-optimal strategy is therefore fundamentally different from the TCMV-optimal strategy, but is indeed feasible to implement as a trading strategy.

We briefly note that each of these approaches to dynamic MV optimization is associated with a different underlying motivational philosophy. In this sense, preference for one strategy over another depends on the MV investor’s investment philosophy and perspective on time-consistency - see Vigna (2016, 2017) for a number of the subtle issues involved. However, for a practical assessment of the relative performance of the different investments strategies, we do not dwell on these philosophical considerations in this paper, and instead only focus on wealth outcomes.

Recently, dynamic MV optimization has received considerable attention in institutional settings, including in pension fund and insurance applications - see for example Chen et al. (2013); Forsyth and Vetzal (2019b); Forsyth et al. (2019); Hojgaard and Vigna (2007); Liang et al. (2014); Lin and Qian (2016); Menoncin and Vigna (2013); Neki (2014); Sun et al. (2016); Vigna (2014); Wang and Chen (2018, 2019); Wei and Wang (2017); Wu and Zeng (2015); Zhao et al. (2016); Zhou et al. (2016), among many others. In particular, we also highlight the popularity of the dTCMV approach in institutional settings, for example in the case of the investment-reinsurance problems faced by insurance providers (Bi and Cai (2019); Li and Li (2013)), investment strategies for pension funds (Liang et al. (2014); Sun et al. (2016); Wang and Chen (2018, 2019)), corporate international investment (Long and Zeng (2016)), and asset-liability management (Peng et al. (2018); Zhang et al. (2017)).

In all of these situations, it is reasonable to argue that the distribution of terminal wealth is of key importance to stakeholders, despite the natural focus in the literature on the mean and variance of terminal wealth. The reason for this is that in any practical setting, the MV investor (or indeed, any investor) is likely to also take into account a number of other measures of risk and investment performance, which might be critical even if only as a result of regulatory considerations (see for example Antolin et al. (2009)).

In the light of these considerations, it is therefore not surprising that there has been significant interest recently in different aspects of the terminal wealth distribution obtained under various investment strategies, including optimal strategies associated with approaches to dynamic MV portfolio optimization - see for example Forsyth and Vetzal (2017a,b, 2019a,b); Forsyth et al. (2019). These papers present a very realistic formulation of the underlying problems, including for example the treatment of withdrawals and contributions, investment constraints, and so on. By necessity, these papers therefore focus on the results obtained from the numerical solutions of the problems under consideration.

In contrast, there seems to be very little available research on the theoretical comparison of terminal wealth distributions in cases where the optimal investment strategies can be expressed analytically. We emphasize that while analytical MV-optimal strategies sometimes call for unacceptably high leverage ratios or unrealistic treatment of insolvency, investment constraints can be incorporated easily in the numerical solution of the MV optimization problem (see for example Cong and Oosterlee (2016a); Dang and Forsyth (2014); Van Staden et al. (2018); Wang and Forsyth (2010, 2011)). However, analytical investment strategies remain very useful, in that an analytical comparison of terminal wealth distributions (i) can provide an additional perspective on some of the implications of the various approaches to dynamic MV optimization that is currently not available in the literature, and (ii) can assist in explaining some of the numerical results recently reported in the literature, such as the results of for example Forsyth and Vetzal (2017b); Forsyth et al. (2019).

The main objective of this paper is therefore a systematic comparison the analytical terminal wealth distributions resulting from the optimal investment strategies associated with the different approaches to dynamic MV optimization in the literature. In order to compare distributions on a fair basis, we assume that the investor remains agnostic as to the philosophical differences underlying the various approaches to MV optimization, and simply wishes to achieve a chosen expected value of terminal wealth regardless of the approach. Our main
contributions are as follows.

- We derive analytical results regarding the terminal wealth distributions that, despite our assumption of no market frictions (in particular, continuous trading with no leverage constraints, no transaction costs and without insolvency/bankruptcy prohibitions), are very effective in explaining the numerical results incorporating realistic investment constraints currently available in the literature.

- For comparison and benchmarking purposes, our analysis includes a simple constant proportion (CP) strategy, whereby the investor invests a fixed proportion of wealth in the risky asset throughout the investment time horizon. The CP strategy is typically not MV-optimal in the sense of any of the other strategies considered, but our analysis proves that it easily outperforms the dTCMV-optimal investment strategy in the general sense of a partial first-order stochastic dominance result we present.

- Our results also show that the dTCMV-optimal strategy performs exceptionally poorly compared to the other MV-optimal investment strategies, with for example the dTCMV-optimal strategy achieving both a higher variance and lower median terminal wealth than the cTCMV strategy. This calls into question the current popularity enjoyed by the dTCMV-optimal strategy in the literature.

- We establish that the cTCMV strategy outperforms the DOMV strategy in a first-order stochastic dominance sense when we consider terminal wealth outcomes below the expected value target. The cTCMV strategy also achieves a lower variance of terminal wealth compared to the DOMV strategy.

- Furthermore, we derive analytical higher-order moment results for the PCMV-optimal wealth distribution which prove that the PCMV strategy results in a terminal wealth distribution with fundamentally different characteristics than any of the other strategies. In particular, the PCMV-optimal strategy achieves the lowest variance of terminal wealth of all the strategies considered, but the negative skewness and large kurtosis of the associated terminal wealth distribution means that the otherwise excellent performance of the PCMV strategy comes at the cost of increased left tail risk for the investor.

- Numerical results, making use of model parameters calibrated to inflation-adjusted, long-term US market data (89 years), are presented to validate and illustrate the implications of our analytical results.

The remainder of the paper is organized as follows. Section 2 describes the underlying dynamics, notational conventions, as well as rigorous definitions of the different approaches to dynamic MV optimization. Subject to certain assumptions, Section 3 presents a number of analytical results, including some new results, regarding the terminal wealth distributions associated with different approaches. In Section 4, we present a rigorous analytical comparison study of terminal wealth distributions associated with different approaches, but all achieving the investor’s chosen expected value target. Numerical results are presented in Section 5, while Section 6 concludes the paper and outlines possible future work.

2 Formulation

For simplicity, our analysis focuses on portfolios consisting of a well-diversified stock index (the risky asset) and a risk-free asset. Since the available analytical solutions for multi-asset PCMV and TCMV approaches (see, for example, Li and Ng (2000) and Zeng and Li (2011)) show that the overall composition of the risky asset basket remains relatively stable over time, it is reasonable to focus on the overall risky asset basket vs. risk-free asset composition of the portfolio as the primary investment question. We leave the extension of our results to multi-asset dynamic MV optimization problems for our future work.

Let $t_0 = 0$ denote the start of the investment time period, and let $T > 0$ denote the fixed investment time horizon or maturity. The controlled wealth, with the control representing some investment strategy, is denoted by $W(t)$, $t \in [0,T]$. Specifically, let $u : (W(t), t) \rightarrow u(t) = u(W(t), t)$, $t \in [0,T]$ be the adapted feedback control representing the amount invested in the risky asset at time $t$ given wealth $W(t)$, and let $A = \{ u(t) = u(w, t) | u : \mathbb{R} \times [0,T] \rightarrow \mathbb{U} \}$ denote the set of admissible controls, where $\mathbb{U} \subseteq \mathbb{R}$ denotes the admissible control space.

We assume that the risky asset follows a geometric Brownian motion (GBM), leaving the treatment of jumps in the risky asset process and alternative model specifications for our future work. While this choice of model may appear to be overly simplistic, we observe the following: (i) The extensive backtesting results presented in Forsyth and Vetzal (2017b) show that the GBM assumption actually performs very well over long investment time horizons, suggesting that more complicated models (including for example incorporating stochastic volatility Ma and Forsyth (2016)) may not offer substantial advantages in this setting. (ii) As...
discussed in more detail below, the analytical results presented in this paper (based on GBM dynamics) are in qualitative agreement with the numerical results presented in Forsyth and Vetzal (2019a); Forsyth et al. (2019) where jump-diffusion models are assumed for the risky asset, indicating that a GBM model appears to be sufficient in capturing the salient characteristics of the different investment strategies.

Therefore, based on the assumption of GBM dynamics for the risky asset, the dynamics of the wealth \( W(t) \) of a self-financing portfolio, with no contributions or withdrawals, is given by (see for example Bjork (2009); Bjork et al. (2014))

\[
dW(t) = [rW(t) + (\mu - r)u(t)]dt + \sigma u(t)dZ, \quad (2.1)
\]

\[
W(t_0) = w_0 > 0. \quad (2.2)
\]

Here, \( w_0 > 0 \) denotes the initial wealth, \( r > 0 \) denotes the continuously compounded risk-free interest rate, \( \mu > r \) and \( \sigma > 0 \) denote the drift and volatility of the dynamics of the risky asset, respectively, while \( Z \) denotes a standard Brownian motion. For subsequent reference, we also define the following combination of parameters,

\[
A = \frac{(\mu - r)^2}{\sigma^2}. \quad (2.3)
\]

Before presenting rigorous definitions of the various approaches to dynamic MV optimization, we introduce a number of notational conventions. Let \( Q^{w,t}[W(T)] \) denote some quantity \( Q \) associated with the terminal wealth \( W(T) \), given wealth \( W(t) = w \) at time \( t \in [0,T] \) and the application of control \( u \in A \) over the time interval \( [t,T] \). Specific examples of the quantity \( Q \) encountered in this paper include the expected value (in which case we set \( Q = E \)), variance \( (Q = Var) \), standard deviation \( (Q = Stdev) \), conditional probability measure \( (Q = P) \), as well as the Value-at-Risk and Conditional Value-at-Risk\(^1\) at level \( \alpha \in (0,1) \), respectively denoted by \( Q = \alpha VaR \) and \( Q = \alpha CVaR \). The optimal control and optimal terminal wealth will be denoted by \( u^*_j \) and \( W_j(T) \), respectively, where the subscript \( j \in \{ p, d, c, cd, cp \} \) is used to distinguish the underlying approach with respect to which \( u^*_j \) and \( W_j(T) \) are optimal. For ease of subsequent reference, the particular association of the subscript \( j \) with the corresponding investment approach is outlined in Table 2.1.

Table 2.1: Summary of notational conventions. The subscript \( j \in \{ p, d, c, cd, cp \} \) is used to identify the approach in terms of which the optimal investment strategy \( u^*_j \) and associated optimal terminal wealth \( W_j(T) \) is obtained. For the sake of simplicity, the constant proportion (CP) strategy is identified using similar notation, but we emphasize that the CP strategy does not represent an MV-optimal strategy in some sense as in the case of the other strategies.

<table>
<thead>
<tr>
<th>Subscript ( j )</th>
<th>Approach</th>
<th>Abbreviation</th>
<th>Optimal control ( u^*_j )</th>
<th>Optimal terminal wealth using control ( u^*_j )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( j = p )</td>
<td>Pre-commitment MV</td>
<td>PCMV</td>
<td>( u^*_p )</td>
<td>( W_p(T) )</td>
</tr>
<tr>
<td>( j = d )</td>
<td>Dynamically-optimal MV</td>
<td>DOMV</td>
<td>( u^*_d )</td>
<td>( W_d(T) )</td>
</tr>
<tr>
<td>( j = c )</td>
<td>Time-consistent MV with constant risk aversion parameter</td>
<td>tTCMV</td>
<td>( u^*_c )</td>
<td>( W_c(T) )</td>
</tr>
<tr>
<td>( j = cd )</td>
<td>Time-consistent MV with wealth-dependent risk aversion parameter</td>
<td>dTCMV</td>
<td>( u^*_{cd} )</td>
<td>( W_{cd}(T) )</td>
</tr>
<tr>
<td>( j = cp )</td>
<td>Constant proportion strategy</td>
<td>CP</td>
<td>( u^*_{cp} )</td>
<td>( W_{cp}(T) )</td>
</tr>
</tbody>
</table>

We now present the definitions of the main approaches to MV portfolio optimization considered in this paper. Using the standard scalarization method for multi-criteria optimization problems (Yu (1971)), a general definition of the dynamic MV optimization problem is given by (see for example Zhou and Li (2000))

\[
\sup_{u \in A} \left( E^w_u[W(T)] - \rho \cdot Var^w_u[W(T)] \right), \quad \rho > 0, \quad (2.4)
\]

where the investor’s level of risk aversion is reflected by the risk aversion (or scalarization) parameter \( \rho > 0 \).

As noted in the Introduction, variance does not satisfy the smoothing property of conditional expectation, therefore dynamic programming cannot be applied directly to (2.4). The first approach to dynamic MV optimization, the pre-commitment MV (PCMV) approach, employs the technique of Li and Ng (2000); Zhou and Li (2000) to embed problem (2.4) in a new optimization problem, often referred to as the embedding problem,

\(^1\)The terms and risk measures are defined rigorously below - see Section 4.
which can be solved using dynamic programming techniques. We follow the convention in literature (see for example Cong and Oosterlee (2017); Dang et al. (2017)) of defining the PCMV optimization problem as the associated MV embedding problem, namely

\[
(\text{PCMV} (\gamma)) : \quad \inf_{u \in \mathcal{A}} \left( E^{w_0, t_0} \left[ \left( W(T) - \frac{\gamma}{2} \right)^2 \right] \right), \quad \gamma \in \mathbb{R}, \tag{2.5}
\]

where \(\gamma\) denotes the embedding parameter. As per Table 2.1, we use the notation \(u^*_p\) and \(W_p(T)\) to denote the optimal control and optimal terminal wealth for problem (2.5), respectively.

