Across-Time Risk-Aware Strategies for Outperforming a Benchmark

Pieter M. van Staden* Peter A. Forsyth† Yuying Li‡

June 13, 2023

Abstract

We propose a novel objective function for constructing dynamic investment strategies with the goal of outperforming an investment benchmark at multiple points of evaluation during the investment time horizon. The proposed objective is intuitive, easy to parameterize, and directly targets a favorable tracking difference of the actively managed portfolio relative to the benchmark. Under stylized assumptions, we derive closed-form optimal investment strategies to guide the intuition in more realistic settings. In the case of discrete rebalancing with investment constraints, optimal strategies are obtained using a neural network-based numerical approach that does not rely on dynamic programming techniques. Compared to the targeting of a favorable tracking difference relative to the benchmark only at some fixed time horizon, our results show that the proposed objective offers a number of advantages: (i) The associated optimal strategies exhibit potentially more attractive asset allocation profiles, in that less extreme positions in individual assets are taken early in the investment time horizon, while achieving a similar terminal terminal wealth distribution. (ii) Across-time risk awareness leads to more robust performance and a higher probability of benchmark outperformance during the investment horizon in out-of-sample testing. The resulting strategies therefore exhibit desirable characteristics for active portfolio managers with periodic reporting requirements.

Keywords: Finance, asset allocation, investment analysis, benchmark outperformance

JEL classification: G11, C61

1 Introduction

Active portfolio managers typically pursue investment strategies with the stated goal of outperforming a pre-specified investment benchmark (Alekseev and Sokolov (2016); Kashyap et al. (2021); Korn and Lindberg (2014); Lehalle and Simon (2021); Zhao (2007)). In the case of pension funds, the benchmark or reference portfolios typically consist of publicly-traded assets held in specified proportions. For example, the Canadian Pension Plan (CPP) makes use of a base reference portfolio of 15% Canadian government bonds and 85% global equity (Canadian Pension Plan (2022)), while the Norwegian government pension plan ("Government Pension Fund Global", or GPFG) uses a benchmark of 70% equities and 30% bonds (Government Pension Fund Global (2022)). With the CPP[1] and GPFG having CAD 540 billion and USD 1.35 trillion in assets under management, respectively, and while performance results (risk and return) are reported relative to the benchmark strategy, the goal of outperforming the benchmark is clearly of immediate practical relevance.

There exists a large literature on the construction of investment strategies for benchmark outperformance, where the objective function often includes utility functions whether implicit or explicit (see for example Al-Aradi and Jaimungal (2018, 2021); Basak et al. (2006); Bernard and Vanduffel (2014); Davis and Lleo (2008); Gerrard et al. (2019, 2022); Lim and Wong (2010); Lu et al. (2016); Nicolosi et al. (2018); Oderda (2015); Tepla (2001)) or aims to penalize underperformance while encouraging outperformance (see Basak et al. (2006); Browne (1999a, 2000); Gaivoronski et al. (2005)). Van Staden et al. (2023) analyzed the optimal dynamic strategies associated with two popular investment objectives, namely maximizing the information ratio, and obtaining a favorable tracking difference relative to the benchmark.

*Cheriton School of Computer Science, University of Waterloo, Waterloo ON, Canada, N2L 3G1, email: pieter.vanstaden@gmail.com

†Cheriton School of Computer Science, University of Waterloo, Waterloo ON, Canada, N2L 3G1, paforsyt@uwaterloo.ca

‡Cheriton School of Computer Science, University of Waterloo, Waterloo ON, Canada, N2L 3G1, yuying@uwaterloo.ca

[1] It is interesting to note that the CPP 2021 annual report(Canadian Pension Plan, 2021) lists personnel costs as CAD 938 million, for 1,936 employees, giving an average cost of CAD 500,000 per employee-year.

The tracking *difference* simply measures the difference between the cumulative returns of the active portfolio and the benchmark over a specific time period (Charteris and McCullough (2020)). This is not to be confused with the tracking *error*, which refers to the volatility of relative returns (Wander (2000)) and is typically to be minimized if the portfolio manager simply wishes to track the benchmark as closely as possible. For long-term investors, however, the tracking difference is recognized as a critical and intuitive metric for performance assessment (Boyde (2021); ETF.com (2021); Hougan (2015); Pastant (2018); Vanguard (2014)), in addition to being recognized by regulators such as European Securities and Markets Authority, who requires its disclosure by certain regulated funds (ESMA (2014)).

An intuitive objective function targeting a favorable tracking difference, which is based on the quadratic deviation (QD) from an elevated benchmark, has been proposed in the literature - see discussion in Van Staden et al. (2023). For illustrative purposes (to be made rigorous below), let $\mathcal{P}$ denote the active investment strategy (or control) taking values in some admissible set $\mathcal{A}$ which encodes the portfolio manager's constraints, and let $W(t)$ and $\hat{W}(t)$ denote the wealth (portfolio value) of the active and benchmark portfolios, respectively, at time $t \in [t_0 = 0, T]$. For performance comparison purposes, set $W(t_0) = \hat{W}(t_0) = w_0$. Investment strategies targeting a favorable tracking difference over $[t_0, T]$ can then be obtained by using the following objective,

$$(QD(\beta)): \qquad \inf_{\mathcal{P}\in\mathcal{A}} E_{\mathcal{P}}^{t_0,w_0}\left[\left(W(T) - e^{\beta T}\hat{W}(T)\right)^2\right], \qquad \beta > 0, \tag{1.1}$$

which is parameterized by a continuously compounded (targeted) outperformance rate of β per year. As β increases, the portfolio manager would be required to take on more risk in order to increase the expected outperformance, while in the limit as $\beta \downarrow 0$, the optimal strategy is simply to invest the benchmark. Some modifications of (1.1) have also been proposed in the literature (Ni et al. (2022)), where underperformance and outperformance are distinguished. While (1.1) is symmetric in the sense that it penalizes both the shortfall $(W(T) < e^{\beta T}\hat{W}(T))$ as well as the excess $(W(T) > e^{\beta T}\hat{W}(T))$ relative to the targeted outperformance, similar results are obtained in the case where only the shortfall $(W(T) < e^{\beta T}\hat{W}(T))$ is penalized (Van Staden et al. (2023)).

However, a possible criticism of the QD objective (1.1) is that it only targets a favourable tracking difference *at* maturity T of the investment time horizon $[t_0, T]$. In practice, due to reporting or regulatory requirements (ESMA (2014)), portfolio managers may wish to target a favorable tracking difference also at some *intermediate* times during $[t_0, T]$.

In this paper we propose an objective function that not only retains the intuitive and transparent structure QD objective, but extends this to the targeting of favourable tracking differences of the active portfolio relative to the benchmark at specified intermediate times during the investment time horizon. Due to its additive structure, we refer to the proposed objective function as the cumulative tracking difference (abbreviated as "CD" for convenience), and in its simplest form it can be formulated as

$$(CD(\delta)): \qquad \inf_{\mathcal{P}\in\mathcal{A}} E_{\mathcal{P}}^{t_0,w_0}\left[\int_{t_0}^{T}\left(W(t) - e^{\delta t}\hat{W}(t)\right)^2 dt\right], \qquad \delta > 0. \tag{1.2}$$

The main contribution of this paper is the analyze the implications of objectives of the form (1.2) for the associated dynamic investment strategies. In more detail, the contributions of this paper are as follows:

(i) The CD problem is solved in closed form using standard assumptions in order to gain intuition regarding the behavior of CD-optimal investment strategies. We also present analytical comparison results for the relative performance of the QD- and CD-optimal investment strategies.

