Dynamic Mean Variance Asset Allocation: Numerics and Backtests

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Outline

1 Dynamic mean variance
   - Embedding result ⇒ quadratic target
   - Removal of spurious points

2 HJB PDE
   - Intuitive discretization
   - Semi-Lagrangian timestepping and explicit control
   - Unconditionally stable, monotone and consistent

3 Calibrate to historical market data (1926-2015)
   - Synthetic market: M-V optimal beats constant proportion
   - Backtests using real historical data: M-V optimal even better!
   - Constant proportion beats any deterministic glide path strategy (lumpsum investment)\(^1\)
     → M-V optimal beats any deterministic glide path strategy

\(^1\)Strategy used in Target Date funds (over $750 billion in US)
Dynamic Mean Variance: Abstract Formulation

Define:

\[ X = \text{Process} \]
\[ \frac{dX}{dt} = \text{SDE} \]
\[ x = (X(t) = x) = \text{State} \]
\[ W(X(t), t) = \text{total wealth} \]

Control \( c(X(t), t) \) is applied to \( X(t) \)

Define admissible set \( \mathcal{Z} \), i.e.

\[ c(x, t) \in \mathcal{Z}(x, t) \]
Mean and Variance under control \( c(X(t), t) \)

Let:

\[
E_{t,x}^{c(\cdot)}[W(T)]
\]

\[\text{Reward}\]
\[= \text{Expectation conditional on } (x, t) \text{ under control } c(\cdot)\]

\[
\text{Var}_{t,x}^{c(\cdot)}[W(T)]
\]

\[\text{Risk}\]
\[= \text{Variance conditional on } (x, t) \text{ under control } c(\cdot)\]

Important:
- mean and variance of \( W(T) \) are as observed at time \( t \), initial state \( x \).
Basic Problem: Find Pareto Optimal Strategy

We desire to find the investment strategy \( c^*(\cdot) \) such that, there exists no other other strategy \( c(\cdot) \) such that

\[
\begin{align*}
E_{t,x}^c [W_T] & \geq E_{t,x}^{c^*(\cdot)} [W_T] \\
\text{Reward under strategy } c(\cdot) & \text{ Reward under strategy } c^*(\cdot)
\end{align*}
\]

\[
\begin{align*}
\text{Var}_{t,x}^c [W_T] & \leq \text{Var}_{t,x}^{c^*(\cdot)} [W_T] \\
\text{Risk under strategy } c(\cdot) & \text{ Risk under strategy } c^*(\cdot)
\end{align*}
\]

and at least one of the inequalities is strict.

Scalarization: For \( \lambda > 0 \), find \( c(\cdot) \) which solves

\[
\inf_{c(\cdot)} \left\{ \lambda \text{Var}_{t,x}^c [W_T] - E_{t,x}^c [W_T] \right\}
\]

Varying \( \lambda \) traces out the efficient frontier.
Pareto optimal points

Let

\[ \mathcal{E} = E^{c(\cdot)}_{t,x}[W_T] ; \quad \mathcal{V} = \text{Var}^{c(\cdot)}_{t,x}[W_T] \]

The achievable set \( \mathcal{Y} \) is

\[ \mathcal{Y} = \{(\mathcal{V}, \mathcal{E}) : c(\cdot) \in \mathcal{Z}\} \]

Given \( \lambda > 0 \), define scalarization set \(^2\)

\[ S_{\lambda}(\mathcal{Y}) = \{(\mathcal{V}, \mathcal{E}) \in \bar{\mathcal{Y}} : \lambda \mathcal{V} - \mathcal{E} = \inf_{(\mathcal{V}_*, \mathcal{E}_*) \in \mathcal{Y}} (\lambda \mathcal{V}_* - \mathcal{E}_*)\} \]

The efficient frontier \( \mathcal{Y}_P \) is

\[ \mathcal{Y}_P = \bigcup_{\lambda > 0} S_{\lambda}(\mathcal{Y}) \]

The efficient frontier is a collection of Pareto points

\(^2\bar{\mathcal{Y}} \) is the closure of \( \mathcal{Y} \).
Scalarization: intuition