**Remark 2.1.** (Time-consistency of PCMV-optimal control \(u^*_p\)) As discussed in detail in Forsyth et al. (2019); Li and Forsyth (2019), there appears to be some confusion in the literature as to whether the PCMV-optimal control \(u^*_p\) is time-consistent or not. This question is of great practical significance, since \(u^*_p\) is typically time-inconsistent (see Basak and Chabakauri (2010); Bjork and Murgoci (2014)) from the perspective of the original MV objective (2.4), which raises questions regarding its feasibility as an implementable trading strategy. This observation is arguably the reason why a number different approaches to dynamic MV optimization has been developed, each with a different underlying philosophy as to how the problem of time-inconsistency with respect to the original objective (2.4) is to be addressed - see Vigna (2016, 2017) for a discussion of the various issues involved.

However, we emphasize that for fixed value of \(\gamma \in \mathbb{R}\), the optimal control \(u^*_p\) of (2.5) is a time-consistent control from the perspective of the quadratic objective function used in (2.5), and is therefore feasible to implement as a trading strategy (see Strub et al. (2019)). It is therefore worth emphasizing that the issue of the time-consistency of \(u^*_p\) is a matter of perspective, and in this paper we always view \(u^*_p\) as the time-consistent strategy minimizing the induced objective function in (2.5). This aligns with our stated objective of comparing terminal wealth distributions from the perspective of an investor who remains agnostic as to the underlying philosophical differences of the various approaches to dynamic MV optimization.

Next, we consider the dynamically-optimal MV (DOMV) approach proposed by Pedersen and Peskir (2017).

Informally, this entails solving an infinite number of problems of the form (2.4) dynamically forward in time. Starting from the initial state and time \((w_0, t_0)\), each new state \((W(t), t)\), \(t \in [0, T]\) attained by the controlled wealth process results in a new problem (2.4) to be solved to obtain the optimal control \(u^*_d(W(t), t) = u^*_d(t)\) applicable at that time instant. In this way, the dynamically optimal control \(u^*_d(t)\) is obtained for all \(t \in [0, T]\), resulting in a DOMV-optimal terminal wealth \(W_d(T)\). More formally, following Pedersen and Peskir (2017), we define the DOMV problem and associated optimal control \(u^*_d\) as follows.

\[
(DOMV (\rho)) : \quad u^*_d \in \mathcal{A} \text{ is dynamically optimal for (2.4) with a given fixed } \rho > 0, \text{ if}
\]

\[
\forall (w, t) \in \mathbb{R} \times [0, T], \text{ and } \forall v \in \mathcal{A} \text{ with } v(w, t) \neq u^*_d(w, t),
\]

\[
\exists u \in \mathcal{A} \text{ satisfying } u(w, t) = u^*_d(w, t) \text{ such that}
\]

\[
E^w_{u}[W(T)] - \rho \cdot \text{Var}^w_{u}[W(T)] \geq E^w_{v}[W(T)] - \rho \cdot \text{Var}^w_{v}[W(T)]. \tag{2.6}
\]

The time-consistent MV (TCMV) approach involves maximizing the objective of (2.4) subject to a time-consistency constraint (see for example Basak and Chabakauri (2010); Bjork et al. (2017); Wang and Forsyth (2011)), so that the resulting optimal control is time-consistent from the perspective of the original MV objective (2.4). As noted in the Introduction, we distinguish two variants of the TCVM approach depending on the treatment of the risk-aversion parameter \(\rho\) in (2.4).

First, using a constant risk-aversion parameter \(\rho > 0\) in (2.4), we define the cTCMV problem as

\[
(c\text{TCMV} (\rho)) : \quad \sup_{u \in \mathcal{A}} \left( E^{w_0, t_0} \left[ W(T) \right] - \rho \cdot \text{Var}^{w_0, t_0}_{u}[W(T)] \right), \quad \rho > 0, \tag{2.7}
\]

s.t. \(u^*_c(t_0; y, v) = u^*_c(t'; y, v), \quad \text{for } v \geq t', t' \in [t_0, T]\), \tag{2.8}

where \(u^*_c(t_0; y, v)\) denotes the optimal control calculated at time \(t_0\) and to be applied at some future time \(v \geq t' \geq t_0\) given future state \(W(v) = y\), while \(u^*_c(t'; y, v)\) denotes the optimal control calculated at some future time \(t' \in [t_0, T]\), also to be applied at the same later time \(v \geq t'\) given the same future state \(W(v) = y\). To lighten notation, as per Table 2.1 we will use the notation \(u^*_c(t)\) to denote the optimal control of the cTCMV problem (2.7)-(2.8).

A popular alternative formulation of the TCVM problem is to specify a risk aversion parameter that is inversely proportional to wealth - see Bjork et al. (2014) for the motivation and a detailed analysis. Specifically, in this formulation, the constant \(\rho\) in (2.7) is replaced by \(\rho(w) = \rho/2w\) for \(\rho > 0\), where \(w\) denotes the current wealth. This results dTCMV problem defined by

\[
\text{(dTCMV)} : \quad \sup_{u \in \mathcal{A}} \left( E^{w_0, t_0} \left[ W(T) \right] - \rho(w) \cdot \text{Var}^{w_0, t_0}_{u}[W(T)] \right), \quad \rho > 0, \tag{2.9}
\]

s.t. \(u^*_c(t_0; y, v) = u^*_c(t'; y, v), \quad \text{for } v \geq t', t' \in [t_0, T]\), \tag{2.10}

where \(\rho(w) = \rho/2w\).
(dTCMV) \( \rho > 0 \),
\[
(dTCMV (\rho)) : \sup_{u \in A} \left( E_{u}^{w_{0}, t_{0}} [W (T)] - \frac{\rho}{2w_{0}} \cdot Var_{u}^{w_{0}, t_{0}} [W (T)] \right),
\]
s.t. \( u_{cd}^{*} (t; y, v) = u_{cd}^{*} (t' ; y, v), \quad \text{for } v \geq t', t' \in [t_{0}, T] \),
\]
where the time-consistency constraint (2.10) has the same interpretation as in (2.8). As per Table 2.1, we denote the dTCMV-optimal control by \( u_{cd}^{*} (t) \) and the associated optimal terminal wealth by \( W_{cd} (T) \).

Finally, for benchmarking and comparison purposes, we also consider the constant proportion (CP) problem, defined as follows.

\[
(CP (\theta_{cp})) : \text{Choose a constant proportion } \theta_{cp} > 0 \text{ of wealth}
\]
to invest in the risky asset, \( \forall t \in [0, T] \), so that
\[
u_{cp}^{*} (t) = \theta_{cp} W (t), \quad \forall t \in [0, T].
\]

As noted in the Introduction, the CP strategy is not designed to be MV-optimal in any sense. However, as per Table 2.1, for convenience we use the notation \( u_{cp}^{*} (t) \) and \( W_{cp} (T) \) to denote the control and terminal wealth associated with the CP problem for a some choice of the constant proportion \( \theta_{cp} \). A concrete example of choosing a value of \( \theta_{cp} \) to achieve a specific goal is given in Section 4.

3 Selected analytical results

In this section, we present analytical results relevant to the terminal wealth distributions obtained under the optimal investment strategies of the problems presented in Section 2. All results in this section are based on the assumption of no market frictions or investment constraints, formally defined as Assumption 3.1.

Assumption 3.1. (No market frictions) Trading continues in the event of insolvency, no transaction costs are applicable, and no leverage constraints are in effect.

Remark 3.1. (Relaxing Assumption 3.1) Since the simultaneous application of multiple realistic investment constraints can be incorporated with relative ease in the numerical solution of dynamic MV optimization problems (see Cong and Oosterlee (2016a); Dang and Forsyth (2014); Van Staden et al. (2018); Wang and Forsyth (2010, 2011), among others), relaxing Assumption 3.1 is not challenging in a practical setting. However, as noted in the Introduction, this paper focuses on a theoretical comparison of optimal terminal wealth distributions in the particular select cases where the optimal investment strategies to dynamic MV optimization problems can be expressed analytically. The two main consequences of Assumption 3.1 are therefore that it (i) ensures that an additional perspective on the implications of the various approaches to dynamic MV optimization can be presented in this paper that is currently missing from the literature, and (ii) assists in explaining some of the numerical results reported in literature (for example Forsyth and Vetzal (2017a,b, 2019a,b); Forsyth et al. (2019)).

Under Assumption 3.1, the optimal controls associated with the dynamic MV optimization problems presented in Section 2 can be expressed analytically, as the following lemma shows.

Lemma 3.2. (Optimal controls) Under Assumption 3.1, the optimal controls of problems PCMV (2.5), DOMV (2.6), cTCMV (2.7)-(2.8) and dTCMV (2.9)-(2.10) are respectively given by

\[
u_{p}^{*} (t) = \frac{A}{(\mu - r)} e^{-r(T-t)} \left[ \frac{\gamma}{2} - W (t) e^{r(T-t)} \right],
\]
\[
u_{d}^{*} (t) = \frac{1}{2 \rho} \cdot \frac{A}{(\mu - r)} e^{(\mu - r)(T-t)},
\]
\[
u_{c}^{*} (t) = \frac{1}{2 \rho} \cdot \frac{A}{(\mu - r)} e^{-r(T-t)},
\]
\[
u_{cd}^{*} (t) = \theta (t) \cdot W (t),
\]

where \( \theta (t) \) in (3.4) is given by the unique solution to the following integral equation:

\[
\theta (t) = -\frac{A}{\rho (\mu - r)} \left\{ e^{-\int_{t}^{T} r(\tau + (\mu - r)\theta (\tau) - \sigma^{2} \theta (\tau)) d\tau} + \rho e^{-\int_{t}^{T} \sigma^{2} \theta (\tau) d\tau} - \rho \right\}.
\]
Proof. See Basak and Chabakauri (2010); Pedersen and Peskir (2017); Zhou and Li (2000) and Bjork et al. (2014).

Including the CP strategy (2.11) in this discussion would therefore result in five different investment strategies under consideration. However, Lemma 3.2 shows that there are only three fundamentally different forms of the resulting controls: (i) The DOMV- and cTCMV-optimal controls ((3.2) and (3.3), respectively) are simply deterministic functions of time, and do not depend on the investor’s wealth. (ii) The CP strategy (2.11) is a special case of the dTCMV-optimal strategy (3.4), since in the case of the dTCMV strategy the proportion \( \theta (t) \) is a deterministic function of time satisfying (3.5). The dTCMV-optimal control (3.4) is therefore an example of a deterministic “glide path” investment strategy (see for example Forsyth and Vetzal (2019b); Graf (2017)), an observation of significant subsequent importance (see Section 4 below). (iii) The PCMV-optimal control (3.1) can be viewed as a linear combination of the TCMV-optimal control (3.3) and the constant proportion strategy (2.11).

Starting from a given initial wealth \( w_0 > 0 \) at time \( t_0 = 0 \), we now assume that the optimal investment strategies from Lemma 3.2, as well as the CP strategy (2.11), are implemented over the investment time horizon \([t_0, T]\). As a result, we obtain the optimal terminal wealth \( W_j (T) \) corresponding to each investment strategy \( j \in \{ p, d, c, cd \} \), as well as the terminal wealth under the CP strategy \( W_{cp} (T) \).