(ii) Using a neural network (NN) approach that does not rely on dynamic programming techniques, the CD problem is solved numerically in the case of discrete portfolio rebalancing and multiple investment constraints, where closed-form solutions cannot be obtained.

(iii) Using empirical market data from 1963 until the end of 2020, with a 10-year investment horizon, we demonstrate the resulting in-sample and out-of-sample investment results associated with the proposed CD objective as well as the QD objective. Data sets are generated using both (i) stochastic differential equations calibrated to historical data and (ii) block bootstrap resampling of historical data (Anarkulova et al., 2022; Cogneau and Zakalmouline, 2013; Politis and Romano, 1994).

(iv) Based on simulations of a parametric model and resampling of market data respectively, our numerical results demonstrate that the CD-optimal strategies require less extreme positions in individual assets early in the investment time horizon compared with the corresponding QD-optimal positions. In addition, distributions of the terminal wealth of CD-optimal and QD-optimal strategies are, somewhat surprisingly, largely indistinguishable in both training and out-of-sample testing results.

The observation that the CD-optimal strategy has a nearly identical terminal wealth distribution to that of the QD-optimal strategy, while the positions of the CD-optimal strategy in the underlying assets exhibit significantly less variation across time, has important implications. It demonstrates that it is potentially insufficient to evaluate the risk in a dynamic strategy based on the statistics (or even the entire distribution) of the terminal wealth alone. Specifically, we further illustrate that while the QD-optimal strategy achieves a higher probability of benchmark outperformance in training, the CD-optimal strategy significantly outperforms the QD-optimal strategy in out-of-sample testing data due to its advantageous risk profile over the investment horizon.

In summary, we demonstrate both theoretically and empirically that targeting a favorable tracking difference directly using objectives of the form (1.2) can be advantageous for the active portfolio manager aiming to outperform a benchmark while being subjected to investment constraints.

The remainder of the paper is organized as follows. Section 2 formulates the problems in general terms, while Section 3 and Section 4 discuss the analytical and numerical solutions of the problems, respectively. Section 5 presents a numerical illustration of the investment results, while Section 6 concludes the paper and outlines possible future work.

2 Formulation

In this section, we formulate the benchmark outperformance problem more rigorously. Let $[t_0 = 0, T]$ denote the investment horizon of the active portfolio manager, for simplicity referred to as the "investor". As above, let $W(t)$ and $\hat{W}(t)$ denote wealth of the investor and benchmark portfolios, respectively, at time $t \in [t_0 = 0, T]$. For performance measurement purposes, we assume $w_0 \coloneqq W(t_0) = \hat{W}(t_0) > 0$. We assume the investor considers investment in N_a candidate assets, while the benchmark is formulated in terms of $\hat{N}_a$ underlying assets. In general, the sets of underlying assets are not required to be identical.

The vector $\hat{\boldsymbol{p}}\left(t, \hat{\boldsymbol{X}}(t)\right) = \left(\hat{p}_i\left(t, \hat{\boldsymbol{X}}(t)\right) : i = 1, .., \hat{N}_a\right) \in \mathbb{R}^{\hat{N}_a}$ denotes the asset allocation of the benchmark at time $t \in [t_0, T]$, where $\hat{p}_i\left(t, \hat{\boldsymbol{X}}(t)\right)$ denotes the proportion of the benchmark wealth $\hat{W}(t)$ invested in asset $i \in \left\{1, .., \hat{N}_a\right\}$, and $\hat{\boldsymbol{X}}(t)$ denotes the state of the system (or informally, the information) taken into account by the benchmark strategy.

Similarly, the vector $\boldsymbol{p}(t, \boldsymbol{X}(t)) = (p_i(t, \boldsymbol{X}(t)) : i = 1, .., N_a) \in \mathbb{R}^{N_a}$ denotes the asset allocation of the investor at time $t \in [t_0, T]$, where $p_i(t, \boldsymbol{X}(t))$ denotes the proportion of the investor's wealth $W(t)$ invested in asset $i \in \{1, .., N_a\}$ and $\boldsymbol{X}(t)$ denotes the information taken into account by the investor in making the asset allocation decision. In the simplest cases, such as in Section 3, we could simply have $\boldsymbol{X}(t) = \left(W(t), \hat{W}(t)\right)$, but additional information can also be incorporated in $\boldsymbol{X}(t)$ in more general scenarios addressed in Section 4.

Let $\mathcal{T} \subseteq [t_0, T]$ denote the set of portfolio rebalancing events. In the case of continuous rebalancing, $\mathcal{T} = [t_0, T]$, while discrete balancing limits the events to the discrete subset $\mathcal{T} \subset [t_0, T]$. The investor and benchmark investment strategies, respectively, are defined by the sets

$$\mathcal{P} = \{\boldsymbol{p}(t, \boldsymbol{X}(t)), \ t \in \mathcal{T}\}, \qquad \text{and} \qquad \hat{\mathcal{P}} = \left\{\hat{\boldsymbol{p}}\left(t, \hat{\boldsymbol{X}}(t)\right), \ t \in \mathcal{T}\right\}. \tag{2.1}$$

The investor's investment constraints are encoded by $\mathcal{A}$ denoting the set of admissible controls, and $\mathcal{Z}$ denote the admissible control space (i.e. the values obtained by the admissible controls). In other words, an admissible investor strategy satisfies $\mathcal{P} \in \mathcal{A}$ if and only if $\mathcal{P} = \{\boldsymbol{p}(t, \boldsymbol{X}(t)) \in \mathcal{Z} : t \in \mathcal{T}\}$.

Finally, let $E_{\mathcal{P}}^{t_0, w_0}[\cdot]$ denote the expectation of some random variable taken with respect to a given initial wealth $w_0 = W(t_0) = \hat{W}(t_0)$ at time $t_0 = 0$, and using control $\mathcal{P} \in \mathcal{A}$ over $[t_0, T]$. The benchmark strategy $\hat{\mathcal{P}}$ that the investor wishes to outperform remains implicit in this notation.

2.1 Directly targeting a favourable tracking difference

As discussed in the Introduction, the following objective function based on minimizing the quadratic deviation (QD) of the investor's terminal wealth from the terminal wealth of an elevated benchmark has been proposed in the literature (see Van Staden et al. (2023)),

$$(QD(\beta)): \qquad \inf_{\mathcal{P}\in\mathcal{A}} E_{\mathcal{P}}^{t_0,w_0}\left[\left(W(T)-e^{\beta T}\hat{W}(T)\right)^2\right], \qquad \beta>0. \tag{2.2}$$

The QD objective directly and intuitively targets the cumulative outperformance of the investor portfolio relative to the benchmark over $[t_0, T]$, i.e. the tracking difference, while parameter β can be interpreted as the annual (continuously compounded) outperformance spread targeted by the investor. It has been demonstrated (Van Staden et al. (2023)) that robust out-of-sample benchmark outperformance can be obtained using the strategies associated with (2.2).