Recall scalarization set:

\[ S_\lambda(\mathcal{Y}) = \{(V, E) \in \bar{\mathcal{Y}} : \lambda V - E = \inf_{(V_*, E_*) \in \mathcal{Y}} (\lambda V_* - E_*) \} \]  \hspace{1cm} (1)

Geometric interpretation:

- Consider the straight line (for fixed \( \lambda \))

\[ \lambda V - E = C_1 \]  \hspace{1cm} (2)

Points in (1)

- Choose \( C_1 \) as small as possible, such that:
  \[ \rightarrow \text{Intersection of } \mathcal{Y} \text{ and straight line (2) has at least one point} \]

\(^3\)We may not get all the Pareto points here if \( \mathcal{Y} \) is not convex
Move dotted lines line $\lambda \mathcal{Y} - \mathcal{E} = C_1$ to the left as much as possible (decrease $C_1$)

Line will touch $\mathcal{Y}$ at Pareto point
Problem

Pareto point

\[ \lambda \mathcal{V} - \mathcal{E} = \inf_{(\mathcal{V}_*, \mathcal{E}_*) \in \mathcal{Y}} (\lambda \mathcal{V}_* - \mathcal{E}_*) \]  \hspace{1cm} (3)

Problem arises from variance

\[ \mathcal{V} = E^c[\mathcal{W}(T)^2] - (E^c[\mathcal{W}(T)])^2 \]

\[ (E^c[\mathcal{W}(T)])^2 \rightarrow \text{problem for dynamic programming} \]

Consider the optimization problem (for fixed \( \gamma \))

\[ \inf_{(\mathcal{V}, \mathcal{E}) \in \mathcal{Y}} \mathcal{V} + \mathcal{E}^2 - \gamma \mathcal{E} \]  \hspace{1cm} (4)

Note that

\[ \mathcal{V} + \mathcal{E}^2 = E^c[\mathcal{W}(T)^2] \]

Minimizing (4) can be done using dynamic programming
Embedded Objective Function Intuition

Examine points \((V, E) \in \mathcal{Y}\) such that (for fixed \(\gamma\))

\[
V + E^2 - \gamma E = \inf_{(V_*, E_*) \in \mathcal{Y}} \ V_* + E_*^2 - \gamma E_*
\]  

(5)

Geometric interpretation:
- Consider the parabola

\[
V + E^2 - \gamma E = C_2
\]  

(6)

Points in (5)
- Choose \(C_2\) as small as possible, such that
  - Intersection of parabola and \(\mathcal{Y}\) has at least one point

Rewriting equation (6)

\[
V = -(E^2 - \gamma E) + C_2 = -(E - \gamma/2)^2 + \gamma^2/4 + C_2
\]

\[
= -(E - \gamma/2)^2 + C_3.
\]

Parabola faces left, symmetric about line \(E = \gamma/2\)
**Embedded Pareto Points**

Suppose \((V_*, E_*) \in \mathcal{Y}_P \rightarrow \exists \lambda > 0, C_1, \text{ s.t.}\)

\[
\lambda V_* - E_* = C_1
\]

Parabola:

\[
V = -(E - \gamma/2)^2 + C_3.
\]

\(\exists \gamma/2, C_3, \text{ such that we can}\)

Move parabola to left \((C_3)\)

Move parabola up/down \((\gamma/2)\)

\(\Rightarrow\) intersect line \(\lambda V - E = C_1\)

at a single point \((V^*, E^*)\).
Tangency Condition

Parabola $\mathcal{V} = -(E - \gamma/2)^2 + C_3$ tangent to line $\lambda \mathcal{V} - E = C_1$ at $(\mathcal{V}_*, E_*)$

\[
\left( \frac{\partial E}{\partial \mathcal{V}} \right)_\text{parabola} = \lambda ; \quad \lambda = \text{slope of dotted lines}
\]

$\quad \Rightarrow \quad \gamma/2 = 1/(2\lambda) + E_*$
Embedding Result

Theorem 1 ((Li and Ng (2000); Zhou and Li (2000))

\[
\lambda V_0 - E_0 = \inf_{(V,E) \in Y} (\lambda V - E), \tag{7}
\]

then

\[
V_0 + E_0^2 - \gamma E_0 = \inf_{(V,E) \in Y} (V + E^2 - \gamma E), \tag{8}
\]

\[
\gamma = \frac{1}{\lambda} + 2E_0
\]