**Theorem 3.3. (Optimal wealth) Under Assumption 3.1, the optimal terminal wealth \( W_j (T) \) corresponding to each investment strategy \( j \in \{ p, d, c, cd \} \), given controlled wealth dynamics (2.1) and optimal controls as in Lemma 3.2, are given by**

\[
W_p (T) = w_0 e^{rT} \cdot \exp \left\{ -\frac{3}{2} \delta T - \sqrt{\delta} \cdot Z (T) \right\} + \frac{\gamma}{2} \cdot A \int_0^T \exp \left\{ -\frac{3}{2} A (T - t) - \sqrt{\delta} \cdot [Z (T) - Z (t)] \right\} dt
\]

\[
W_d (T) = w_0 e^{rT} - \frac{1}{2\rho} (1 - e^{-\delta T}) + \frac{1}{2\rho} \sqrt{\delta} \int_0^T e^{\delta (T-t)} dZ,
\]

\[
W_c (T) = w_0 e^{rT} + \frac{1}{2\rho} \delta T + \frac{1}{2\rho} \sqrt{\delta} \cdot Z (T),
\]

\[
W_{cd} (T) = w_0 e^{rT} \cdot \exp \left\{ \int_0^T \left[ (\mu - r) \theta (t) - \frac{1}{2} \sigma^2 \theta^2 (t) \right] dt + \int_0^T \sigma \theta (t) dZ \right\}.
\]

The terminal wealth \( W_{cp} (T) \) under a CP strategy \( u_{cp} (t) = \theta_{cp} W (t) \) is given by

\[
W_{cp} (T) = w_0 e^{rT} \cdot \exp \left\{ \left[ (\mu - r) \theta_{cp} - \frac{1}{2} \sigma^2 \theta_{cp}^2 \right] T + \sigma \theta_{cp} Z (T) \right\}.
\]

**Proof.** To prove (3.6), we substitute the PCMV-optimal control \( u_p \) given by (3.1) into the controlled wealth dynamics (2.1) to obtain the PCMV-optimal wealth dynamics as

\[
dW_p (t) = \left[ (r - A) W_p (t) + \frac{\gamma}{2} \beta e^{-\beta (T-t)} \right] dt + \sqrt{\beta} \left[ \frac{\gamma}{2} \beta e^{-\beta (T-t)} - W_p (t) \right] dZ.
\]

Define auxiliary processes \( X_{p,1} \) and \( X_{p,2} \) with the following dynamics on \([t_0, T]\),

\[
dX_{p,1} (t) = (r - A) \cdot X_{p,1} (t) dt - \sqrt{\beta} \cdot X_{p,1} (t) dZ, \quad t \in (t_0, T],
\]

\[
X_{p,1} (t_0) = 1,
\]

\[
dX_{p,2} (t) = \frac{1}{X_{p,1} (t)} \left[ \gamma A e^{-\gamma (T-t)} dt + \frac{\gamma}{2} \sqrt{\beta} e^{-\gamma (T-t)} dZ \right], \quad t \in (t_0, T],
\]

\[
X_{p,2} (t_0) = w_0.
\]

Then it follows from Ito’s lemma that \( W_p (T) \) can be expressed as the following product,

\[
W_p (T) = X_{p,1} (T) X_{p,2} (T).
\]

Solving the SDEs (3.12)-(3.13) analytically to obtain \( X_{p,1} (T) \) and \( X_{p,2} (T) \), the product (3.14) simplifies to the
result reported in (3.6). The proof of (3.7)-(3.10) is straightforward, and therefore omitted.

Based on the results of Theorem 3.3, the distribution of terminal wealth can be identified easily in all cases except for the PCMV-optimal terminal wealth \( W_p(T) \), as the following lemma confirms.

**Lemma 3.4.** (Distribution of terminal wealth under the DOMV, cTCMV, dTCMV, CP strategies) Under Assumption 3.1, the terminal wealth under the optimal controls of problems DOMV and cTCMV are normally distributed. Specifically, \( W_d(T) \sim N(\bar{\mu}_d, \bar{\sigma}_d^2) \), where

\[
\bar{\mu}_d := E^{u_0^d}_{u_2^d} [W_d(T)] = w_0e^{\bar{\theta}T} + \frac{1}{2\rho} (e^{\bar{\theta}T} - 1),
\]

(3.15)

\[
\bar{\sigma}_d^2 := Var^{u_0^d}_{u_2^d} [W_d(T)] = \frac{1}{2} \left( \frac{1}{2\rho} \right)^2 (e^{2\bar{\theta}T} - 1),
\]

(3.16)

while \( W_c(T) \sim N(\bar{\mu}_c, \bar{\sigma}_c^2) \) with

\[
\bar{\mu}_c := E^{u_0^c}_{u_2^c} [W_c(T)] = w_0e^{\bar{\theta}T} + \frac{1}{2\rho} \bar{\theta}T,
\]

(3.17)

\[
\bar{\sigma}_c^2 := Var^{u_0^c}_{u_2^c} [W_c(T)] = \left( \frac{1}{2\rho} \right)^2 \bar{\theta}T.
\]

(3.18)

The terminal wealth under the dTCMV-optimal and CP investment strategies are lognormally distributed. In particular, \( W_{cd}(T) \sim Logn(\bar{\mu}_{cd}, \bar{\sigma}_{cd}^2) \), where

\[
\bar{\mu}_{cd} := E^{u_0^d}_{u_2^d} [\log W_{cd}(T)] = \log w_0 + rT + \int_0^T \left( (\mu - r) \theta(t) - \frac{1}{2} \bar{\sigma}_{cd}^2 \right) dt,
\]

(3.19)

\[
\bar{\sigma}_{cd}^2 := Var^{u_0^d}_{u_2^d} [\log W_{cd}(T)] = \int_0^T \bar{\sigma}_{cd}^2 \theta^2(t) dt,
\]

(3.20)

while \( W_{cp}(T) \sim Logn(\bar{\mu}_{cp}, \bar{\sigma}_{cp}^2) \) with

\[
\bar{\mu}_{cp} := E^{u_0^d}_{u_2^d} [\log W_{cp}(T)] = \log w_0 + rT + \left[ (\mu - r) \theta_{cp} - \frac{1}{2} \bar{\sigma}_{cp}^2 \right] T,
\]

(3.21)

\[
\bar{\sigma}_{cp}^2 := Var^{u_0^d}_{u_2^d} [\log W_{cp}(T)] = \bar{\sigma}_{cp}^2 \theta(T).
\]

(3.22)

Proof. The results follow directly from the results of Theorem 3.3.

It is clear from the results of Theorem 3.3 that the distribution of the PCMV-optimal terminal wealth \( W_p(T) \) is significantly more complex than any of the results presented in Lemma 3.4, as it appears not to conform to any of the commonly encountered probability distributions. However, the following theorem gives us access to all of the moments of the distribution of \( W_p(T) \).

**Theorem 3.5.** (Distribution of PCMV-optimal terminal wealth: Non-central moments) Under Assumption 3.1, the non-central moments of the PCMV-optimal terminal wealth \( W_p(T) \) can be expressed as

\[
E^{u_0_n}_{u_2_n} [W_p^n(T)] = m_p^{(n)}(T), \quad n \in \mathbb{N},
\]

(3.23)

where for each \( n \in \mathbb{N} \), the function \( t \to m_p^{(n)}(t) \), \( t \in [0,T] \) can be obtained using the following recursive scheme,

\[
m_p^{(n)}(t) = w_0^n \cdot e^{\frac{n(r-A)}{2} + \frac{1}{2} n(n-1)A} + n(n-1)A \int_0^t e^{-\frac{n(r-A)}{2} + \frac{1}{2} n(n-1)A} d\tau \]

\[
+ \frac{1}{2} n(n-1)A \int_0^t e^{-\frac{n(r-A)}{2} + \frac{1}{2} n(n-1)A} \theta(T-t) m^{(n-1)}_p(\tau) d\tau,
\]

(3.24)

with initial values

\[
m_p^{(1)}(t) := m_p^0(t) \equiv 1, \quad \forall t \in [0,T].
\]

(3.25)

Proof. Given the PCMV-optimal wealth dynamics in (3.11), Ito’s lemma can be used to obtain the dynamics of \( W_p^n(t) \) for \( n \in \mathbb{N} \). Taking expectations and using the definitions \( m_p^{(n)}(\tau) = E^{u_0}_n [W^n(\tau)] \), \( \tau \in [t_0,T] \), as
well as (3.25), we obtain the following initial value problem for \( m_p^{(n)} (\tau) \) for any \( n \in \mathbb{N} \),

\[
\frac{dm_p^{(n)} (\tau)}{d\tau} = \left[ n (n-A) + \frac{1}{2} n (n-1) A \right] m_p^{(n)} (\tau) + \left[ n - n (n-1) A \left( \frac{\gamma}{2} \right) \right] e^{-\tau (T-\tau)} m_p^{(n-1)} (\tau) \\
+ \frac{1}{2} n (n-1) A \left( \frac{\gamma}{2} \right)^2 e^{-2\tau (T-\tau)} m_p^{(n-2)} (\tau), \quad \tau \in (\tau_0, T],
\]

\[
m_p^{(n)} (0) = w_0^0.
\]

Solving the ordinary differential equation (3.26)-(3.27) and simplifying gives the results (3.23)-(3.25).

The first four non-central moments of the distribution of the PCMV-optimal terminal wealth plays an important role in Section 4, and are given by the following lemma.

**Lemma 3.6. (Distribution of PCMV-optimal terminal wealth: First four non-central moments)** Under Assumption 3.1, the first four non-central moments of the distribution of \( W_p (T) \) are given by \( E_{\omega^p, t_0}^{w_\omega^p, t_0} [W_p^{(n)} (T)] = m_p^{(n)} (T) \), \( n \in \{1, 2, 3, 4\} \), where

\[
m_p^{(1)} (T) = w_0 e^{AT} + e^{-AT} (e^{AT} - 1) \left[ \frac{\gamma}{2} - w_0 e^{AT} \right],
\]

\[
m_p^{(2)} (T) = \left[ m_p^{(1)} (T) \right]^2 + e^{-2AT} (e^{AT} - 1) \left[ \frac{\gamma}{2} - w_0 e^{AT} \right]^2,
\]

\[
m_p^{(3)} (T) = 3 \left[ m_p^{(1)} (T) \right]^3 - 2 \left[ m_p^{(1)} (T) \right] \left[ m_p^{(2)} (T) \right] \\
- e^{-3AT} \left[ (e^{AT} - 1)^3 + 3 (e^{AT} - 1)^2 \left[ \frac{\gamma}{2} - w_0 e^{AT} \right] \right],
\]

\[
m_p^{(4)} (T) = 4 \left[ m_p^{(1)} (T) \right]^4 - 6 \left[ m_p^{(1)} (T) \right]^2 \left[ m_p^{(2)} (T) \right] + 3 \left[ m_p^{(1)} (T) \right]^4 \\
+ (e^{2AT} - 4e^{-AT} + 6e^{-2AT} - 3e^{-4AT}) \left[ \frac{\gamma}{2} - w_0 e^{AT} \right]^4.
\]

**Proof.** The results follow from executing the recursive scheme of Theorem 3.5 for \( n \in \{1, 2, 3, 4\} \), and then simplifying and factorizing the results.

Up to this point, we made no reference to any particular choices made by the investor regarding the risk aversion parameters \( \rho > 0 \), embedding parameter \( \gamma \in \mathbb{R} \), or constant proportion \( \theta_{cp} > 0 \). In the next section (Section 4), we introduce specific choices for these parameters that, when substituted into the results presented in this section, allows the investor to consider the resulting terminal wealth distributions on a comparable basis.

### 4 Comparison of terminal wealth distributions

The analytical results presented in Section 3 are used in this section to compare the terminal wealth distributions resulting from implementing the various investment strategies under consideration.

Throughout this discussion, we assume that the investor remains agnostic as to the philosophical perspectives underlying the different approaches to dynamic MV optimization. Specifically, we assume that the investor considers the resulting optimal controls in Lemma 3.2 as well as the CP strategy (2.11) simply as different candidate investment strategies, each resulting in a terminal wealth distribution that can be assessed according to various pre-specified risk and return criteria.

In order to compare the resulting terminal wealth distributions on a fair basis, we introduce the following practical assumption.

**Assumption 4.1. (Expected value target for terminal wealth)** We assume that, regardless of investment strategy \( j \in \{ p, d, c, cd, cp \} \), the investor sets a particular target value \( \mathcal{E} > w_0 e^{AT} \) for the expected value of terminal wealth. In other words, the investor requires

\[
E_{\omega_j, t_0}^{w_\omega_j, t_0} [W_j^* (T)] \equiv \mathcal{E}, \quad \text{with} \ \mathcal{E} > w_0 e^{AT}, \quad \text{for all} \ j \in \{ p, d, c, cd, cp \},
\]

where \( w_\omega_j^* \) denotes the optimal control for investment strategy \( j \) achieving the optimal terminal wealth \( W_j^* (T) \) with expected value \( \mathcal{E} \). We will refer to \( W_j^* (T) \) as the target terminal wealth, and its distribution as the target terminal wealth distribution.
Using the results of Section 3, the targeted expected value \((4.1)\) is achieved as follows. For investment strategies \(j \in \{p, d, c, cd\}\), the strategy \(\gamma^c_j\) is found by choosing the appropriate value of \(\gamma\) or \(p\) in Lemma 3.2, while \(\gamma^d_j\) is found by choosing the appropriate proportion \(\theta^p\) in (2.11). Specifically, for \(j \in \{p, d, c, cd, cp\}\), we respectively set \(\gamma \equiv \gamma^c_p\), \(\rho \equiv \rho^d_p\), \(\rho \equiv \rho^e_c\), \(\rho \equiv \rho^e_cd\) and \(\theta^p \equiv \theta^p_{cp}\), where

\[
PCMV \ (\gamma \equiv \gamma^c_p) : \quad \gamma^c_p = 2w_0 e^{rT} + \frac{2e^{rT}}{(e^{rT}-1)} (E - w_0 e^{rT}), \tag{4.2}
\]

\[
DOMV \ (\rho \equiv \rho^0_d) : \quad \rho^d_p = \frac{(e^{rT} - 1)}{2 (E - w_0 e^{rT})}, \tag{4.3}
\]

\[
cTCMV \ (\rho \equiv \rho^e_c) : \quad \rho^e_c = \frac{AT}{2(E - w_0 e^{rT})}, \tag{4.4}
\]

\[
dTCMV \ (\rho \equiv \rho^e_cd) : \quad \rho^e_cd \text{ together with the function } t \to \theta^p(t) \text{ determined numerically}
\]

\[
CP \ (\theta^p \equiv \theta^p_{cp}) : \quad \theta^p_{cp} = \frac{\log(E/w_0) - rT}{(\mu - \tau) \cdot T}. \tag{4.6}
\]

Using the results of Lemma 3.4 and Lemma 3.6, it is straightforward to verify that the choices (4.2)-(4.6) result in the terminal wealth distributions with the required expected value target \(E\).