Since active portfolio managers may also wish to target a favourable tracking difference at intermediate times $t \in [t_0, T]$, instead of only considering the tracking difference *at* the maturity T as in the case of (2.2), we propose the following investment objective in this paper:

$$(CD(\delta)): \quad \begin{cases} \inf_{\mathcal{P}\in\mathcal{A}} E_{\mathcal{P}}^{t_0,w_0}\left[\int_{t_0}^{T}\left(W(t)-e^{\delta t}\hat{W}(t)\right)^2 dt\right], & \delta>0, \quad \text{if } \mathcal{T}=[t_0,T], \qquad (2.3)\\ \inf_{\mathcal{P}\in\mathcal{A}} E_{\mathcal{P}}^{t_0,w_0}\left[\sum_{t\in\mathcal{T}\cup T}\left(W(t)-e^{\delta t}\hat{W}(t)\right)^2\right], & \delta>0, \quad \text{if } \mathcal{T}\subseteq[t_0,T], \mathcal{T} \text{ discrete.} \qquad (2.4)\end{cases}$$

We subsequently refer to (2.3)-(2.4) simply as the CD problem, or as problem $CD(\delta)$ if the value of the parameter δ is to be emphasized. We make the following observations:

(i) Definition (2.3) is subsequently used where, in order to gain the necessary intuition regarding the theoretical properties of the associated optimal investment strategies, the CD problem is analyzed under the assumptions of continuous rebalancing with no investment constraints (Section 3). In contrast, Definition (2.4) is used in more practical settings when there are discrete rebalancing and multiple investment constraints (Section 4). Note that in (2.4), the terminal time T is explicitly included in the objective function, since it is typical for T not to be a rebalancing time ($T \notin \mathcal{T}$) in discrete rebalancing settings. For convenience, we assumed that the tracking difference assessment times in (2.4) correspond to the set of portfolio rebalancing times, although this assumption can be relaxed without difficulty.

(ii) The definition of the CD problem retains the intuitive aspects of the QD problem, with the tracking difference being the quantity of interest that is directly and transparently targeted.

(iii) We intuitively expect a close connection between the QD and CD problems, since as $t_0 \to T$, the results associated with the $CD(\delta)$ problem are expected to resemble the corresponding results of the $QD(\beta)$ problem, provided that $\beta = \delta$. The closed-form solutions of Section 3 confirm this intuition.

(iv) The formulation (2.3)-(2.4) can be extended to allow for different levels of relevance/importance to be attached to the tracking differences at different times in the investment time horizon. For example, considering just the case of continuous rebalancing for the moment, we could define a non-negative deterministic function of time $t \to \varpi(t) \geq 0, \forall t \in [t_0, T]$ giving the "weight" assigned to the tracking difference at time t, and replace the integral in (2.3) with

$$\int_{t_0}^{T} \varpi(t)\cdot\left(W(t)-e^{\delta t}\hat{W}(t)\right)^2 dt, \tag{2.5}$$

along with the corresponding modifications in the case of discrete rebalancing (2.4). The generalization (2.5) might be valuable in certain settings, for example if the investor places more value on the tracking differences closer to maturity T, in which case a function $\varpi(t)$ that is strictly increasing might be used. However, for purposes of concreteness and simplicity, we continue with the proposed definition (2.3), with the numerical results of Section 5 confirming that it yields promising investment results. As a result, we leave further generalizations such as (2.5) for future work.

The remaining sections are devoted to exploring both the analytical properties and practical implications of using the CD problem formulation (2.3)-(2.4) to obtain investment strategies for benchmark outperformance, and comparing the results associated with the QD and CD problems.

Remark 2.1 (Relation between QD and CD). Note that if $\varpi(t) = \mathcal{D}(t-T)$ in equation (2.5), where $\mathcal{D}(t)$ is the Dirac function, then we recover the QD objective function from equation (2.5).

3 Closed-form solutions

To gain insight into CD-optimal investment strategies, in this section we present the closed-form solution to the CD problem (2.3)-(2.4), as well as comparison results for the QD and CD problems, under idealized assumptions. Note that these assumptions are relaxed in Section 4, where a data-driven neural network (numerical) solution approach is presented. However, as subsequently observed (Section 5), the closed-form solutions of this section remain extremely valuable for gaining intuition regarding the behavior of optimal strategies when the stylized assumptions are relaxed. In this section, we specify parametric dynamics for the underlying assets, and in particular we allow for jumps in the risky asset processes and cash contributions to the portfolio, aspects which are not frequently considered in the current benchmark outperformance literature (Bo et al. (2021); Nicolosi et al. (2018),Al-Aradi and Jaimungal (2018); Basak et al. (2006); Browne (1999a,b, 2000); Davis and Lleo (2008); Lim and Wong (2010); Oderda (2015); Tepla (2001); Yao et al. (2006); Zhang and Gao (2017); Zhao (2007)).

We start by summarizing the main assumptions for obtaining closed-form results in this section. These assumptions are typically required in order to obtain closed form solutions for multi-period portfolio optimization (Zhou and Li, 2000). We emphasize that these assumptions, as well as the assumption of parametric dynamics for the underlying assets, are not required in the case of the numerical solution approach in Section 4.

Assumption 3.1. *(Underlying assets, continuous rebalancing, no market frictions) The investor and benchmark invest in the same set of N_a underlying assets, consisting of one risk-free asset and N_a^r risky assets ($N_a = N_a^r + 1$). The investor and benchmark portfolios are rebalanced continuously, so that the set of rebalancing times is $\mathcal{T} = [t_0, T]$. We assume that trading continues in the event of insolvency (i.e. trading continues if $W(t) < 0$ for some $t \in [t_0, T]$). No transaction costs are applicable, no investment constraints (such as leverage or short-selling restrictions) are in effect, and cash is contributed at a constant rate of $q \geq 0$ per year to the investor and benchmark portfolios.*

Remark 3.1. (Trading if insolvent) It is, of course, unrealistic to suppose that an investor can continue to trade and borrow if insolvent. However, this assumption is typically required to obtain closed form solutions, see Zhou and Li (2000) for the case of multi-period mean-variance asset allocation.

Identical cash contributions to the investor and benchmark portfolios as per Assumption 3.1 ensure that the performance of the two portfolios remains meaningfully comparable.

Given the underlying assets as described in Assumption 3.1, we define the proportional allocations to the *risky* assets at time $t \in [t_0, T]$ for the investor and benchmark strategies, respectively, as the vectors $\boldsymbol{\varrho}(t, \boldsymbol{X}(t)) = \left(\varrho_1(t, \boldsymbol{X}(t)), ..., \varrho_{N_a^r}(t, \boldsymbol{X}(t))\right) \in \mathbb{R}^{N_a^r}$ and $\hat{\boldsymbol{\varrho}}\left(t, \hat{\boldsymbol{X}}(t)\right) = \left(\hat{\varrho}_1\left(t, \hat{\boldsymbol{X}}(t)\right), ..., \hat{\varrho}_{N_a^r}\left(t, \hat{\boldsymbol{X}}(t)\right)\right) \in \mathbb{R}^{N_a^r}$. Specifically, $\varrho_i(t, \boldsymbol{X}(t))$ denotes the proportion of the investor's wealth $W(t)$ invested in risky asset $i \in \{1, ..., N_a^r\}$ at time t given information $\boldsymbol{X}(t)$, while $\hat{\varrho}_i\left(t, \hat{\boldsymbol{X}}(t)\right)$ denotes the proportion of benchmark wealth $\hat{W}(t)$ invested in the same asset i at time t given information $\hat{\boldsymbol{X}}(t)$.

We introduce the following assumption regarding the benchmark strategy for the purposes of deriving the closed-form results of this section.