Implication

- We can determine all the Pareto points from (7) by solving problem (8)
Value function

Note:

\[ V + E^2 - \gamma E = E_{t,x}^c [ (W(T) - \frac{\gamma}{2})^2 ] + \frac{\gamma^2}{4}, \]

Define value function\(^4\) (ignore \(\gamma^2/4\) term when minimizing)

\[ V(x, t) = \inf_{c(\cdot) \in Z} E_{t,x}^c [(W(T) - \gamma/2)^2] \quad (9) \]

**Key Result:** Given point \((V^*, E^*)\) on the efficient frontier, generated by control \(c^*(\cdot)\), then \(\exists \gamma\) s.t.

\[ \rightarrow c^*(\cdot) \text{ is an optimal control for (9)} \]

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\(^4\)Precommitment MV optimal \(\equiv\) quadratic target optimal. Precommitment

\[ \rightarrow \text{choose target wealth } \gamma/2 \text{ at time zero} \]
Spurious points

But, converse not necessarily true: i.e. there may be some \( \gamma \in (-\infty, +\infty) \) s.t. \( c^*(\cdot) \) which solves

\[
V(x, t) = \inf_{c(\cdot) \in \mathbb{Z}} E_{t,x}^c(\cdot) [(W(T) - \gamma/2)^2]
\]  

(10)

does not correspond to a point on the efficient frontier.
Technical Point: Precommitment vs. time consistent

We are solving for the optimal *precommitment* policy

- This is not *time-consistent*, since $\gamma/2$ (the target) depends on the initial state.
- However, a way to think about this is as follows:
  - At $t = 0$ we determine where we want to be on the efficient frontier. This fixes $\gamma/2$.
  - At $t > 0$, we can think of this policy as the optimal time consistent strategy which minimizes quadratic loss w.r.t. fixed $\gamma/2$.
  - This is intuitive and easy to explain to pension plan investors (Vigna (2014))
    - The target amount $\gamma/2$ is the amount needed to fund retirement.
Basic Algorithm

Discretize the parameter $\gamma$

$$\gamma \in \Gamma^k = \left[-\gamma_{\text{max}}^k, -\gamma_{\text{max}}^k + h_k, \ldots, |\gamma_{\text{max}}^k|\right]$$

$$h_k \to 0 ; \; \gamma_{\text{max}}^k \to \infty ; \; k \to \infty$$

For each $\gamma_i$,

- Determine optimal control $c_{\gamma_i}^*(\cdot)$ by solving the embedded problem (solve HJB equation, store control)
- Using this control, compute $E_{t,x}^{c_{\gamma_i}^*(\cdot)}[(W_T)]$, $\text{Var}_{t,x}^{c_{\gamma_i}^*(\cdot)}[(W_T)]$ via Monte Carlo (one point on the frontier)

Does this converge to *true* efficient frontier as $k \to \infty$?
Problems

1. Controls which minimize $E_{t,x}^c[(W(T) - \gamma/2)^2]$ (from numerical solve)
   - May generate spurious points (e.g. non-convex $\mathcal{Y}$)

2. The control which minimizes

   $$E_{t,x}^c[(W(T) - \gamma/2)^2]$$  \hspace{1cm} (13)

   may not be unique.
   - Numerical HJB solve for fixed $\gamma/2$
     $\rightarrow$ picks out only one control $c^*(\cdot)$
   - Does the control we compute correspond to a point in $\mathcal{Y}_p$?
Convergent Algorithm\textsuperscript{5}

For $k = 0, 1, \ldots$

- Solve value function $\forall \gamma_i \in \Gamma^k$
- Generate set of candidate points on the efficient frontier $\mathcal{A}^k$
- Determine upper left convex hull $S(\mathcal{A}^k)$
- Approximate points on efficient frontier: $\mathcal{A}^k \cap S(\mathcal{A}^k)$

\textsuperscript{5}Tse, Forsyth, Li (2014, SIAM Cont. Opt.); Dang, Forsyth, Li (2016, Numerische Mathematik)
Convergence result

Recall def’n of scalarization set:

\[ S_\lambda(X) = \left\{ (V_\ast, E_\ast) \in X : \lambda V_\ast - E_\ast = \inf_{(V, E) \in X} \lambda V - E \right\}, \tag{14} \]