Figure 4.1 illustrates the probability density functions (PDFs) of the distributions of \(W_j(T)\), \(j \in \{p, d, c, cd, cp\}\) for the particular choices (4.2)-(4.6), all with the same expected value \(E = 250\). In the case of \(j \in \{d, c, cd, cp\}\), these PDFs can be obtained analytically by appropriately substituting (4.3)-(4.6) into the corresponding results of Lemma 3.4. In the case of PCMV \((j = p)\), the simulated PDF of \(W^e_p(T)\) can be obtained using the expression (3.5) such that \(E_{w_0, to}^{u_{cd}} [W^e_{cd}(T)] \equiv E\), using (3.5) in Lemma 3.3 with \(\gamma = \gamma^e_p\) as per (4.2).

The rest of this section is devoted to a quantitative analysis of the differences in the distributions of \(W_j(T)\) for investment strategies \(j \in \{p, d, c, cd, cp\}\), illustrated by Figure 4.1.

![Figure 4.1: Probability density functions (PDFs) of the target terminal wealth \(W_j(T)\), for \(j \in \{p, d, c, cd, cp\}\), all with the same expected value \(E = 250\). \(w_0 = 100\), \(T = 10\), other parameters as Section 5. Note that the same scale is used on the x-axis.](image)

As an introductory result, the following lemma gives a relationship between the parameters of the target terminal wealth distributions in the case of the CP and dTCMV strategies that turns out to have far-reaching consequences.

**Lemma 4.1.** *(Parameters of the distribution of \(W_j(T)\), \(j \in \{cd, cp\}\): CP vs dTCMV) Assume that the conditions of Assumption 3.1 and Assumption 3.1 are satisfied. For any target value \(E\) satisfying (4.1), the parameters \(\mu^e_j\) and \(\sigma^e_j\) of the lognormally distributed target terminal wealth distributions, \(W^e_j(T) \sim \text{Logn} \left( \mu^e_j, \left( \sigma^e_j \right)^2 \right)\), \(j \in \{cp, cd\}\), satisfy the following relationships:

\[
\hat{\mu}^{cp} \geq \hat{\mu}^{cd}, \quad \hat{\sigma}^{cp} \leq \hat{\sigma}^{cd}. \tag{4.7}
\]

![Diagram](image)
Proof. By Lemma 3.4, \( \hat{\mu}_{cp}^\varepsilon = \log(\mathcal{E}) - \frac{1}{2} (\hat{\sigma}_{cp}^\varepsilon)^2 \) and \( \hat{\mu}_{cd}^\varepsilon = \log(\mathcal{E}) - \frac{1}{2} (\hat{\sigma}_{cd}^\varepsilon)^2 \), so we only need to prove that \( \hat{\sigma}_{cp}^\varepsilon \leq \hat{\sigma}_{cd}^\varepsilon \), where

\[
\hat{\sigma}_{cp}^\varepsilon = \frac{1}{\sqrt{AT}} \left[ \log(\mathcal{E}/w_0) - rt \right], \quad \hat{\sigma}_{cd}^\varepsilon = \sigma \cdot \left( \int_0^T |\theta^\varepsilon(t)|^2 \, dt \right)^{\frac{1}{2}}.
\] (4.8)

To ensure that \( W_{cp}(T) \) has the required mean \( \mathcal{E} \), the function \( t \to \theta^\varepsilon(t) \) and risk aversion parameter \( \rho^\varepsilon \) in (4.5) are solved numerically using the integral equation (3.5) to guarantee that

\[
\int_0^T \theta^\varepsilon(t) \, dt = \frac{\log(\mathcal{E}/w_0) - rt}{(\mu - r)}.
\] (4.9)

With \( \theta_{cp}^\varepsilon \) defined as the constant proportion in (4.6), we recognize that \( \theta_{cp}^\varepsilon \cdot \mathcal{E} = \int_0^T \theta^\varepsilon(t) \, dt \). Furthermore, the Cauchy-Schwarz inequality implies that

\[
\frac{1}{T} (\theta_{cp}^\varepsilon)^2 = \frac{1}{T} \left( \int_0^T \theta^\varepsilon(t) \, dt \right)^2 \leq \int_0^T (\theta^\varepsilon(t))^2 \, dt.
\] (4.10)

Therefore, (4.8) and (4.10) implies that we always have \( \hat{\sigma}_{cp}^\varepsilon \leq \hat{\sigma}_{cd}^\varepsilon \), regardless of the target \( \mathcal{E} > w_0 e^{rT} \).

As noted before, the dTCMV-optimal strategy is an example of a deterministic “glide path” strategy typically encountered in the pension fund literature, and in that particular context the result (4.10) used in the proof of Lemma 4.1 is a known result (see for example Forsyth and Vetzal (2019b); Graf (2017)). However, it is worth emphasizing the result (4.7) in this paper for two reasons.

First, in the specific case of the dTCMV problem, the conclusion of Lemma 4.1 enables the comparison of the distributions of \( W_{cp}^\varepsilon(T) \) and \( W_{dp}^\varepsilon(T) \) without resorting to the numerical solution of the function \( t \to \theta^\varepsilon(t) \) using the cumbersome integral equation (3.5). In particular, note that the exact form of the function \( t \to \theta^\varepsilon(t) \) does not matter; the only relevant fact regarding \( \theta^\varepsilon(t) \) is that its integral satisfies (4.9), which is just a constant multiple of the value of \( \theta_{cp}^\varepsilon \) in (4.6). Second, the result (4.7) turns out to be sufficient to prove a number of very interesting results, not just limited to mean and variance, but also including a first-order stochastic dominance result (see Theorem 4.10 below). This follows since we have a complete description of the relevant distributions under the stated assumptions.

We now return to our comparison of the distributions of the target terminal wealth \( W_j^\varepsilon(T) \), for investment strategies \( j \in \{p, d, c, cd, cp\} \). First, since a MV investor is by definition primarily concerned with the mean and variance of terminal wealth, and all the target terminal wealth distributions \( W_j^\varepsilon(T) \) have the same mean \( \mathcal{E} \) as per (4.1), we start by considering the variance of \( W_j^\varepsilon(T) \) obtained for each investment strategy \( j \).

**Lemma 4.2. (Variance: Target terminal wealth distribution)** Assume that the conditions of Assumption 3.1 and Assumption 4.1 are satisfied. The variance of the target terminal wealth \( W_j^\varepsilon(T) \), for \( j \in \{p, d, c, cd, cp\} \), is given by the following expressions:

\[
Var_{w_{d}^\varepsilon} [W_j^\varepsilon(T)] = \frac{1}{e^{AT} - 1} \left( \mathcal{E} - w_0 e^{rT} \right)^2, \quad Var_{w_{c}^\varepsilon} [W_j^\varepsilon(T)] = \frac{1}{e^{AT} + 1} \left( \mathcal{E} - w_0 e^{rT} \right)^2, \quad Var_{w_{cp}^\varepsilon} [W_j^\varepsilon(T)] = \frac{1}{e^{AT} - 1} \left( \mathcal{E} - w_0 e^{rT} \right)^2, \quad j \in \{d, c, cd\},
\] (4.11)

\[
Var_{w_{p}^\varepsilon} [W_j^\varepsilon(T)] = \frac{1}{e^{AT} + 1} \left( \mathcal{E} - w_0 e^{rT} \right)^2, \quad Var_{w_{cp}^\varepsilon} [W_j^\varepsilon(T)] = \mathcal{E}^2 \cdot \left( \sigma^\varepsilon \right)^2 \cdot \left( \sigma^\varepsilon \right)^2 - 1, \quad j \in \{cd, cp\},
\] (4.12)

where \( \delta_j^\varepsilon, j \in \{c, cd\} \) are given by (4.8).

Proof. The results follow from Lemma 3.4, Lemma 3.6 and (4.2)-(4.6).

The following lemma compares the variances of the target terminal wealth distributions.

**Lemma 4.3. (Comparison: Variance)** Assume that the conditions of Assumption 3.1 and Assumption 4.1 are satisfied. The variance of the target wealth distributions for investment strategies \( j \in \{p, d, c, cd, cp\} \) are related as follows.

\[
Var_{w_{d}^\varepsilon} [W_j^\varepsilon(T)] \leq Var_{w_{c}^\varepsilon} [W_j^\varepsilon(T)] \leq Var_{w_{cp}^\varepsilon} [W_j^\varepsilon(T)] \leq Var_{w_{p}^\varepsilon} [W_j^\varepsilon(T)] \leq Var_{w_{cd}^\varepsilon} [W_j^\varepsilon(T)].
\] (4.13)

\[
Var_{w_{d}^\varepsilon} [W_j^\varepsilon(T)] < Var_{w_{c}^\varepsilon} [W_j^\varepsilon(T)] < Var_{w_{cp}^\varepsilon} [W_j^\varepsilon(T)] < Var_{w_{p}^\varepsilon} [W_j^\varepsilon(T)] < Var_{w_{cd}^\varepsilon} [W_j^\varepsilon(T)].
\] (4.14)
Proof. Inequality (4.13) is obvious from the variance results (4.11)-(4.12) in Lemma 4.2. Considering (4.14), we first observe that \((x - 2) e^x + x + 2 > 0, \forall x > 0\). Since \(A > 0\) (recall that \(\mu > r, \sigma > 0\)) and \(T > 0, AT > 0\), we the following inequality which turns out to be very useful for proving some of the subsequent results,

\[
AT > \frac{2(e^{AT} - 1)}{(e^{AT} + 1)}, \quad \forall A, T > 0. \tag{4.15}
\]

Considering the results of Lemma 4.2, the inequality (4.15) implies that \(\text{Var}_{u_d^T} W_f^T (T) < \text{Var}_{u_d^T} W_f^T (T)\). Next, observing that \(\log x \geq [1 - (1/x)], \forall x > 1\), and \(\exp(y \cdot \log^2 x) > [1 + y \cdot \log^2 x], \forall x, y > 0\), it follows that

\[
\exp \left( y \cdot \log^2 x \right) - y \left(1 - \frac{1}{x} \right)^2 - 1 > 0, \quad \forall x > 1, y > 0. \tag{4.16}
\]

Since \(\mathcal{E} (u_0 e^{rT}) > 1\) by (4.1) and \(AT > 0\), (4.16) implies that we also have \(\text{Var}_{u_d^T} W_f^T (T) < \text{Var}_{u_d^T} W_f^T (T)\). Finally, the conclusion \(\text{Var}_{u_d^T} W_f^T (T) \leq \text{Var}_{u_d^T} W_f^T (T)\) follows from (4.12) and (4.7).

Lemma 4.3 therefore shows that a hypothetical MV investor who is only narrowly interested in the mean and variance of terminal wealth and agnostic as to the philosophical differences underlying the various approaches to dynamic MV optimization, would conclude the following: (i) the PCMV strategy always outperforms all the other strategies, (ii) the cTCMV strategy outperforms both the DOMV and CP strategies, and (iii) as expected based on the result of Lemma 4.1, the CP strategy outperforms the dTCMV strategy. Our analytical results therefore confirm and assist in explaining the conclusions from numerical tests regarding the relative performance of the PCMV and the CP strategies in Forsyth and Vetzal (2017b), as well as the performance comparison of the PCMV, cTCMV, dTCMV, and CP strategies presented in Forsyth and Vetzal (2019b).

As noted in the Introduction, in any practical setting (see for example Forsyth et al. (2019)) an MV investor is unlikely to take a narrow view purely focused on the mean and variance of terminal wealth. We therefore extend our analysis of the target terminal wealth distributions to include a number of other considerations.

Back et al. (2018) observes that there is evidence indicating that investors are concerned with higher-order moments, not just the mean and variance. In the subsequent results we focus on the skewness and (excess) kurtosis of the target wealth distribution, since these are the quantities typically included in portfolio optimization programs that generalize MV optimization to include higher-order moments - see for example Aracioglu et al. (2011); Jondeau and Rockinger (2006); Jurczenko et al. (2012); Lai et al. (2006); Maringer and Parpas (2009).

Lemma 4.4 compares the skewness\(^2\) of the target terminal wealth distributions.