Assumption 3.2. *(Closed-form solutions: Information known about the benchmark strategy) For the closed-form solutions of this section, we assume that the benchmark's risky asset allocation strategy is an adapted feedback control of the form $\hat{\boldsymbol{\varrho}}\left(t, \hat{\boldsymbol{X}}(t)\right) = \hat{\boldsymbol{\varrho}}\left(t, \hat{W}(t)\right), t \in [t_0, T]$, and that the investor is limited to investing in the same set of underlying assets as the benchmark. We also assume that the investor can instantaneously observe the vector $\hat{\boldsymbol{\varrho}}\left(t, \hat{W}(t)\right)$ at each $t \in [t_0, T]$, so that the investor wishes to derive $\boldsymbol{\varrho}(t, \boldsymbol{X}(t)) = \boldsymbol{\varrho}\left(t, W(t), \hat{W}(t), \hat{\boldsymbol{\varrho}}\left(t, \hat{W}(t)\right)\right), t \in [t_0, T]$, the adapted feedback control representing the fraction of the investor's wealth $W(t)$ invested in each risky asset at time t according to the investor's strategy.*

Recalling from the Introduction that constant proportion (i.e. deterministic) benchmark strategies are commonly used in practice by pension funds, it is clear that Assumption 3.2 is sufficiently general, since it allows for any adapted feedback control to serve as the benchmark strategy.

Combining definition (2.1) with Assumption 3.2, for the purposes of this section we therefore consider investor and benchmark strategies, respectively, of the following form,

$$\mathcal{P} = \left\{ \boldsymbol{p}\left(t, \boldsymbol{X}\left(t\right)\right) = \left(1 - \sum_{i=1}^{N_a^r} \varrho_i\left(t, \boldsymbol{X}\left(t\right)\right), \varrho_1\left(t, \boldsymbol{X}\left(t\right)\right), ..., \varrho_{N_a^r}\left(t, \boldsymbol{X}\left(t\right)\right)\right) : t \in [t_0, T] \right\},$$

$$\hat{\mathcal{P}} = \left\{ \hat{\boldsymbol{p}}\left(t, \hat{W}\left(t\right)\right) = \left(1 - \sum_{i=1}^{N_a^r} \hat{\varrho}_i\left(t, \hat{W}\left(t\right)\right), \hat{\varrho}_1\left(t, \hat{W}\left(t\right)\right), ..., \hat{\varrho}_{N_a^r}\left(t, \hat{W}\left(t\right)\right)\right) : t \in [t_0, T] \right\}, \quad (3.1)$$

where $\boldsymbol{X}(t) = \left(W(t), \hat{W}(t), \hat{\boldsymbol{\varrho}}\left(t, \hat{W}(t)\right)\right)$. In this section, the *risky* asset allocations $\boldsymbol{\varrho}(t, \boldsymbol{X}(t))$ and $\hat{\boldsymbol{\varrho}}\left(t, \hat{W}(t)\right)$ will informally be referred to as the investor and benchmark strategies, respectively, due to the form of (3.1). However, in more general settings (e.g. the numerical results of Section 4), the formal definition (2.1) will be used.

Given Assumption 3.1 and Assumption 3.2, the investor's set of admissible controls is given in terms of the risky asset allocation $\boldsymbol{\varrho}$ as

$$\mathcal{A}_0 = \left\{ \boldsymbol{\varrho}\left(t, w, \hat{w}, \hat{\boldsymbol{\varrho}}\left(t, w\right)\right) \middle| \, \boldsymbol{\varrho} : [t_0, T] \times \mathbb{R}^{N_a^r + 2} \to \mathbb{R}^{N_a^r} \right\}, \quad (3.2)$$

so that the investment problems analyzed in this section are given by

$$(QD(\beta)) : \quad \inf_{\boldsymbol{\varrho} \in \mathcal{A}_0} E_{\boldsymbol{\varrho}}^{t_0, w_0}\left[\left(W(T) - e^{\beta T}\hat{W}(T)\right)^2\right], \quad \beta > 0, \quad (3.3)$$

$$(CD(\delta)) : \quad \inf_{\boldsymbol{\varrho} \in \mathcal{A}_0} E_{\boldsymbol{\varrho}}^{t_0, w_0}\left[\int_{t_0}^{T} \left(W(t) - e^{\delta t}\hat{W}(t)\right)^2 dt\right], \quad \delta > 0. \quad (3.4)$$

Note that we use definition (2.3) of the CD problem since $\mathcal{T} = [t_0, T]$ by Assumption 3.1.

3.1 Wealth dynamics for closed-form solutions

The closed-form solutions of (3.3)-(3.4) require the specification of underlying dynamics. The risk-free asset is assumed to have unit value $S_0(t)$ with dynamics in terms of the risk-free rate $r > 0$ given by

$$dS_0(t) = rS_0(t)\, dt. \quad (3.5)$$

In the case of the risky assets, the vector $\boldsymbol{S}(t) = \left(S_i(t) : i = 1, ..., N_a^r\right)^\top$ has ith component $S_i(t)$ which denotes the unit value of the risky asset i at time $t \in [t_0, T]$. The superscript "$\top$" denotes the transpose. For the dynamics of $S_i(t)$, in this section we allow for any of the popular finite-activity jump-diffusion models in finance (see for example Kou (2002); Merton (1976)).

Let $\boldsymbol{\xi} = \left(\xi_i : i = 1, ..., N_a^r\right)^\top$, where ξ_i denotes the random variable with corresponding probability density function (pdf) $f_{\xi_i}(\xi_i)$ representing the jump multiplier associated with the ith risky asset. Let

$$\kappa_i^{(1)} = \mathbb{E}\left[\xi_i - 1\right], \qquad \kappa_i^{(2)} = \mathbb{E}\left[(\xi_i - 1)^2\right], \qquad i = 1, ..., N_a^r, \quad (3.6)$$

and define $\boldsymbol{\kappa}^{(1)} = \left(\kappa_i^{(1)} : i = 1, ..., N_a^r\right)^\top$ and $\boldsymbol{\kappa}^{(2)} = \left(\kappa_i^{(2)} : i = 1, ..., N_a^r\right)^\top$. If a jump occurs in the dynamics of risky asset i at time t, its value jumps from $S_i(t^-)$ to $S_i(t) = \xi_i \cdot S_i(t^-)$, where, given any functional $\psi(t), t \in [t_0, T]$, we use the notation $\psi(t^-)$ and $\psi(t^+)$ as shorthand for the one-sided limits $\psi(t^-) = \lim_{\epsilon \downarrow 0} \psi(t - \epsilon)$ and $\psi(t^+) = \lim_{\epsilon \downarrow 0} \psi(t + \epsilon)$, respectively. For ease of exposition, we assume that $\boldsymbol{\xi}$ has independent components, i.e. the jump components of the different risky asset processes are independent, while dependence will be introduced via the diffusion components. Note that the assumption of independent jumps can be relaxed without any technical difficulty (Kou (2007)) at the cost of significantly increasing the notational complexity.