Suppose \( S_\lambda(Y) \neq \emptyset, \lambda > 0 \) (i.e. \( S_\lambda(Y) \) are points on the efficient frontier for fixed \( \lambda \))

**Theorem 2**

*Suppose \( \Gamma^k \) is systematically refined \(^6\) as \( k \to \infty \), and let \( (V_k, E_k) \in S_\lambda(A^k) \). Let \( (V_\ast, E_\ast) \) be a limit point of \( \{(V_k, E_k)\} \). Then \( (V_\ast, E_\ast) \) is on the original efficient frontier.*

**Remark 1**

*All points on the approximate efficient frontier \( A^k \cap S(A^k) \) are valid points on the true efficient frontier as \( k \to \infty \). \(^7\)

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\(^6\)Any reasonable refinement satisfies this condition

\(^7\)There may some gaps in the approximate frontier if there are 3 or more points on a straight line segment.
Asset allocation: risk free bond, stock index

Risk free bond $B$

$$dB = rB \, dt$$

$r =$ risk-free rate

*Amount* in risky stock index $S$ *(jump diffusion)*

$$dS = (\mu - \rho \kappa)S \, dt + \sigma S \, dZ + (J - 1)S \, dq$$

$\mu = \mathbb{P}$ measure drift ; $\sigma =$ volatility

d$Z =$ increment of a Wiener process

$$dq = \begin{cases} 
0 & \text{with probability } 1 - \rho \, dt \\
1 & \text{with probability } \rho dt, 
\end{cases}$$

$log J \sim$ double exponential. ; $\kappa = E[J - 1]$
Optimal Control

Define:

\[ X = (S(t), B(t)) = \text{Process} \]
\[ x = (S(t) = s, B(t) = b) = (s, b) = \text{State} \]
\[ (s + b) = \text{total wealth} \]

Let \((s, b) = (S(t^-), B(t^-))\) be the state of the portfolio the instant before applying a control.

The control \(c(s, b) = (d, B^+)\) generates a new state:

\[ b \rightarrow B^+ \]
\[ s \rightarrow S^+ \]

\[ S^+ = (s + b) - B^+ - d \]

Note: we allow cash withdrawals of an amount \(d \geq 0\) at a rebalancing time.
Optimal de-risking (free cash flow)

Let

\[ F(t) = \frac{\gamma}{2} e^{-r(T-t)} = \text{discounted target wealth} \]

Proposition 1 (Dang and Forsyth (2016))

If \( W_t > F(t) \), \( t \in [0, T] \), an optimal MV strategy is

- Withdraw cash \( d = W_t - F(t) \) from the portfolio
- Invest the remaining amount \( F(t) \) in the risk-free asset.

We will refer to the amount withdrawn as a free cash flow. \(^8\)

---

Constraints on the strategy

The investor can continue trading only if solvent

\[ W(s, b) = s + b > 0 . \]

\( W(s, b) = s + b > 0 \).

Solvency condition

In the event of bankruptcy, the investor must liquidate \(^9\)

\[ S^+ = 0 ; \quad B^+ = W(s, b) ; \quad \text{if} \quad W(s, b) \leq 0 . \]

bankruptcy

Leverage is also constrained

\[ \frac{S^+}{W^+} \leq q_{max} \]

\[ W^+ = S^+ + B^+ = \text{Total Wealth} \]

\(^9\)The No Donald Trump trading condition.
HJB PIDE

Find optimal control $c(\cdot) \Rightarrow$ solve for value function

$$V(x, t) = \inf_{c \in \mathcal{Z}} \left\{ E_{t,x}^c \left[ (W(T) - \gamma/2)^2 \right] \right\},$$

Define:

$$\mathcal{L}V \equiv \frac{\sigma^2 s^2}{2} V_{ss} + (\mu - \rho \kappa) s V_s - \rho V,$$

$$\mathcal{J}V \equiv \int_0^\infty p(\xi) V(\xi s, b, \tau) \, d\xi$$

$p(\xi) = \text{jump size density}$ ; $\rho = \text{jump intensity}$

and the intervention operator $\mathcal{M}(c) \, V(s, b, t)$

$$\mathcal{M}(c) \, V(s, b, t) = V(S^+(s, b, c), B^+(s, b, c), t)$$
HJB PIDE II

Value function, control \( c(\cdot) \Rightarrow \) solve impulse control HJB equation

\[
\max \left[ V_t + \mathcal{L}V + rbV_b + \mathcal{J}V, V - \inf_{c \in \mathcal{Z}} (\mathcal{M}(c) \ V) \right] = 0
\]