**Lemma 4.4.** *(Comparison: Skewness)* Assume that the conditions of Assumption 3.1 and Assumption 4.1 are satisfied. The skewness of the target wealth distributions, \(\text{Skew}_{u_p^T} [W_f^T (T)], j \in \{p, d, c, cd, cp\}\), are related as follows,

\[
\text{Skew}_{u_p^T} [W_f^T (T)] < 0 = \text{Skew}_{u_c^T} [W_f^T (T)] = \text{Skew}_{u_d^T} [W_d^T (T)] \tag{4.17}
\]

\[
< \text{Skew}_{u_c^T} [W_f^T (T)] \tag{4.18}
\]

\[
\leq \text{Skew}_{u_d^T} [W_d^T (T)]. \tag{4.19}
\]

**Proof.** From Lemma 3.6, it follows that

\[
\text{Skew}_{u_p^T} [W_f^T (T)] = - (e^{AT} - 1)^{\frac{1}{2}} \left[ (e^{AT} - 1) + 3 \right] < 0, \quad \forall A, T > 0, \tag{4.20}
\]

which together with Lemma 3.4 implies (4.17). It follows from Lemma 4.1 that

\[
\text{Skew}_{u_j^T} [W_f^T (T)] = \left[ \{e(s_f^j)\}^2 + 2 \cdot \{e(s_f^j)\} - 1 \right]^{\frac{3}{2}}, j \in \{cd, cp\}, \tag{4.21}
\]

which implies (4.18), and together with (4.7) also implies (4.19).

Before discussing the implications of Lemma 4.4, we present the comparison of the excess kurtosis of the target terminal wealth distributions.

\(^2\)We use the standard definition of Pearson's moment coefficient of skewness, which in this context is simply given by

\[
\text{Skew}_{u_p^T} [W_f^T (T)] = \frac{E_{s_f^T} \left( (W_f^T (T) - \mathcal{E}) \right)^3}{\left[ \text{Var}_{s_f^T} [W_f^T (T)] \right]^{3/2}}.
\]
Lemma 4.5. (Comparison: Excess kurtosis) Assume that the conditions of Assumption 3.1 and Assumption 4.1 are satisfied. The excess kurtosis of the target wealth distributions, \( \text{Kurt}^{\beta_j_{c_t}}_{u_{c_t}}[W^\beta_j(T)], j \in \{p,d,cd,cp\} \), are related as follows.

\[
0 = \text{Kurt}^{\beta_j_{c_t}}_{u_{c_t}}[W^\beta_c(T)] = \text{Kurt}^{\beta_j_{c_t}}_{u_{c_t}}[W^\beta_d(T)] = \begin{cases} 
\text{Kurt}^{\beta_j_{c_t}}_{u_{c_t}}[W^\beta_p(T)], \\
\text{Kurt}^{\beta_j_{c_t}}_{u_{c_t}}[W^\beta_{cp}(T)] \leq \text{Kurt}^{\beta_j_{c_t}}_{u_{c_t}}[W^\beta_{cd}(T)].
\end{cases}
\]

(4.22)

(4.23)

Proof. (4.22) follows from Lemma 3.4. Noting the following factorization,

\[
e^{2AT} - 4e^{-AT} + 6e^{-3AT} - 3e^{-4AT}
= e^{-4AT} (e^{AT} - 1)^2 \left[ (e^{AT} - 1)^4 + 6 (e^{AT} - 1)^3 + 15 (e^{AT} - 1)^2 + 16 (e^{AT} - 1) + 3 \right].
\]

Lemma 3.6 implies that the excess kurtosis of \( W^\beta_p(T) \) is always positive,

\[
\text{Kurt}^{\beta_j_{c_t}}_{u_{c_t}}[W^\beta_p(T)] = (e^{AT} - 1) \left[ (e^{AT} - 1)^3 + 6 (e^{AT} - 1)^2 + 15 (e^{AT} - 1) + 16 \right] > 0.
\]

(4.24)

In the case of CP and dTCMV, Lemma 4.1 implies that

\[
\text{Kurt}^{\beta_j_{c_t}}_{u_{c_t}}[W^\beta_j(T)] = e^{4(\beta_j^c)^2} + 2e^{3(\beta_j^c)^2} + 3e^{2(\beta_j^c)^2} - 6 > 0, \quad j \in \{cd,cp\},
\]

(4.25)

which together with (4.7) implies (4.23).

Considering the results of Lemma 4.4 and Lemma 4.5, we note that there is overwhelming evidence in the literature that investors prefer positive skewness under very general assumptions - see for example Agren (2006); Back et al. (2018); Barberis et al. (2016); Barberis and Huang (2005); Boyer et al. (2010); Goetzmann and Kumar (2008); Hagestande and Wittussen (2016); Heuson et al. (2016); Kumar (2009); Maringer and Parpas (2009); Mitton and Vorkink (2007); Omed and Song (2014), among many others. This appears to follow from an investor preference for the possibility of a large gain (Agren (2006)), which may not be entirely rational (Omed and Song (2014)). In contrast, the evidence on kurtosis preferences is far more complicated 3 - see for example Haas (2007). However, when portfolio optimization with higher-order moments is performed (see for example Jurczenko et al. (2012)), kurtosis is usually minimized, suggesting that lower kurtosis is preferred (Maringer and Parpas (2009)).

Based on these observations, the results of Lemma 4.4 and Lemma 4.5 indicate that the excess kurtosis and especially the negative skewness associated with the PCMV-optimal strategy are at least somewhat undesirable from the perspective of an investor concerned with higher-order moments. The desirable variance result reported in Lemma 4.3 for the PCMV strategy therefore come at the cost of other potentially undesirable shape characteristics. These results therefore explain the numerical results reported in Forsyth and Vetzal (2019b) where the increased left tail risk of the PCMV strategy compared to the cTCMV and CP strategies is observed.

We also observe that the dTCMV strategy results in the largest (positive) skewness, but is also associated with the largest variance and the largest excess kurtosis. The normally distributed terminal wealth of the DOMV and cTCMV strategies result in zero skewness and excess kurtosis, as expected. Therefore, for an investor concerned with the first four moments, the cTCMV strategy is always to be preferred to the DOMV strategy, since the associated target terminal wealth distributions have the same mean (Assumption 4.1), the same skewness and kurtosis (Lemma 4.4 and Lemma 4.5), but the cTCMV strategy has a lower variance (Lemma 4.3).

Finally, we note the interesting fact that the skewness and kurtosis results for the CP and dTCMV strategies depend on the target \( \mathcal{E} \), but this is not the case for PCMV, cTCMV or DOMV strategies. As discussed in Section 5, this has some interesting consequences.

Given the preceding results on skewness and kurtosis, and the fact that as per Assumption 4.1 all the target distributions considered in this section have identical means \( \mathcal{E} \), the comparison of the median terminal wealth outcomes, given in the following lemma, is instructive. All else being equal, investors are expected to prefer larger median values (Forsyth et al. (2019)).

---

3 As Haas (2007) notes, “while risk aversion implies that investors dislike large losses more than they like large profits, kurtosis aversion requires that they dislike fat tails more than they like high peaks.”
**Lemma 4.6.** (Comparison: Median) Assume that the conditions of Assumption 3.1 and Assumption 4.1 are satisfied. The median of the target wealth distributions, \( Med_{w_j^o}^{w_0, t_0} [W_j^x (T)] \), \( j \in \{p, d, c, cd, cp\} \), are related as follows.

\[
\begin{align*}
\text{Med}_{w_d^o}^{w_0, t_0} [W_d^x (T)] & \leq \text{Med}_{w_c^o}^{w_0, t_0} [W_c^x (T)] \\
& < \text{Med}_{w_d^o}^{w_0, t_0} [W_d^x (T)] = Med_{w_c^o}^{w_0, t_0} [W_c^x (T)] = \mathcal{E}.
\end{align*}
\]

Proof. Since \( Med_{w_d^o}^{w_0, t_0} [W_d^x (T)] = \mathcal{E} \cdot \exp \left\{ -\frac{1}{2} (\sigma_j^x)^2 \right\} \) for \( j \in \{cd, cp\} \), the results follow from Lemma 3.4 and Lemma 4.1. \( \square \)

On the basis of median terminal wealth, Lemma 4.6 shows that the investor would prefer the CP strategy to the dTCMV strategy, and prefer either the cTCMV and DOMV strategies to the CP strategy. This conclusion therefore provides an analytical explanation of the numerically calculated median results reported in Forsyth and Vetzal (2019b).

Note that Lemma 4.6 does not include a result for the PCMV strategy, since it is non-trivial to find the median, or for example to prove that the PCMV target terminal wealth distribution is unimodal, based only on the results of Theorem 3.3 and Theorem 3.5. However, numerical results (see for example the PDF in Figure 4.1) together with the skewness result in Lemma 4.4 suggests that the median wealth for the PCMV strategy is expected to exceed the mean \( \mathcal{E} \), which combined with (4.26)-(4.27) indicates that we would expect the PCMV strategy to outperform all the other strategies on the basis of median terminal wealth. These observations provide further intuition as to why the PCMV strategy performs so well on the basis of median terminal wealth in the numerical results presented in Forsyth and Vetzal (2019b); Forsyth et al. (2019).

The remaining results of this section rely on our ability to obtain analytical expressions of the cumulative distribution functions (CDFs) of \( W_j^x (T) \). As a result, the PCMV-optimal target terminal wealth \( W_p^x (T) \) is excluded from these results, since recovering the CDF from a complete description of all of its moments (see Theorem 3.5) is a non-trivial problem (Mnatsakanov and Hakobyan (2009)). The remaining results in this section therefore focus on the strategies \( j \in \{d, c, cd, cp\} \), in other words, all the strategies except the PCMV-optimal strategy, and we leave further analysis of the CDF of \( W_p^x (T) \) for our future work.

The following lemma gives the CDFs of \( W_j^x (T) \), for \( j \in \{d, c, cd, cp\} \).

**Lemma 4.7.** (CDFs: Target terminal wealth distributions) Assume that the conditions of Assumption 3.1 and Assumption 4.1 are satisfied, and let \( \Phi (\cdot) \) denote the standard normal CDF. Then the CDFs of the distributions of the target terminal wealth \( W_j^x (T) \), for \( j \in \{d, c, cd, cp\} \), are as follows.

\[
\begin{align*}
\mathbb{P}_{w_d^o}^{w_0, t_0} [W_d^x (T) \leq w] & = \Phi \left( \frac{(w - \mathcal{E})}{\mathcal{E} - w_0 e^{rT}} \cdot \sqrt{\frac{2 (e^{AT} - 1)}{(e^{AT} + 1)}} \right), \quad w \in \mathbb{R}, \\
\mathbb{P}_{w_c^o}^{w_0, t_0} [W_c^x (T) \leq w] & = \Phi \left( \frac{(w - \mathcal{E})}{\mathcal{E} - w_0 e^{rT}} \cdot \sqrt{AT} \right), \quad w \in \mathbb{R}, \\
\mathbb{P}_{w_j^o}^{w_0, t_0} [W_j^x (T) \leq w] & = \Phi \left( \frac{\log(w / \mathcal{E}) + \frac{1}{2} (\sigma_j^x)^2}{\sigma_j^x} \right), \quad w > 0, \quad j \in \{cd, cp\}.
\end{align*}
\]

Proof. Follows from Lemma 3.4, as well as the definitions (4.1) and (4.8). \( \square \)

We now recall the concept of first-order stochastic dominance by applying the definition given in Joshi and Paterson (2013) in our setting.

**Definition 4.8.** (First-order stochastic dominance) \( W_j^x (T) \) has first-order stochastic dominance over \( W_k^x (T) \) for some \( j, k \in \{p, d, c, cd, cp\} \) if

\[
\begin{align*}
\mathbb{P}_{w_j^o}^{w_0, t_0} [W_j^x (T) \leq w] & \leq \mathbb{P}_{w_k^o}^{w_0, t_0} [W_k^x (T) \leq w], \quad \text{for all} \ w, \\
\mathbb{P}_{w_j^o}^{w_0, t_0} [W_j^x (T) \leq w] & < \mathbb{P}_{w_k^o}^{w_0, t_0} [W_k^x (T) \leq w], \quad \text{for some} \ w.
\end{align*}
\]

We observe that Definition 4.8 is a very general result, since it implies that any investor preferring more wealth to less wealth (i.e. any investor with an increasing utility function) would prefer \( W_j^x (T) \) over \( W_k^x (T) \) if (4.31)-(4.32) are satisfied.
However, the conditions of Definition 4.8 can be impossible to satisfy if we limit our attention to non-trivial investment strategies such as the dynamic MV strategies considered in this paper. As a result, we give the following weaker definition, which is sufficient for our purposes.

**Definition 4.9.** (Partial first-order stochastic dominance relative to a level $\ell$) Let $j, k \in \{p, d, c, cd, cp\}$. We define $W^\ell_j (T)$ as having partial first-order stochastic dominance over $W^\ell_k (T)$ relative to a level $\ell$, if

$$
\mathbb{P}^\nu_{W^\ell_k} [W^\nu_j (T) \leq w] \leq \mathbb{P}^\nu_{W^\ell_k} [W^\nu_j (T) \leq w], \quad \forall w < \ell.
$$

Note that Definition 4.9 focuses on “downside risk”, in that (4.33) is only concerned with the behavior of the CDFs below the given level $\ell$. In what follows, we typically set $\ell$ equal to the investor’s expected value target $\mathcal{E}$. In other words, we assume that the investor is primarily concerned with the possibility of underperforming the expected value target, while considering the “upside” of outcomes above $\mathcal{E}$ as a satisfying windfall, but not critical for investment strategy comparison purposes. We argue that this treatment is reasonable given the popularity of dynamic MV strategies in institutional settings, especially in the case of pension funds and insurance companies who are likely to take a keen interest in avoiding the underperformance of expectations.