Let $\boldsymbol{Z}(t) = \left(Z_i(t) : i = 1, ..., N_a^r\right)^\top$ denote a standard N_a^r-dimensional Brownian motion, while $\boldsymbol{\mu} = \left(\mu_i : i = 1, ..., N_a^r\right)^\top$ denote the drift coefficients of the risky assets under the objective (or real-world) probability measure and $\boldsymbol{\sigma} = \left(\sigma_{i,j}\right)_{i,j=1,...,N_a^r} \in \mathbb{R}^{N_a^r \times N_a^r}$ denotes the volatility matrix. Let $\boldsymbol{\pi}(t) = \left(\pi_i(t) : i = 1, ..., N_a^r\right)^\top$ denote a vector of N_a^r independent Poisson processes, with each $\pi_i(t)$ having the corresponding intensity $\lambda_i \geq 0$, and define $\boldsymbol{\lambda} = \left(\lambda_i : i = 1, ..., N_a^r\right)^\top$. We assume that ξ_i, $\pi_j(t)$ and $Z_k(t)$ are mutually independent for all

$i,j,k \in \{1,...,N_a^r\}$. Define the matrices

$$\boldsymbol{\Sigma} = \boldsymbol{\sigma}\boldsymbol{\sigma}^\top, \qquad \boldsymbol{\Lambda} = \text{diag}\left(\lambda_i \kappa_i^{(2)} : i = 1,...,N_a^r\right). \tag{3.7}$$

We make the standard assumptions that $\mu_i > r$, for all i, and assume that the covariance matrix $\boldsymbol{\Sigma} = \boldsymbol{\sigma}\boldsymbol{\sigma}^\top$ is positive definite (see for example Bjork (2009); Zhou and Li (2000)). We also define the following combinations of parameters from the underlying asset dynamics,

$$\boldsymbol{\alpha} = \left(\mu_i - r - \lambda_i \kappa_i^{(1)} : i = 1,...,N_a^r\right)^\top, \qquad \tilde{\boldsymbol{\mu}} = \left(\mu_i - r : i = 1,...,N_a^r\right)^\top, \tag{3.8}$$

$$\eta = \tilde{\boldsymbol{\mu}}^\top \cdot (\boldsymbol{\Sigma} + \boldsymbol{\Lambda})^{-1} \cdot \tilde{\boldsymbol{\mu}}. \tag{3.9}$$

The dynamics of $S_i(t)$ is therefore assumed to be of the form

$$\frac{dS_i(t)}{S_i(t^-)} = \left(\mu_i - \lambda_i \kappa_i^{(1)}\right) \cdot dt + \sum_{j=1}^{N_a^r} \sigma_{ij} \cdot dZ_j(t) + d\left(\sum_{k=1}^{\pi_i(t)} \left(\xi_i^{(k)} - 1\right)\right), \qquad i = 1,...,N_a^r, \tag{3.10}$$

where $\xi_i^{(k)}$ are i.i.d. random variables with the same distribution as ξ_i. To simplify notation, define the vector $d\boldsymbol{\mathcal{N}}(t) = \left(\int_0^\infty (\xi_i - 1) N_i(dt, d\xi_i) : i = 1,...,N_a^r\right)^\top$, where N_i is the Poisson random measure (Oksendal and Sulem (2019)) corresponding to the dynamics of $S_i(t)$ in (3.10).

Recalling that $q \geq 0$ denotes the constant rate (per year) at which cash is contributed to each portfolio (Assumption 3.1), the investor and benchmark wealth processes for the purposes of obtaining closed-form solutions are as follows,

$$\begin{aligned} dW(t) &= \left\{W(t^-) \cdot \left[r + \boldsymbol{\alpha}^\top \boldsymbol{\varrho}(t, \boldsymbol{X}(t^-))\right] + q\right\} \cdot dt + W(t^-) \left(\boldsymbol{\varrho}(t, \boldsymbol{X}(t^-))\right)^\top \boldsymbol{\sigma} \cdot d\boldsymbol{Z}(t) \\ &\quad + W(t^-) \left(\boldsymbol{\varrho}(t, \boldsymbol{X}(t^-))\right)^\top \cdot d\boldsymbol{\mathcal{N}}(t), \end{aligned} \tag{3.11}$$

$$\begin{aligned} d\hat{W}(t) &= \left\{\hat{W}(t^-) \cdot \left[r + \boldsymbol{\alpha}^\top \hat{\boldsymbol{\varrho}}\left(t, \hat{W}(t^-)\right)\right] + q\right\} \cdot dt + \hat{W}(t^-) \left(\hat{\boldsymbol{\varrho}}\left(t, \hat{W}(t^-)\right)\right)^\top \boldsymbol{\sigma} \cdot d\boldsymbol{Z}(t) \\ &\quad + \hat{W}(t^-) \left(\hat{\boldsymbol{\varrho}}\left(t, \hat{W}(t^-)\right)\right)^\top \cdot d\boldsymbol{\mathcal{N}}(t), \end{aligned} \tag{3.12}$$

for $t \in (t_0, T]$, where $W(t) = \hat{W}(t) = w_0$ and $\boldsymbol{X}(t) = \left(W(t), \hat{W}(t), \hat{\boldsymbol{\varrho}}\left(t, \hat{W}(t)\right)\right)$.

3.2 Closed-form solution: $QD(\beta)$ problem

For subsequent reference, the following proposition recalls the closed-form solution for the QD-optimal control available in the literature.

Proposition 3.3. *(QD-optimal control) Suppose that Assumption 3.1, Assumption 3.2 and wealth dynamics (3.11)-(3.12) are applicable. Then the optimal fraction of the investor's wealth to be invested in risky asset* $i \in \{1,...,N_a^r\}$ *for problem* $QD(\beta)$ *in (3.3) is given by the* i^{th} *component of the vector* $\boldsymbol{\varrho}_{qd}^*\left(t, \boldsymbol{X}_{qd}^*(t^-;\beta);\beta\right)$, *where*

$$\begin{aligned} W_{qd}^*(t^-;\beta) \cdot \boldsymbol{\varrho}_{qd}^*\left(t, \boldsymbol{X}_{qd}^*(t^-;\beta);\beta\right) &= \left[h_{qd}(t;\beta,q) - \left(W_{qd}^*(t^-;\beta) - e^{\beta T}\hat{W}(t^-)\right)\right] \cdot (\boldsymbol{\Sigma} + \boldsymbol{\Lambda})^{-1} \tilde{\boldsymbol{\mu}} \\ &\quad + e^{\beta T}\hat{W}(t^-) \cdot \hat{\boldsymbol{\varrho}}\left(t, \hat{W}(t^-)\right), \end{aligned} \tag{3.13}$$

with $W_{qd}^*(t;\beta)$ *denoting the investor's wealth process (3.11) under the QD(*β*)-optimal control* $\boldsymbol{\varrho}_{qd}^*$, *and* $\boldsymbol{X}_{qd}^*(t;\beta) = \left(W_{qd}^*(t;\beta), \hat{W}(t), \hat{\boldsymbol{\varrho}}\left(t, \hat{W}(t)\right)\right)$. *Here,* h_{qd} *is the following deterministic function,*

$$h_{qd}(t;\beta,q) \coloneqq q\left(e^{\beta T} - 1\right) \cdot \int_t^T e^{-r(u-t)} du = \frac{q}{r}\left(e^{\beta T} - 1\right)\left(1 - e^{-r(T-t)}\right), \quad t \in [t_0, T]. \tag{3.14}$$

Proof. See Van Staden et al. (2023). ☐

As shown in Van Staden et al. (2023), implementing (3.13) can be viewed as pursuing (at time t) a targeted level of wealth given by $e^{\beta T}\hat{W}(t)$. In other words the wealth target is a *multiple* ($e^{\beta T}$) of the benchmark wealth $\hat{W}(t)$, and for subsequent reference we observe that the multiplier $e^{\beta T}$ remains constant throughout the time horizon $[t_0, T]$. Previous work showed that the QD-optimal strategy delivers excellent performance out-of-sample relative to maximizing the information ratio (IR), which is another popular objective in practice (Goetzmann et al. (2002, 2007); Van Staden et al. (2023)).

3.3 Closed-form solution: $CD(\delta)$ problem

We now derive the closed-form solution of the CD problem (3.4), starting with the HJB partial integro-differential equation (PIDE) satisfied by its value function.