Discretize computational domain \((s, b) \in [0, \infty) \times (-\infty, +\infty)\)

\[
\{s_1, s_2, \ldots, s_{i_{\text{max}}}\} ; \quad \{b_1, \ldots, b_{j_{\text{max}}}\}
\]

Constant timesteps, discretize control

\[
\Delta \tau = \tau^{n+1} - \tau^n ; \quad B^+ \in \{b_1, \ldots, b_{j_{\text{max}}}\}
\]

Discretization parameter \( h \)

\[
\max_i(s_{i+1} - s_i) = \max_j(b_{j+1} - b_j) = \max_n(\tau^{n+1} - \tau^n) = O(h)
\]
Computational Domain$^{10}$

If $\mu > r$ it is never optimal to short $S$

\[(S,B) \in [0, \infty] \times [-\infty, +\infty]\]
Intuitive Derivation of Discretization

Consider a set of discrete rebalancing times \( \{t_1, t_2, \ldots \} \)

Define

\[
t_m^+ = t_m + \epsilon \quad ; \quad t_m^- = t_m - \epsilon \quad ; \quad \epsilon \to 0^+
\]  

(16)

At \( t = t_m^+ \), \( s = S(t) \) and \( b = B(t) \)

Step \( [t_m^+, t_{m+1}^-] \) (bond amount constant)

- The value function \( V(s, b, t) \) evolves according to the PIDE

\[
V_t + \mathcal{L}V + \mathcal{J}V = 0,
\]
Evolution over $[t_{m+1}^-, t_{m+1}^+]$

Step $[t_{m+1}^-, t_{m+1}^+]$ (Stock amount constant)

- Pay interest earned in $[t_m^+, t_{m+1}^-]$

$$V(s, b, t_{m+1}^-) = V(s, be^{r\Delta t}, t_{m+1}) ; \text{ by no-arbitrage}$$

$$\Delta t = t_{m+1} - t_m$$

Step $[t_{m+1}, t_{m+1}^+]$

- Optimal rebalance

$$V(s, b, t_{m+1}) = \min_c V(S^+(s, b, c), B^+(s, b, c), t_{m+1}^+)$$
Backwards time: discrete solution

Now, we write these steps down in backwards time \( \tau = T - t \)

- Define \( V_{i,j}^n \equiv \) discrete solution \( V_h(s_i, b_j, \tau^n) \)

\[
\tilde{V}_{i,j}^n = \min_{c \in \mathbb{Z}_h} V_h(S^+(s_i, b_j e^{r \Delta \tau}, c), B^+(s_i, b_j e^{r \Delta \tau}, c), \tau^n)
\]

\[
\frac{V_{i,j}^{n+1}}{\Delta \tau} - \mathcal{L}_h V_{i,j}^{n+1} - \mathcal{J}_h V_{i,j}^{n+1} = \frac{\tilde{V}_{i,j}^n}{\Delta \tau}
\]

Formally: Semi-Lagrangian timestepping and explicit impulse control
Discretization Properties

1. Positive coefficient method used to discretize $\mathcal{P}$.
2. Jump term: fixed point iteration + FFT for dense matrix-vector product.
3. Linear interpolation used to approximate $V_h$ at off grid points (needed for optimal control).

Assume strong comparison property holds:
- Consistent, $\ell_\infty$ stable, monotone
  $\rightarrow$ Convergence to viscosity solution
Example Asset Allocation: Constant Proportions

According to Benjamin Graham\textsuperscript{11}, defensive investors should

\begin{itemize}
  \item Pick a fraction $p$ of wealth to invest in a diversified equity fund (e.g. $p = 1/2$).
  \item Invest $(1 - p)$ in bonds
  \item Rebalance to maintain this asset mix
\end{itemize}

$\rightarrow$ i.e. a constant proportion strategy

How does this strategy compare with standard target date funds, which follow a deterministic glide path over time $T$?