Using Definition 4.9, the following theorem gives one of the key results of this paper.

**Theorem 4.10.** (Partial first-order stochastic dominance for underperforming expectations) Assume that the conditions of Assumption 3.1 and Assumption 4.1 are satisfied. We have the following relationships between the CDFs of $W^\nu_j (T)$, for $j \in \{d, c, cd, cp\}$.

$$
\mathbb{P}^{\nu_{W^\ell_k}} [W^\nu_j (T) \leq w] < \mathbb{P}^{\nu_{W^\ell_k}} [W^\nu_j (T) \leq w], \quad \forall w < \mathcal{E},
$$

and

$$
\mathbb{P}^{\nu_{W^\ell_k}} [W^\nu_c (T) \leq w] < \mathbb{P}^{\nu_{W^\ell_k}} [W^\nu_c (T) \leq w], \quad \forall w \in (w^0_{cp,c}, \mathcal{E}).
$$

Furthermore, there exists a unique value of terminal wealth $w^0_{cp,c} \in (0, \mathcal{E})$, with the upper bound

$$
w^0_{cp,c} < \frac{\mathcal{E} - w_0 e^T}{\log (\mathcal{E}/w_0) - rT},
$$

such that

$$
\mathbb{P}^{\nu_{W^\ell_k}} [W^\nu_c (T) \leq w] < \mathbb{P}^{\nu_{W^\ell_k}} [W^\nu_c (T) \leq w], \quad \forall w \in (w^0_{cp,c}, \mathcal{E}).
$$

**Proof.** Result (4.34) follows from (4.28)-(4.29), the relationship (4.15), and the fact that $\Phi$ is strictly increasing. To prove (4.35), we first note that

$$
x \log (z) - \frac{1}{2} \sigma^2 z - \frac{1}{2} x^2 y \leq 0, \quad \forall x \geq 0, y \geq 0, z \leq 1.
$$

The result (4.35) follows from setting $y = \tilde{\sigma}_c^2$, $x = \tilde{\sigma}_d - \tilde{\sigma}_c^2$ (so that $x \geq 0$, by (4.7)) and $z = w/\mathcal{E}$, noting the definition (4.30) and using the fact that $\Phi$ is strictly increasing. Next, let $x^0_{cp,c}$ be the unique root in the interval $(0, 1)$ of the function $x \rightarrow f_{pc,c} (x; c_1, c_2)$, defined by

$$
f_{pc,c} (x; c_1, c_2) = \left[ c_1 \right] \cdot \log (x) - \left[ \frac{c_1 e^{c_2}}{e^{c_2} - 1} \right] (x - 1) + \frac{1}{2} c_2, \quad x \in (0, 1), (c_1 > 0, c_2 > 0).
$$

Then (4.36)-(4.38) follows by setting $w^0_{cp,c} = \mathcal{E} \cdot x^0_{cp,c}, c_1 = AT$ and $c_2 = (\log (\mathcal{E}/w_0) - rT)$.

The results of Theorem 4.10 are illustrated in Figure 4.2 and Figure 4.3 below, and provides theoretical support for the qualitatively similar observations regarding the numerical results presented in Forsyth and Vetzal (2019b). We make the following observations regarding our analytical results.

First, subject to the stated assumptions, any investor who is agnostic about the philosophy underlying the different MV optimization approaches and simply concerned about the risk of underperforming the expectation $\mathcal{E}$, would never choose the DOMV or the dTCMV strategies, since better results can be obtained using the

---

4See for example Alla et al. (2016); Bi and Cai (2019); Liang et al. (2014); Liang and Song (2015); Lin and Qian (2016); Sun et al. (2016); Vigna (2014); Wu and Zeng (2015), among many others.

5The numerical results in Forsyth and Vetzal (2019b) does not include the DOMV-optimal strategy.
cTCMV or the CP strategies, respectively. Note that, as in the case of (4.34), we typically have strict inequality in (4.35) as well, since in typical applications it is the case that $\bar{\sigma}_{ed}^2 > \bar{\sigma}_{cp}^2$ in (4.7).

Second, (4.37)-(4.38) indicates that the CP strategy is preferred to the cTCMV strategy if we set the level $\ell \leq w_{cp,c}^0$ in Definition 4.9. Note that the upper bound (4.36) on $w_{cp,c}^0$ is strictly (and often substantially) less than $\bar{\sigma}$, so this bound can be very useful for a quick assessment depending on the critical value of $w$ under consideration in (4.37)-(4.38). This behavior is to be expected, since wealth can assume negative values in the case of the cTCMV strategy but not in the case of the CP strategy (see Lemma 3.4). However, the skewness results of the target wealth distribution in the case of the CP strategy (see Lemma 4.4 and Lemma 4.6) means that it starts (in aggregate probability) underperforming the cTCMV strategy fairly quickly as $\bar{\sigma}$ is approached from below - see Figure 4.3.

For illustrative purposes, Figure 4.3 also includes the simulated CDF of the PCMV target terminal wealth distribution. Compared to the CP and cTCMV strategies, it is clear that the negative skewness (Lemma 4.4) and excess kurtosis (Lemma 4.5) in this case combines to imply that the PCMV-optimal strategy holds substantial downside risks, as noted above.

Figure 4.2: Illustration of the results of Theorem 4.10: CDFs of $W_j^e(T)$, $j \in \{d,c,cd,cp\}$, all with the same expected value $\bar{\sigma} = 250$. $w_0 = 100$, $T = 10$, other parameters as Section 5.

Figure 4.3: Illustration of the results of Theorem 4.10: CDFs of $W_j^e(T)$, $j \in \{p,c,cp\}$, all with the same expected value $\bar{\sigma} = 250$. $w_0 = 100$, $T = 10$, other parameters as in Section 5. The CDF result for PCMV was estimated numerically using 30 million Monte Carlo simulations of $W_p^e(T)$. The value of $w_{pc,c}^0$ in (4.37)-(4.38) is indicated in both figures.

Up to this point, we have only focused on the expectation $\bar{\sigma}$ of the target terminal wealth distribution. However, the expectation conditional on $W_j^e(T)$ being below the risk-free investment outcome $w_0 e^{T}$ or sim-
ply conditional on underperforming the expectation target $E$ is also likely to be of particular interest to the investor. The following lemma summarizes the conditional expectation results for the investment strategies $j \in \{d,c,cd,cp\}$.

**Lemma 4.11.** (Conditional expectations of target terminal wealth distributions) Assume that the conditions of Assumption 3.1 and Assumption 4.1 are satisfied, and let $\phi(\cdot)$ and $\Phi(\cdot)$ be the probability density function and CDF of the standard normal distribution, respectively. The conditional expectations of $W_j^T (T)$, given that $W_j^T (T) \leq w$, for $j \in \{d,c,cd,cp\}$, are as follows.

\[
E_{u^T_{w_j}} [W_d^T (T) | W_d^T (T) \leq w] = E - \sqrt{\frac{(e^{AT} + 1)}{2(e^{AT} - 1)}} \cdot (E - w_0 e^{T}) \phi \left( \frac{(w - E)}{\sqrt{w_0 e^{T}}} \right) \cdot \sqrt{\frac{2(e^{AT} - 1)}{(e^{AT} + 1)}} \cdot \Phi \left( \frac{(w - E)}{\sqrt{w_0 e^{T}}} \right), \quad (4.39)
\]

\[
E_{u^T_{w_j}} [W_c^T (T) | W_c^T (T) \leq w] = E - \frac{1}{\sqrt{AT}} \cdot (E - w_0 e^{T}) \phi \left( \frac{(w - E)}{\sqrt{w_0 e^{T}}} \right) \cdot \sqrt{AT} \cdot \Phi \left( \frac{(w - E)}{\sqrt{w_0 e^{T}}} \right), \quad (4.40)
\]

\[
E_{u^T_{w_j}} [W^{jc)} (T) | W^{jc)} (T) \leq w] = E - \frac{1}{\sqrt{AT}} \cdot (E - w_0 e^{T}) \phi \left( \frac{(w - E)}{\sqrt{w_0 e^{T}}} \right) \cdot \sqrt{AT} \cdot \Phi \left( \frac{(w - E)}{\sqrt{w_0 e^{T}}} \right), \quad (4.41)
\]

**Proof.** Follows from Lemma 3.4 and Assumption 4.1.

We now use the results of Lemma 4.11 to compare the expectations of the target terminal wealth distributions conditional on $W_j^T (T) \leq w$, for any $w < E$, where $j \in \{d,c,cd,cp\}$. The results, given in Lemma 4.12, are intuitively expected given the results up to this point.

**Lemma 4.12.** (Comparison: Conditional expectations for underperforming target $E$) Assume that the conditions of Assumption 3.1 and Assumption 4.1 are satisfied. The conditional expected values of $W_j^T (T)$, conditional on $W_j^T (T) \leq w$, where $w < E$ and $j \in \{d,c,cd,cp\}$, satisfies the following.

\[
E_{u^T_{w_j}} [W_d^T (T) | W_d^T (T) \leq w] < E_{u^T_{w_j}} [W_c^T (T) | W_c^T (T) \leq w], \quad \forall w < E, \quad (4.42)
\]

\[
E_{u^T_{w_j}} [W^{jc)} (T) | W^{jc)} (T) \leq w] < E_{u^T_{w_j}} [W^{jc)} (T) | W^{jc)} (T) \leq w], \quad \forall w \in (0,E). \quad (4.43)
\]

**Proof.** The inverse Mills ratio $\lambda(x) := \phi(x) / \Phi(x)$ is strictly decreasing for all $x \in \mathbb{R}$, with $\lambda(x) \in (-1,0)$, $\forall x$. Since $\lambda'(x) = -\lambda(x) [x + \lambda(x)]$ and $\lambda(x) > 0$ for all $x$, we have in particular, $x + \lambda(x) > 0$ for all $x < 0$. Therefore, we have

\[
\frac{d}{dx} \left[ -\frac{1}{x} \lambda(x) \right] = -\frac{1}{x^2} [x + \lambda(x)] < 0, \quad \forall x < 0, \quad (4.44)
\]

so the function $\frac{1}{x} \lambda(x)$ is strictly decreasing for all $x < 0$. Considering (4.39) and (4.40), together with the requirement that $w < E$ and the inequality (4.15), this is sufficient to conclude (4.42). To prove (4.43), we fix some constant $c \geq 0$ and consider the auxiliary function $x \to f_\Phi (x; c)$ defined by

\[
f_\Phi (x; c) = \frac{\Phi \left( \frac{-c}{x} - \frac{1}{x} \right)}{\Phi \left( \frac{-c}{x} + \frac{1}{x} \right)}, \quad x \geq 0, (c \geq 0), \quad (4.45)
\]

We observe that $f_\Phi \geq 0$, and $f_\Phi (x; c) \leq 0$ if and only if

\[
\left[ \frac{c}{x^2} - \frac{1}{2} \right] \cdot \lambda \left( -\frac{c}{x} - \frac{1}{x} \right) \leq \left[ \frac{c}{x^2} + \frac{1}{2} \right] \cdot \lambda \left( \frac{c}{x} + \frac{1}{x} \right), \quad x \geq 0, (c \geq 0). \quad (4.46)
\]

If $\left[ \frac{c}{x^2} - \frac{1}{2} \right] \leq 0$, then (4.46) holds since $\lambda(x)$ is positive and decreasing for all $x \in \mathbb{R}$. If $\left[ \frac{c}{x^2} - \frac{1}{2} \right] > 0$, or equivalently $c > \frac{1}{2} x^2$, the inequality (4.46) also holds since $y \to \frac{1}{2} \lambda(y), \forall y < 0$ is decreasing as a result of (4.44). Therefore, since $f_\Phi (x; c)$ is decreasing in $x \geq 0$ for any fixed $c \geq 0$, the relationship (4.7) and expressions (4.41) imply the result (4.43).

The results of Lemma 4.12, while not making as general a statement as Theorem 4.10, is arguably of more practical relevance to investors since its conclusions are simple and intuitive to interpret. Informally, (4.42)-(4.43) simply states that when the investor is primarily concerned with outcomes underperforming the target
\( \mathcal{E} \), the DOMV and dTCMV strategies always lead to worse underperformance on average than the cTCMV and the CP strategies, respectively.

Note that Lemma 4.12 does not also provide a comparison of the conditional expectations in the case of CP and cTCMV. The reason is that such a comparison depends on the process and investment parameters in a fairly complicated way, and we instead explore the relationship between CP and cTCMV outcomes in more detail in the VaR results below. Here we simply observe that since the cTCMV strategy can result in negative wealth outcomes, we do know that for some sufficiently small value \( w_3 > 0 \) we have

\[
E_{u_c^0,t_0}^{w_3} [W_c^e(T)|W_c^e(T) \leq w] < E_{u_c^0,t_0}^{w_3} [W_{cp}^e(T)|W_{cp}^e(T) \leq w] \quad \text{for } w \in (0,w_3],
\]

which turns out to be sufficient to explain the numerical results observed in Section 5.