Theorem 3.4. *(CD problem: Verification theorem) Fix $\delta > 0$. Suppose that for all $(t, w, \hat{w}, \hat{\boldsymbol{\varrho}}) \in [t_0, T] \times \mathbb{R}^{N_a^r+2}$, there exist functions $V_{cd}(t, w, \hat{w}, \hat{\boldsymbol{\varrho}}) : [t_0, T] \times \mathbb{R}^{N_a^r+2} \to \mathbb{R}$ and $\boldsymbol{\varrho}^*_{cd}(t, w, \hat{w}, \hat{\boldsymbol{\varrho}}; \delta) : [t_0, T] \times \mathbb{R}^{N_a^r+2} \to \mathbb{R}^{N_a^r}$ with the following two properties. (i) V_{cd} and $\boldsymbol{\varrho}^*_{cd}$ are sufficiently smooth and solve the HJB PIDE (3.15)-(3.16), and (ii) the function $\boldsymbol{\varrho}^*_{cd}(t, w, \hat{w}, \hat{p}; \delta)$ attains the pointwise supremum in (3.15).*

$$\frac{\partial V_{cd}}{\partial t} + \left(w - e^{\delta t}\hat{w}\right)^2 + \inf_{\boldsymbol{\varrho} \in \mathbb{R}^{N_a^r}} \left\{ \mathcal{H}(\boldsymbol{\varrho}; t, w, \hat{w}, \hat{\boldsymbol{\varrho}}) \right\} = 0, \tag{3.15}$$

$$V_{cd}(T, w, \hat{w}, \hat{\boldsymbol{\varrho}}) = 0, \tag{3.16}$$

where

$$\begin{aligned}
\mathcal{H}(\boldsymbol{\varrho}; t, w, \hat{w}, \hat{\boldsymbol{\varrho}}) = {} & \left(w \cdot \left[r + \boldsymbol{\alpha}^\top \boldsymbol{\varrho}\right] + q\right) \cdot \frac{\partial V_{cd}}{\partial w} + \left(\hat{w} \cdot \left[r + \boldsymbol{\alpha}^\top \hat{\boldsymbol{\varrho}}\right] + q\right) \cdot \frac{\partial V_{cd}}{\partial \hat{w}} - \left(\sum_{i=1}^{N_a^r} \lambda_i\right) \cdot V_{cd} \\
& + \frac{1}{2} w^2 \cdot \left(\boldsymbol{\varrho}^\top \boldsymbol{\Sigma} \boldsymbol{\varrho}\right) \cdot \frac{\partial^2 V_{cd}}{\partial w^2} + \frac{1}{2} \hat{w}^2 \cdot \left(\hat{\boldsymbol{\varrho}}^\top \boldsymbol{\Sigma} \hat{\boldsymbol{\varrho}}\right) \cdot \frac{\partial^2 V_{cd}}{\partial \hat{w}^2} + w\hat{w} \cdot \left(\boldsymbol{\varrho}^\top \boldsymbol{\Sigma} \hat{\boldsymbol{\varrho}}\right) \cdot \frac{\partial^2 V_{cd}}{\partial w \partial \hat{w}} \\
& + \sum_{i=1}^{N_a^r} \lambda_i \int_0^\infty V_{cd}\left(w + \varrho_i w(\xi_i - 1), \hat{w} + \hat{\varrho}_i \hat{w}(\xi_i - 1), t\right) f_{\xi_i}(\xi_i)\, d\xi_i.
\end{aligned} \tag{3.17}$$

Then under Assumption 3.1, Assumption 3.2 and wealth dynamics (3.11)-(3.12), V_{cd} is the value function and $\boldsymbol{\varrho}^*_{cd}$ is the optimal control for the $CD(\delta)$ problem (3.4).

Proof. See Appendix A.1. □

We proceed to solve the HJB PIDE (3.15)-(3.16), with Proposition 3.5 presenting the CD-optimal control.

Proposition 3.5. *(CD-optimal control) Suppose that Assumption 3.1, Assumption 3.2 and wealth dynamics (3.11)-(3.12) are applicable. Then the optimal fraction of the investor's wealth to be invested in risky asset $i \in \{1, ..., N_a^r\}$ for problem $CD(\delta)$ with continuous rebalancing (3.4) is given by the i^{th} component of the vector $\boldsymbol{\varrho}^*_{cd}(t, \boldsymbol{X}^*_{cd}(t^-; \delta); \delta)$, where*

$$\begin{aligned}
W^*_{cd}(t^-; \delta) \cdot \boldsymbol{\varrho}^*_{cd}\left(t, \boldsymbol{X}^*_{cd}(t^-; \delta); \delta\right) = {} & \left[h_{cd}(t; \delta, q) - \left(W^*_{cd}(t^-; \delta) - g_{cd}(t; \delta)\hat{W}(t^-)\right)\right] \cdot (\boldsymbol{\Sigma} + \boldsymbol{\Lambda})^{-1} \tilde{\boldsymbol{\mu}} \\
& + g_{cd}(t; \delta)\hat{W}(t^-) \cdot \hat{\boldsymbol{\varrho}}\left(t, \hat{W}(t^-)\right),
\end{aligned} \tag{3.18}$$

with $W^*_{cd}(t; \delta)$ denoting the investor's wealth process (3.11) under the $CD(\delta)$-optimal control $\boldsymbol{\varrho}^*_{cd}$, and $\boldsymbol{X}^*_{cd}(t; \delta) = \left(W^*_{cd}(t; \delta), \hat{W}(t), \hat{\boldsymbol{\varrho}}\left(t, \hat{W}(t)\right)\right)$. Here, h_{cd} and g_{cd} are the following deterministic functions of time,

$$h_{cd}(t; \delta, q) = -\frac{F(t; \delta, q)}{2A(t)}, \qquad g_{cd}(t; \delta) = -\frac{D(t; \delta)}{2A(t)}, \tag{3.19}$$

where A, D and F are respectively given by the following functions,

$$A(t) = \frac{e^{(2r-\eta)(T-t)} - 1}{(2r - \eta)}, \qquad D(t; \delta) = \frac{2e^{\delta T}}{(2r - \eta + \delta)} \left[e^{-\delta(T-t)} - e^{(2r-\eta)(T-t)}\right], \tag{3.20}$$

and

$$F(t;\delta,q) = \frac{2q}{2r-\eta}\left[\frac{e^{(2r-\eta)(T-t)}-e^{(r-\eta)(T-t)}}{r}-\frac{e^{(r-\eta)(T-t)}-1}{r-\eta}\right]$$
$$+\frac{2qe^{\delta T}}{2r-\eta+\delta}\left[\frac{e^{(r-\eta)(T-t)}-e^{-\delta(T-t)}}{r-\eta+\delta}-\frac{\left(e^{(2r-\eta)(T-t)}-e^{(r-\eta)(T-t)}\right)}{r}\right]. \quad (3.21)$$

Proof. See Appendix A.2. □

We therefore observe that the CD-optimal control (3.18) has a similar functional form to the QD-optimal control (3.13). Specifically, (3.18) involves a multiple $g_{cd}(t;\delta)$ of the benchmark wealth $\hat{W}(t)$, which is now time-dependent (and therefore non-constant), while the role of the contributions remains limited to the term $h_{cd}(t;\delta,q)$. Analyzing the properties of the functions $g_{cd}(t;\delta)$ and $h_{cd}(t;\delta,q)$ in (3.19) is therefore not only helpful for the purposes of gaining intuition regarding the behavior of the CD-optimal control (3.18), but also for rigorously proving the subsequent comparison results. For convenience, we summarize some of the key properties of $g_{cd}(t;\delta)$ and $h_{cd}(t;\delta,q)$ to aid the intuition, with further details provided in Appendix A.3 and Appendix A.4:

(i) Summary of the properties of $g_{cd}(t;\delta)$: For any $t\in[t_0=0,T]$, $\delta\to g_{cd}(t;\delta)$ is strictly increasing on $\delta\in(0,\infty)$, while for any fixed $\delta>0$, $t\to g_{cd}(t;\delta)$ is strictly increasing on $t\in[t_0,T]$ to a maximum of $g_{cd}(T;\delta)=e^{\delta T}$. In fact, we have the bounds

$$e^{\delta t} < g_{cd}(t;\delta) < e^{\delta T}, \qquad \forall t\in[t_0,T). \quad (3.22)$$

See Appendix A.3 for a proof of these properties.