Typical deterministic glide path strategy\textsuperscript{12}

\[ p(t) = \frac{(110 - \textit{your age})}{100} \]

\textsuperscript{11}Benjamin Graham, \textit{The Intelligent Investor}
\textsuperscript{12}This used to be $(100 - \textit{your age})$ but people are living longer
Lumpsum Investment: ineffectiveness of glide paths

Consider any \textit{deterministic} glide path strategy \( p(t) \)

\[
p(t) = \text{fraction of wealth invested in equities}
\]

Define a constant weight strategy \( p^* \) where

\[
p^* = \frac{1}{T} \int_0^T p(s) \, ds
\]

\[
= \text{time average fraction in equities}
\]

Let \( W \) denote total wealth. We can prove (GBM + jumps) \(^{13}\)

\[
\overbrace{E[W(T)]}^{\text{constant weight}} = \overbrace{E[W(T)]}^{\text{glide path}}; \quad \overbrace{\text{Var}[W(T)]}^{\text{constant weight}} \leq \overbrace{\text{Var}[W(T)]}^{\text{glide path}} \quad (17)
\]

Backtests on historical data and MC simulations\(^{14}\) indicates (17) holds in general \(\rightarrow\) constant proportion beats deterministic glide path

\(^{13}\)Graf (2016), Forsyth and Vetzal (2016)

Technical Point: lumpsum vs. periodic contributions

Constant proportion beats any deterministic glide path for a lumpsum investment.

- This is not true for periodic contributions (accumulation) or withdrawals (decumulation)
- However, for $T > 20$ years
  - Numerical tests show that the optimal deterministic MV glide path strategy is only slightly better than a constant proportions strategy.
  - In practice, deterministic glide path strategies are devised using heuristics.
Monte Carlo Simulation Results

- Inflation-adjusted equity: jump diffusion\textsuperscript{15} model estimated using CRSP\textsuperscript{16} total return index and CPI data (1926 to 2015)
- Inflation-adjusted bonds: average real 3M T-bills (1926 to 2015)

<table>
<thead>
<tr>
<th>Strategy</th>
<th>Expected Value</th>
<th>Standard Deviation</th>
<th>Prob(W(T)) &lt; 300</th>
<th>Prob(W(T)) &lt; 400</th>
</tr>
</thead>
<tbody>
<tr>
<td>Constant Proportion $p = 0.5$</td>
<td>417</td>
<td>299</td>
<td>0.41</td>
<td>0.60</td>
</tr>
<tr>
<td>M-V Optimal Control</td>
<td>417</td>
<td>117</td>
<td>0.13</td>
<td>0.22</td>
</tr>
</tbody>
</table>

Table: Investment horizon $T = 30$ years. Initial investment $W(0) = 100$. Optimal de-risking; no trading if insolvent; maximum leverage = 1.5, rebalancing once/year.

**Standard deviation reduced by 250%, shortfall probability reduced by 3×**

\textsuperscript{15}Jump size had double exponential distribution (Kou, 2002)
\textsuperscript{16}Capitalization weighted index of all stocks traded on major US exchanges.
Cumulative Distribution Function: IRR

\[ E[W(T)] = 417 \text{ same for both strategies} \]

Optimal policy: Contrarian:
when market goes down → increase stock allocation;
when market goes up → decrease stock allocation

Optimal allocation gives up gains \( \gg \) target in order to reduce variance and probability of shortfall.

Investor must pre-commit to target wealth

MV optimal beats constant proportion, consequently it also beats any deterministic glide path!

\[ \text{Internal rate of return (i.e. effective rate of return) } = \frac{\log(W(T)/W(0))}{T} \]
Strategy Heat Map

Fraction in Risky Asset

Red: maximum leverage

Blue: 100% bond

\[ W_0 = 100 \]

\[ E[ W_T ] = 417 \]
Back Testing

M-V optimal performance on historical data

- Compute and store strategy based on estimated parameters for entire historical period (January 1, 1926 - December 31, 2014).
- \( E[W(T)] \) same as for constant proportion strategy \((p = .5)\), for this set of average parameters.
- Select starting date
- Compare:
  - Optimal MV strategy (based on average parameters, not tuned to this period)
  - Constant proportion strategy
Back Test, Real Returns: Jan 1, 1930 - Dec 31, 1959

Note *Falling Knife* effect in 1932

Can we fix this: regime switching plus machine learning?