We introduce the following definition of the \( \alpha \text{VaR} \) and \( \alpha \text{CVaR} \), which has been adapted from the definition given in Forsyth et al. (2019) to our setting. Note that depending on application, slightly different formulations are used in literature (for example, focusing on the “loss distribution” instead - see Miller and Yang (2017); Rockafellar and Uryasev (2002)), but all these definitions have same qualitative content.

**Definition 4.13.** (\( \alpha \text{VaR} \) and \( \alpha \text{CVaR} \)) Fix a level \( \alpha \in (0,1) \). The Value-at-Risk at level \( \alpha \), or \( \alpha \text{VaR} \), is defined as the terminal wealth value \( \alpha \text{VaR}_{u_j^{w_3,t_0}}^{w_3,t_0} \), where

\[ \alpha \text{VaR}_{u_j^{w_3,t_0}}^{w_3,t_0} := w_\alpha, \quad \text{such that } \alpha \equiv \mathbb{P}_{u_j^{w_3,t_0}}^{w_3,t_0} [W_{f_j}^e(T) \leq w_\alpha], \quad j \in \{p,d,c,cd,cp\}. \]

The Conditional Value-at-Risk (also known as the Expected Shortfall) at level \( \alpha \), or \( \alpha \text{CVaR} \), is the expected value of terminal wealth \( W_{f_j}^e(T) \) given that it is below the level of the associated \( \alpha \text{VaR} \). In other words,

\[ \alpha \text{CVaR}_{u_j^{w_3,t_0}}^{w_3,t_0} := E_{u_j^{w_3,t_0}}^{w_0} [W_{f_j}^e(T) | W_{f_j}^e(T) \leq \alpha \text{VaR}_{u_j^{w_3,t_0}}^{w_3,t_0}], \quad j \in \{p,d,c,cd,cp\}. \]

Note that according to Definition 4.13, all else being equal, smaller values of \( \alpha \text{VaR}_{u_j^{w_3,t_0}}^{w_3,t_0} \) and \( \alpha \text{CVaR}_{u_j^{w_3,t_0}}^{w_3,t_0} \) represent a worse outcome for the investor than larger values. This qualitative interpretation is of course the opposite in those examples in literature where these quantities are defined in terms of the loss distribution.

Typical values of \( \alpha \) used in Definition 4.13 are fairly small, for example \( \alpha = 0.05 \) (5%) or \( \alpha = 0.01 \) (1%). However, the following lemma compares the \( \alpha \text{VaR} \) results for any choice of \( \alpha \in (0,0.5) \), since this interval is wide enough to allow that all likely values of interest of \( \alpha \) will be included.

**Lemma 4.14.** (Comparison: \( \alpha \text{VaR} \)) Assume that the conditions of Assumption 3.1 and Assumption 4.1 are satisfied. Fix a level \( \alpha \in (0,0.5) \). The following comparison results hold for \( \alpha \text{VaR}_{u_j^{w_3,t_0}}^{w_0,t_0}, j \in \{d,c,cd,cp\}. \)

\[
\begin{align*}
\alpha \text{VaR}_{u_j^{w_3,t_0}}^{w_0,t_0} &< \alpha \text{VaR}_{u_j^{w_3,t_0}}^{w_0,t_0} \quad \forall \alpha \in (0,0.5), \\
\alpha \text{VaR}_{u_j^{w_3,t_0}}^{w_0,t_0} &\leq \alpha \text{VaR}_{u_j^{w_0,t_0}}^{w_0,t_0} \quad \forall \alpha \in (0,0.5).
\end{align*}
\]

**Proof.** Follows from the results of Theorem 4.10. However, a direct proof is instructive due to the key role played by \( \alpha \text{VaR} \) in the risk management literature (Jorion (2009)). We start by noting that the definition (4.48) together with the results of Lemma 4.7 implies that

\[
\begin{align*}
\alpha \text{VaR}_{u_j^{w_3,t_0}}^{w_3,t_0} &= \mathcal{E} + \sqrt{\frac{(e^{AT}+1)}{2(e^{AT}-1)}} \left( \mathcal{E} - w_0e^{rT} \right) \cdot \Phi^{-1}(\alpha), \\
\alpha \text{VaR}_{u_j^{w_3,t_0}}^{w_3,t_0} &= \mathcal{E} + \frac{1}{\sqrt{AT}} \left( \mathcal{E} - w_0e^{rT} \right) \cdot \Phi^{-1}(\alpha), \\
\alpha \text{VaR}_{u_j^{w_3,t_0}}^{w_3,t_0} &= \mathcal{E} \cdot \exp \left\{ \frac{\sigma_j^e \cdot \Phi^{-1}(\alpha) - \frac{1}{2} \left( \sigma_j^e \right)^2}{\sigma_j^e} \right\}, \quad j \in \{cd,cp\},
\end{align*}
\]

The result (4.50) therefore follows from (4.52)-(4.53), the inequality (4.15), together with the fact that \( \Phi^{-1}(\alpha) < 0, \forall \alpha < 0.5 \). Next, we observe that if \( \sigma_{cd} > \sigma_{cp} \) then it is clear that \( \alpha \text{VaR}_{u_j^{w_3,t_0}}^{w_0,t_0} = \alpha \text{VaR}_{u_j^{w_3,t_0}}^{w_3,t_0} \). Assume therefore that \( \sigma_{cp} < \sigma_{cd} \). Then (4.54) implies that \( \alpha \text{VaR}_{u_j^{w_3,t_0}}^{w_0,t_0} < \alpha \text{VaR}_{u_j^{w_3,t_0}}^{w_0,t_0} \) for all \( \alpha > 0 \) such that \( \alpha < \Phi \left( \frac{1}{2} \left[ \sigma_{cd} + \sigma_{cp} \right] \right) \).

Observing that \( 0.5 < \Phi \left( \frac{1}{2} \left[ \sigma_{cd} + \sigma_{cp} \right] \right) \), the result (4.51) also holds.

\footnote{The value of \( w_3 \) should be sufficiently small in context of all the investment and process parameters. For example, in Section 5 we give an example where \( w_3 > w_0e^{rT} \).}
Given the results of Theorem 4.10, Lemma 4.14 as well as the fact that the αVaR might be of particular interest to investors, we analyze the αVaR results for the CP and cTCMV strategies in more detail. To this end, we give the following simple initial result.

**Lemma 4.15. (Comparison: α VaR for CP and cTCMV, a simple condition)** Assume that the conditions of Assumption 3.1 and Assumption 4.1 are satisfied. Then

\[ αVaR_{w^ε_{cu}}^{u_{cp},T_0} < αVaR_{w^ε_{cu}}^{u_{cp},T_0}, \quad \text{if } α < \Phi \left( -\frac{E}{(E - w_0e^T)} \cdot \sqrt{AT} \right). \]  

(4.55)

**Proof.** By Lemma 3.4, \( W^ε_c(T) \) can assume negative values, but \( W^ε_{cp}(T) \) cannot. Therefore, if α is chosen such that \( αVaR_{w^ε_{cu}}^{u_{cp},T_0} < 0 \), then it necessarily follows that \( αVaR_{w^ε_{cu}}^{u_{cp},T_0} < αVaR_{w^ε_{cu}}^{u_{cp},T_0} \). The condition on α in (4.55) follows from the expression for \( αVaR_{w^ε_{cu}}^{u_{cp},T_0} \) in (4.53), ensuring that \( αVaR_{w^ε_{cu}}^{u_{cp},T_0} < 0 \).

The result of Lemma 4.15 is useful in that it is easy to verify, and if α is small the condition (4.55) is often easily satisfied; for example, it is sufficient to explain the 1%VaR results for CP and cTCMV reported in Section 5. However, if we consider more general values for α ∈ (0, 0.5), the comparison results of αVaR for CP and cTCMV are more involved, as the following lemma shows. Specifically, we give two conditions on the process and investment parameters, either of which can be used to obtain more specific comparison results regarding αVaR for CP and cTCMV.

**Lemma 4.16. (Comparison: α VaR for CP and cTCMV)** Assume that the conditions of Assumption 3.1 and Assumption 4.1 are satisfied. Furthermore, assume that the wealth process (2.1) and investment parameters are such that Condition C1 or Condition C2 is satisfied, where

\[ C1 : \quad \log^2 \left( \frac{E}{w_0e^T} \right) \cdot \exp \left\{ -\frac{1}{2AT} \log^2 \left( \frac{E}{w_0e^T} \right) \right\} > \frac{2}{5} \sqrt{AT} \left( \frac{E - w_0e^T}{E} \right), \]

(4.56)

\[ C2 : \quad \frac{1}{\sqrt{AT}} \left[ \log \left( \frac{E}{w_0} \right) - rT \right]^2 \cdot \exp \left\{ -\frac{1}{2AT} \left[ \log \left( \frac{E}{w_0} \right) - rT \right]^2 \right\} > \frac{2}{5} \left( \frac{E - w_0e^T}{E} \right). \]

(4.57)

Then there exists a unique value \( α_{cp,c} ∈ (0, 0.5) \) such that

\[ αVaR_{w^ε_{cu}}^{u_{cp},T_0} < αVaR_{w^ε_{cu}}^{u_{cp},T_0}, \quad \forall α ∈ (0, α_{cp,c}), \]

(4.58)

\[ αVaR_{w^ε_{cu}}^{u_{cp},T_0} < αVaR_{w^ε_{cu}}^{u_{cp},T_0}, \quad \forall α ∈ (α_{cp,c}, 0.5), \]

(4.59)

while the difference \( [αVaR_{w^ε_{cu}}^{u_{cp},T_0} - αVaR_{w^ε_{cu}}^{u_{cp},T_0}] \) attains a maximum at \( α^* ∈ (α_{cp,c}, 1) \) given by

\[ α^* = \Phi \left( \frac{\sqrt{AT}}{\log(E/w_0) - rT} \cdot \log \left( 1 - \frac{E}{w_0e^T} \right) - \frac{1}{2} \cdot \log \left( E/w_0 \right) - rT \right). \]

(4.60)

**Proof.** From Lemma 4.15, we know that \( αVaR_{w^ε_{cu}}^{u_{cp},T_0} < αVaR_{w^ε_{cu}}^{u_{cp},T_0} \) provided α is sufficiently small. From the results (4.53)-(4.54), it is clear that \( αVaR_{w^ε_{cu}}^{u_{cp},T_0} < αVaR_{w^ε_{cu}}^{u_{cp},T_0} \) if α = 0.5, and by continuity therefore also for some \( ε \)-neighborhood of α = 0.5. It is straightforward to show that either of the relatively simple conditions (4.56)-(4.57) are sufficient to ensure that the function α → \( [αVaR_{w^ε_{cu}}^{u_{cp},T_0} - αVaR_{w^ε_{cu}}^{u_{cp},T_0}] \) is strictly concave, so that the results (4.58)-(4.60) follow.

The results of Lemma 4.16 are useful in providing an explanation of the numerical results presented in Section 5, where we encounter a particular example where both conditions (4.56)-(4.57) are satisfied and \( α_{cp,c} ∈ (0.05, 0.1) \).

Given the recent interest in using αCVaR as a risk measure in dynamic portfolio optimization applications (see for example Forsyth (2019); Miller and Yang (2017)), the following lemma compares the αCVaR results for investment strategies \( j ∈ \{d, c, cd, cp\} \), for any choice α ∈ (0, 1). We highlight that while the conditional expectation comparison (Lemma 4.12) compares the results below a fixed wealth level regardless of the associated percentile, the αCVaR comparison in Lemma 4.17 considers the conditional expectations of wealth outcomes below a fixed percentile (see Definition 4.13).
Lemma 4.17. (Comparison: $\alpha CVaR$) Assume that the conditions of Assumption 3.1 and Assumption 4.1 are satisfied. Fix a level $\alpha \in (0, 1)$. The following comparison results hold for $\alpha CVaR_{u_j^0, t_0}^{w_0, t_0}$, $j \in \{d, c, cd, cp\}$.

$$\alpha CVaR_{u_c^0, t_0}^{w_0, t_0} < \alpha CVaR_{u_d^0, t_0}^{w_0, t_0}, \quad \forall \alpha \in (0, 1), \tag{4.61}$$

$$\alpha CVaR_{u_d^0, t_0}^{w_0, t_0} \leq \alpha CVaR_{u_c^0, t_0}^{w_0, t_0}, \quad \forall \alpha \in (0, 1). \tag{4.62}$$

Proof. Given Definition 4.13, the results of Lemma 4.12 and the results for $\alpha CVaR_{u_j^0, t_0}^{w_0, t_0}$ in (4.52)-(4.54), we have the following expressions for $\alpha CVaR_{u_j^0, t_0}^{w_0, t_0}$, $j \in \{d, c, cd, cp\}$:

$$\alpha CVaR_{u_c^0, t_0}^{w_0, t_0} = \mathcal{E} - \frac{\sqrt{e^{AT} + 1}}{2(e^{AT} - 1)} \cdot (\mathcal{E} - w_0 e^{rT}) \cdot \frac{\phi(\Phi^{-1}(\alpha))}{\alpha}, \tag{4.63}$$

$$\alpha CVaR_{u_d^0, t_0}^{w_0, t_0} = \mathcal{E} - \frac{1}{\sqrt{AT}} (\mathcal{E} - w_0 e^{rT}) \cdot \frac{\phi(\Phi^{-1}(\alpha))}{\alpha}, \tag{4.64}$$

$$\alpha CVaR_{u_d^0, t_0}^{w_0, t_0} = \mathcal{E} \cdot \frac{\Phi(\Phi^{-1}(\alpha) - \hat{\sigma}_j^0)}{\alpha}, \quad j \in \{cd, cp\}. \tag{4.65}$$

Since $\phi(x) > 0, \forall x$ and $\alpha > 0$, the result (4.61) follows from the inequality (4.15) together with (4.63)-(4.64). Secondly, (4.62) follows from (4.63) together with (4.7) and the fact that $\Phi$ is strictly increasing. \hfill \qed

The results of Lemma 4.17 are intuitively expected given the results of Lemma 4.12 and Lemma 4.14. We do not provide a comparison of $\alpha CVaR$ in the case of CP and cTCMV, since such a comparison too cumbersome to be of much practical use - this can be seen by comparing the requirement of Definition 4.13 with the $\alpha CVaR$ results in Lemma 4.16.