(ii) Summary of the properties of $h_{cd}(t;\delta,q)$: If $q=0$, it is clear that $h_{cd}(t;\delta,q)\equiv 0$, while we always have $h_{cd}(T;\delta,q)=0$. For any $t\in[t_0=0,T]$ and $\delta>0$, $q\to h_{cd}(t;\delta,q)$ is strictly increasing on $q\in[0,\infty)$, with $\delta\to h_{cd}(t;\delta,q)$ being strictly increasing on $\delta\in(0,\infty)$. In addition, h_{cd} satisfies the bounds

$$0 \le h_{cd}(t;\delta,q) \le h_{qd}(t;\beta=\delta,q), \qquad \forall t\in[t_0=0,T], \quad (3.23)$$

where $h_{qd}(t;\beta,q)$ is given by (3.14). See Appendix A.4 for a proof of these properties.

For the purposes of interpreting the subsequent results, the key intuition is that the CD investor implementing (3.18) can be viewed as pursuing (at time t) a targeted level of $W^*_{cd}(t;\delta)$ given by $g_{cd}(t;\delta)\cdot\hat{W}(t)$, qualitatively similar to the case of the QD investor pursuing a targeted level of $e^{\beta T}\cdot\hat{W}(t)$. However, unlike the QD investor implementing a constant multiplier, the CD investor uses a multiplier $g_{cd}(t;\delta)$ that increases over time up to a maximum of $e^{\delta T}$, always remaining within the bounds (3.22). Therefore, if we were to compare the $CD(\delta)$ and $QD(\beta=\delta)$ optimal controls, (3.22) shows that the CD investor has a smaller implicit benchmark outperformance target throughout the investment time horizon, with the difference likely to be especially pronounced early in the investment time horizon (t close to $t_0=0$).

With this intuition in mind, we now present some closed-form comparison results for the $QD(\beta)$- and $CD(\delta)$-optimal investment strategies.

3.4 Comparison of investment strategies

To lighten notation for the analysis of this subsection, we suppress the dependence of the optimal controls on $\boldsymbol{X}^*_k, k\in\{cd,qd\}$, and denote the optimal allocation to the risky assets simply by

$$\boldsymbol{\varrho}^*_{qd}\left(t,\boldsymbol{X}^*_{qd}(t;\beta);\beta\right) := \boldsymbol{\varrho}^*_{qd}(t;\beta)=\left(\varrho^*_{qd,k}(t;\beta):k=1,...,N^r_a\right), \quad (3.24)$$
$$\boldsymbol{\varrho}^*_{cd}\left(t,\boldsymbol{X}^*_{cd}(t;\delta);\delta\right) := \boldsymbol{\varrho}^*_{cd}(t;\delta)=\left(\varrho^*_{cd,k}(t;\delta):k=1,...,N^r_a\right). \quad (3.25)$$

Similarly, for the benchmark, we suppress dependence on $\hat{W}(t)$ and use the notation $\hat{\boldsymbol{\varrho}}\left(t,\hat{W}(t)\right)=\left(\hat{\varrho}_k(t):k=1,...,N^r_a\right)$. We emphasize that this is just for convenience, as the benchmark strategy is not required to be deterministic (see Assumption 3.2).

For the subsequent analysis, it is helpful to define the total allocation by each strategy to the risky asset *basket* as

$$\mathcal{R}_{qd}^{*}(t;\beta)=\sum_{k=1}^{N_a^r}\varrho_{qd,k}^{*}(t;\beta), \qquad \mathcal{R}_{cd}^{*}(t;\delta)=\sum_{k=1}^{N_a^r}\varrho_{cd,k}^{*}(t;\delta), \qquad \hat{\mathcal{R}}(t)=\sum_{k=1}^{N_a^r}\hat{\varrho}_k(t). \tag{3.26}$$

In the case of continuous-time mean-variance optimization (i.e. without a benchmark present), it can be shown that the optimal risky basket composition does not depend on the state of the system (Zhou and Li (2000)), whereas the QD-optimal risky asset basket composition is only weakly dependent on the state in that particular ratios involving the risky asset allocations remain constant (Van Staden et al. (2023)). The following corollary can be interpreted as showing that in the case of the CD-optimal investment strategy, the risky asset basket composition is also only weakly dependent on the state of the system.

Corollary 3.6. *(Risky asset basket ratios) Let Assumption 3.1, Assumption 3.2 and wealth dynamics (3.11)-(3.12) hold. Note that* $W_{cd}^{*}(t;\delta), W_{qd}^{*}(t;\beta)$ *and* $\hat{\mathcal{R}}(t)$ *represent information known to the investor at time* t*. For any values of* $\beta,\delta>0$*, the total optimal risky asset basket allocations* $\mathcal{R}_{cd}^{*}(t;\delta)$ *and* $\mathcal{R}_{qd}^{*}(t;\beta)$ *can be obtained from the following constant ratios,*

$$\begin{aligned}\frac{W_{cd}^{*}(t;\delta)\cdot\mathcal{R}_{cd}^{*}(t;\delta)-g_{cd}(t;\delta)\hat{W}(t)\cdot\hat{\mathcal{R}}(t)}{\left[g_{cd}(t;\delta)\hat{W}(t)+h_{cd}(t;\delta,q)\right]-W_{cd}^{*}(t;\delta)} &= \frac{W_{qd}^{*}(t;\beta)\cdot\mathcal{R}_{qd}^{*}(t;\beta)-e^{\beta T}\hat{W}(t)\cdot\hat{\mathcal{R}}(t)}{\left[e^{\beta T}\hat{W}(t)+h_{qd}(t;\beta,q)\right]-W_{qd}^{*}(t;\beta)}\\ &= \sum_{k=1}^{N_a^r}\left[(\boldsymbol{\Sigma}+\boldsymbol{\Lambda})^{-1}\tilde{\boldsymbol{\mu}}\right]_k.\end{aligned} \tag{3.27}$$