---

\(^{19} W(1930) = 100. \) Maximum leverage 1.5. Optimal MV strategy computed using parameters for 1926-2015 period. Yearly rebalancing.
Bootstrap Resampling: 1926-2015

More Scientific Test: Resampling

Use real historical data, monthly returns

- Randomly draw 30 years of returns (with replacement) from historical returns (blocksize 10 years)
- 10,000 simulations, each block starts at random month

<table>
<thead>
<tr>
<th>Strategy</th>
<th>Expected Value</th>
<th>Standard Deviation</th>
<th>$Pr(W(T)) &lt; 300$</th>
<th>Expected Free Cash</th>
</tr>
</thead>
<tbody>
<tr>
<td>Constant Proportion $p = 0.5$</td>
<td>385</td>
<td>183</td>
<td>0.38</td>
<td>0.0</td>
</tr>
<tr>
<td>M-V Optimal Control</td>
<td>431</td>
<td>84</td>
<td>0.07</td>
<td>40</td>
</tr>
</tbody>
</table>

Table: $T = 30$ years. $W(0) = 100$. Yearly rebalancing. Optimal de-risking; no trading if insolvent; maximum leverage = 1.5.

Performs even better on actual historical data than on synthetic market data!
Resampled Cumulative Distribution Function: IRR

Internal rate of return, (i.e. effective rate of return) = \log\left(\frac{W(T)}{W(0)}\right) / T
Technical Point: Bootstrap resampling

Data is *wrapped around* to avoid end effects (i.e. 1930s appears more often).

To minimize blocksize end effects, blocksize is selected randomly from a geometric distribution.

<table>
<thead>
<tr>
<th>Time series</th>
<th>Optimal Expected Block size (months)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Real 90-day T-bills</td>
<td>50.1</td>
</tr>
<tr>
<td>Real 10 year treasury</td>
<td>4.7</td>
</tr>
<tr>
<td>Real CRSP index (cap weight)</td>
<td>1.8</td>
</tr>
<tr>
<td>Real CRSP index (equal weight)</td>
<td>10.4</td>
</tr>
</tbody>
</table>

Table: Optimal expected blocksize $1/p$ where the blocksize is distributed according to a geometric distribution $Pr(b = k) = (1 - p)^{k-1} p$. The algorithm in (Politis et al (2009)) is used.
Technical Point: Bootstrap resampling vs rolling quarters

A common backtest is to use *rolling quarters*

- This amounts to starting the investment at each historical quarter, and then seeing how it performed over the next 30 years.
- Summary statistics of probability of failure are then quoted
- However, there are not enough 30 year rolling blocks to get reasonable samples → investment results are too good!

<table>
<thead>
<tr>
<th>Strategy</th>
<th>Expected Value</th>
<th>$Pr(W(T)) &lt; 300$</th>
<th>$Pr(W(T)) &lt; 400$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Constant Proportion $p = 0.5$</td>
<td>351</td>
<td>0.31</td>
<td>0.73</td>
</tr>
<tr>
<td>M-V Optimal Control</td>
<td>453</td>
<td>0.0</td>
<td>0.07</td>
</tr>
</tbody>
</table>

Table: $T = 30$ years. $W(0) = 100$. Yearly rebalancing. Optimal de-risking; no trading if insolvent; maximum leverage $= 1.5$. Rolling quarters, 30 year blocksize, data is wrapped-around.
Conclusions

- M-V strategy is very robust
  - Insensitive to calibration ambiguity
  - MC tests: insensitive to random perturbations of synthetic market SDE parameters
  - Stochastic volatility: typical parameters, insignificant for long term investors
  - 10 year treasuries (instead of 3-M) similar results
  - Good results on historical backtests

- Similar results for accumulation, decumulation
- M-V beats constant proportion, i.e. probability of shortfall $2 - 3 \times$ smaller
  - Constant proportion beats any deterministic glide path
- M-V optimal equivalent to minimizing quadratic loss w.r.t. wealth target
  - Optimal strategy is M-V optimal and quadratic loss optimal

- More sophisticated models
  - Regime switching? (machine learning approach being investigated)