In the next section, we present numerical results illustrating the analytical results presented in this section.

5 Numerical results

To obtain the numerical results presented in this section, we assume a fixed initial wealth of $w_0 = 100$ and an investment time horizon of $T = 10$ years. The wealth dynamics (2.1) is parameterized using the same calibration data and calibration techniques as detailed in Dang and Forsyth (2016); Forsyth and Vetzel (2017a), which we now briefly summarize. In terms of the empirical data sources, the risky asset data is based on inflation-adjusted daily total return data (including dividends and other distributions) for the period 1926-2014 from the CRSP’s VWD index, which is a capitalization-weighted index of all domestic stocks on major US exchanges. The risk-free rate is based on 3-month US T-bill rates over the period 1934-2014, and has been augmented with the NBER’s short-term government bond yield data for 1926-1933 to incorporate the impact of the 1929 stock market crash. Prior to calculations, all time series were inflation-adjusted using data from the US Bureau of Labor Statistics. Standard maximum likelihood techniques are used to calibrate the GBM dynamics - see Dang and Forsyth (2016); Forsyth and Vetzel (2017a) for more information regarding the relevant details. As a result, we obtain the following parameters for use throughout this section,

$$\mu = 0.0816, \quad \sigma = 0.1863, \quad r = 0.00623. \tag{5.1}$$

Table 5.1 presents the numerical results on various aspects of the target terminal wealth distributions for two expected value targets, $\mathcal{E} = 125$ and $\mathcal{E} = 250$. Note that investing all wealth in the risk-free asset over the entire time period $[0, T]$ results in a terminal wealth of $w_0 e^{rT} = 106.43$. Therefore, the strategies associated with the target $\mathcal{E} = 125$ are quite risk-averse, but not to the extent that all wealth is invested in the risk-free asset. In contrast, a target of $\mathcal{E} = 250$ requires a substantial investment in the risky asset during at least a significant portion of the investment time period.

We make the following observations regarding the results in Table 5.1:

Calculation were based on data from the Historical Indexes 2015©, Center for Research in Security Prices (CRSP), The University of Chicago Booth School of Business. Wharton Research Data Services was used in preparing this article. This service and the data available thereon constitute valuable intellectual property and trade secrets of WRDS and/or its third party suppliers.

Data has been obtained from See http://research.stlouisfed.org/fred2/series/TB3MS.


The annual average CPI-U index, which is based on inflation data for urban consumers, was used - see http://www.bls.gov/cpi.
• The role of the expected value target in shaping the results is worth highlighting. Specifically, the larger the expected value target, the larger the investment required in the risky asset, which magnifies the differences between the investment strategies, as expected. As a result, for purposes of clarity we focus mostly on the results for the target $E = 250$ in the subsequent discussion.

• The first-order stochastic dominance results of Theorem 4.10 are illustrated quite dramatically in Table 5.1. It is clear from the results that, subject to the stated assumptions under which these results were derived, no rational investor purely interested in the terminal wealth distributions would pursue the DOMV-optimal or the dTCMV-optimal strategies, since the cTCMV-optimal and CP strategies perform respectively much better.

• The performance of the dTCMV-optimal strategy can be exceptionally poor. Of course, while this has been established convincingly by the results presented in Section 4, the sheer degree of the underperformance can be quite dramatic, as the case of $E = 250$ highlights. Observe for example that in this case, the standard deviation of $W^E_{cd}(T)$ is more than double that of $W^E_{cp}(T)$, about four times that of $W^E_{c}(T)$, and more than six times that of $W^E_{p}(T)$. The median of $W^E_{cd}(T)$ is also exceptionally poor, and there is a 45% chance that $W^E_{cd}(T)$ is below $w_0e^{rT}$. Arguably the only redeeming feature of $W^E_{cd}(T)$ is the role of its lognormal distribution in limiting the downside tail risk in the most extreme cases; this is illustrated by the 1%VaR and 1%CVaR results. However, the same can be said of the corresponding CP strategy, which as per Theorem 4.10 performs much better overall the dTCMV strategy. Since the poor performance of the dTCMV strategy has also been confirmed in Forsyth and Vetzal (2019b) using numerical experiments for the case where multiple realistic investment constraints are applied simultaneously, the popularity of applying the dTCMV approach in institutional settings in the literature (see for example Bi and Cai (2019); Li and Li (2013); Liang et al. (2014); Sun et al. (2016); Wang and Chen (2018, 2019); Liang et al. (2014); Sun et al. (2016); Wang and Chen (2018, 2019); Long and Zeng (2016); Peng et al. (2018); Zhang et al. (2017)) raises some concerns.

• The cTCMV-optimal strategy performs very well compared to the CP strategy by a number of the measures considered, for example standard deviation and the probability that the terminal wealth will fall below $w_0e^{rT}$ or the target $E$. However, the CP strategy performs better where the extreme left tail of the distribution is concerned (for example, the $\alpha$VaR and $\alpha$CVaR for $\alpha \in \{1\%,5\\%\}$), which agrees with the numerical results presented in Forsyth and Vetzal (2019b), and also confirms the analytical conclusions of Section 4, especially Theorem 4.10.

• The PCMV-optimal strategy is the best performing strategy in terms of the standard deviation (Lemma 4.3) and also in terms of the median wealth. While the PCMV-optimal median wealth was not be included in the analytical results presented in Lemma 4.6, the excellent performance in terms of the median wealth outcomes has also been confirmed by the experiments of Forsyth et al. (2019). However, as observed in Forsyth and Vetzal (2019b), this performance comes at the cost of increased left tail risk, as confirmed by our negative skewness and excess kurtosis results for the distribution of $W^E_{p}(T)$ - see Lemma 4.4 and Lemma 4.5. The implication in this example is that the resulting 1%VaR and 1%CVaR is the worst of all the strategies considered. However, this is only true for very extreme tail outcomes, since already the 5%VaR and 5%CVaR associated with $W^E_{p}(T)$ are the best of all the strategies considered.

Finally, we note that while the numerical results presented in Table 5.1 illustrate the analytical results of Section 4, and are therefore also subject to Assumption 3.1 and Assumption 4.1, the qualitative observations regarding the relative performance of the different strategies are in agreement with the observations from the relevant numerical results available in the literature. In particular, we refer the reader to Forsyth and Vetzal (2017b, 2019a,b); Forsyth et al. (2019), where the portfolio optimization problems are solved numerically subject to multiple realistic investment constraints being applied simultaneously. This illustrates that our analytical results, while obtained under stylized assumptions regarding trading in the underlying market, are nevertheless of practical use in explaining the performance of dynamic MV-optimal investment strategies in a realistic setting.

6 Conclusion

In this paper, we compared the terminal wealth distributions obtained by implementing the optimal investment strategies associated with the different approaches to dynamic MV optimization available in the literature. In particular, we considered the pre-commitment MV (PCMV) approach, the dynamically optimal MV (DOMV)
Table 5.1: Numerical results related to the target terminal wealth distributions for two expected value targets, $\mathcal{E} = 125$ and $\mathcal{E} = 250$. Initial wealth $w_0 = 100$, $T = 10$ years. “Prob. $\leq k$” refers to the probability $P_{w_0}^{\mu_\mathcal{E}, \sigma_\mathcal{E}}[W^j_\mathcal{E}(T) \leq k]$, and “CExp. $\leq k$” to the conditional expectation $E_{w_0}^{\mu_\mathcal{E}, \sigma_\mathcal{E}}[W^j_\mathcal{E}(T)|W^j_\mathcal{E}(T) \leq k]$, respectively, for $j \in \{p, d, c, cd, cp\}$. Where necessary, results for PCMV were estimated numerically using 30 million Monte Carlo simulations of $W^p_\mathcal{E}(T)$. Numbers rounded to nearest integer except where doing so would obscure relevant information.

<table>
<thead>
<tr>
<th>Quantity</th>
<th>Target expected value $\mathcal{E} = 125$</th>
<th>Target expected value $\mathcal{E} = 250$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>PCMV</td>
<td>DOMV</td>
</tr>
<tr>
<td>Mean</td>
<td>125</td>
<td>125</td>
</tr>
<tr>
<td>Median</td>
<td>127</td>
<td>125</td>
</tr>
<tr>
<td>Stddev</td>
<td>9</td>
<td>16</td>
</tr>
<tr>
<td>Skewness</td>
<td>-15</td>
<td>0</td>
</tr>
<tr>
<td>Ex.Kurtosis</td>
<td>1042</td>
<td>0</td>
</tr>
<tr>
<td>1% VaR</td>
<td>91</td>
<td>88</td>
</tr>
<tr>
<td>5% VaR</td>
<td>113</td>
<td>99</td>
</tr>
<tr>
<td>10% VaR</td>
<td>119</td>
<td>105</td>
</tr>
<tr>
<td>1% CVaR</td>
<td>63</td>
<td>82</td>
</tr>
<tr>
<td>5% CVaR</td>
<td>97</td>
<td>92</td>
</tr>
<tr>
<td>10% CVaR</td>
<td>107</td>
<td>97</td>
</tr>
<tr>
<td>Prob. $\leq w_{\mu_\mathcal{E}}+\epsilon^2$</td>
<td>3%</td>
<td>12%</td>
</tr>
<tr>
<td>Prob. $\leq \mathcal{E}$</td>
<td>26%</td>
<td>50%</td>
</tr>
<tr>
<td>CExp. $\leq w_{\mu_\mathcal{E}}+\epsilon^2$</td>
<td>87</td>
<td>99</td>
</tr>
<tr>
<td>CExp. $\leq \mathcal{E}$</td>
<td>117</td>
<td>112</td>
</tr>
</tbody>
</table>

approach, as well as the time-consistent MV approach with a constant risk aversion parameter (cTCMV) and
wealth-dependent risk aversion parameter (dTCMV), respectively. For comparison and benchmarking purposes,
a constant proportion (CP) strategy was also considered.

We introduced some simplifying assumptions regarding the underlying market in order to analytically com-
pare the resulting terminal wealth distributions on a fair basis. Specifically, we assumed that the investor is
agnostic about the philosophical differences underlying the various approaches to MV optimization, and simply
wishes to achieve a chosen expected value of terminal wealth regardless of the approach. We also assumed that
the investor faced no leverage constraints or transaction costs, and could trade continuously in the market.

Subject to these assumptions, we presented first-order stochastic dominance results proving that for wealth
outcomes below the chosen expected value target, the cTCMV strategy always outperforms the DOMV strategy,
and the CP strategy always outperforms the dTCMV strategy. We also show that the dTCMV strategy performs
exceptionally poorly among the strategies considered according to a number of criteria, including variance and
median of terminal wealth, raising concerns regarding the popularity of the dTCMV in the literature applying
this strategy in institutional settings. Furthermore, we derived higher-order moment results for the PCMV-
optimal wealth distribution which proved that the PCMV strategy results in a terminal wealth distribution
with fundamentally different characteristics than any of the other strategies.

Our analytical results, while derived under simplifying assumptions, nonetheless proves effective in explaining
the numerical results incorporating realistic investment constraints currently available in the literature
Finally, we leave further analysis of the PCMV-optimal target terminal wealth distribution, extension of our
results to solutions for multiple risky assets, and treatment of alternative model specifications (e.g. jumps in
the risky asset process and alternative model specifications) for our future work.

References

Alia, I., F. Chighoub, and A. Sohail (2016). A characterization of equilibrium strategies in continuous-time mean–variance
Antolin, P., S. Blome, D. Karim, S. Payet, G. Scheuenstuhl, and J. Yermo (2009). Investment regulations and defined