For any values of $\beta,\delta>0$*, the allocation within each risky asset basket to asset* $i\in\{1,...,N_a^r\}$ *can be determined from the following constant ratios,*

$$\begin{aligned}\frac{W_{cd}^{*}(t;\delta)\cdot\varrho_{cd,i}^{*}(t;\delta)-g_{cd}(t;\delta)\hat{W}(t)\hat{\varrho}_i(t)}{W_{cd}^{*}(t;\delta)\cdot\mathcal{R}_{cd}^{*}(t;\delta)-g_{cd}(t;\delta)\hat{W}(t)\cdot\hat{\mathcal{R}}(t)} &= \frac{W_{qd}^{*}(t;\beta)\cdot\varrho_{qd,i}^{*}(t;\beta)-e^{\beta T}\hat{W}(t)\cdot\hat{\varrho}_i(t)}{W_{qd}^{*}(t;\beta)\cdot\mathcal{R}_{qd}^{*}(t;\beta)-e^{\beta T}\hat{W}(t)\cdot\hat{\mathcal{R}}(t)}\\ &= \frac{\left[(\boldsymbol{\Sigma}+\boldsymbol{\Lambda})^{-1}\tilde{\boldsymbol{\mu}}\right]_i}{\sum_{k=1}^{N_a^r}\left[(\boldsymbol{\Sigma}+\boldsymbol{\Lambda})^{-1}\tilde{\boldsymbol{\mu}}\right]_k}.\end{aligned} \tag{3.28}$$

Proof. In (3.27)-(3.28), $[\boldsymbol{v}]_k$ denotes the kth component of the vector $\boldsymbol{v}$. The results follow from combining the results of Proposition 3.3, Proposition 3.5 and (3.26). □

Proposition 3.27 shows that as a result of the constant ratios (3.27)-(3.28), it is sufficient to consider a *single* well-diversified stock index (i.e. a single "risky asset") when illustrating the analytical solutions in Subsection 5.2, as this would give the necessary intuition regarding the behavior of these closed-form optimal strategies. This intuition is helpful for understanding the behavior of the strategies when the conditions of Proposition 3.27 no longer hold, such as when applying multiple investment constraints in the results of Subsection 5.3, where multiple risky assets are considered.

While the results of Corollary 3.6 are general in that, subject to the stated assumptions, (3.27)-(3.28) hold for any values of $\beta,\delta>0$, properties such as (3.22) and (3.23) suggest that the QD- and CD-optimal investment strategies exhibit a number of differences over time. To analyze the strategies in more detail, a reasonable basis for the comparison is required (i.e. specific choices of the values of β and δ), and two possibilities are immediately available:

(i) Comparing investment strategies on the basis of equal expectation of terminal wealth: the parameters $\delta^{\mathcal{E}}$ and $\beta^{\mathcal{E}}$ are selected for the $CD\left(\delta=\delta^{\mathcal{E}}\right)$ and $QD\left(\beta=\beta^{\mathcal{E}}\right)$ problems, respectively, such that

$$E_{\boldsymbol{\varrho}_{cd}^{*}}^{t_0,w_0}\left[W_{cd}^{*}\left(T;\delta=\delta^{\mathcal{E}}\right)\right]\equiv E_{\boldsymbol{\varrho}_{qd}^{*}}^{t_0,w_0}\left[W_{qd}^{*}\left(T;\beta=\beta^{\mathcal{E}}\right)\right]\equiv\mathcal{E}. \tag{3.29}$$

While (3.29) provides a very intuitive basis for comparing strategies and wealth distributions (see for example Van Staden et al. (2021)), we show in Appendix B.1 (Proposition B.2) that (3.29) also implies that

$$\delta^{\mathcal{E}} \quad > \quad \beta^{\mathcal{E}}. \tag{3.30}$$

However, in general we have to solve numerically for the parameter values $\delta^{\mathcal{E}}$ and $\beta^{\mathcal{E}}$ satisfying (3.29). Hence obtaining analytically tractable comparison results on the basis of (3.29) is very challenging.

(ii) Comparing investment strategies on the basis of equal parameters $\delta = \beta$: while comparing the results of the $CD(\delta)$ and $QD(\beta = \delta)$ problems are also intuitive due to the role of these parameters in their respective objective functions, we show in Appendix B.1 (Proposition B.1) that setting $\delta = \beta$ implies that

$$E^{t_0,w_0}_{\boldsymbol{\varrho}^*_{cd}}\left[W^*_{cd}(t;\delta)\right] \;<\; E^{t_0,w_0}_{\boldsymbol{\varrho}^*_{qd}}\left[W^*_{qd}(t;\beta=\delta)\right], \qquad \forall t \in (t_0, T]. \tag{3.31}$$

As a result, (3.31) implies that for example the comparison of terminal wealth distributions will be significantly less intuitive if we simply set $\delta = \beta$.

Since these are clearly distinct but reasonable possibilities for comparing strategies, we proceed as follows: since the assumption of equal parameters ($\delta = \beta$) makes the comparison of investment strategies amenable to analysis, we set $\delta = \beta$ in the derivation of analytical comparison results in the remainder of this section. However, we use $\delta^{\mathcal{E}} > \beta^{\mathcal{E}}$ to compare results on the basis of (3.29) in the numerical results of Section 5 below. Finally, in Appendix B, we combine both possibilities by comparing the results of the $CD\left(\delta^{\mathcal{E}}\right)$, $CD\left(\delta=\beta^{\mathcal{E}}\right)$ and $QD\left(\beta^{\mathcal{E}}\right)$ problems, concluding that the difference $\left(\delta^{\mathcal{E}} - \beta^{\mathcal{E}}\right) > 0$ in (3.30) is typically sufficiently small such that the conclusions from the analytical results (obtained by setting $\delta = \beta$) still remain qualitatively accurate regardless of the basis of comparison.

Proposition 3.7 compares the $CD(\delta)$-optimal and $QD(\beta = \delta)$-optimal risky asset basket allocations at the two endpoints of the investment time horizon ($t = t_0 \equiv 0$ and $t = T$). As will be discussed in Section 5, Proposition 3.7 is particularly helpful in explaining the respective asset allocation profiles over time, as well as the resulting out-of-sample investment results.

Proposition 3.7. *(Comparison - allocation to risky asset basket: $CD(\delta)$ and $QD(\beta=\delta)$) Suppose that Assumption 3.1, Assumption 3.2 and wealth dynamics (3.11)-(3.12) are applicable. Recall the investment time horizon is given by $t \in [t_0 = 0, T]$.*

At time $t = t_0$, if the total benchmark risky asset basket allocation satisfies $\hat{\mathcal{R}}(t_0) = \sum_{k=1}^{N_a^r} \hat{\varrho}_k(t_0, w_0) \geq 0$, we have

$$\mathcal{R}^*_{qd}(t_0;\beta=\delta) \;>\; \mathcal{R}^*_{cd}(t_0;\delta). \tag{3.32}$$

At time $t = T$, we have

$$E^{t_0,w_0}_{\boldsymbol{\varrho}^*_{qd}}\left[W^*_{qd}(T;\beta=\delta)\cdot\mathcal{R}^*_{qd}(T;\beta=\delta)\right] \;<\; E^{t_0,w_0}_{\boldsymbol{\varrho}^*_{cd}}\left[W^*_{cd}(T;\delta)\cdot\mathcal{R}^*_{cd}(T;\delta)\right]. \tag{3.33}$$

Proof. See Appendix B.2. □

Note that Proposition 3.7 does not require any information regarding the functional form of the benchmark strategy $\hat{\boldsymbol{\varrho}}\left(t, \hat{W}(t)\right)$, while (3.32) specifies only a very weak condition, namely that $\hat{\mathcal{R}}(t_0) > 0$.

Proposition 3.7 therefore shows that compared to the QD($\beta = \delta$)-optimal strategy, the CD(δ)-optimal strategy allocates *less* wealth to the risky asset basket early in the investment time horizon ($t = t_0$), but is expected to allocate *more* wealth to the risky asset basket at maturity ($t = T$). Comparing the QD- and CD-optimal allocations to individual risky assets, we have the following corollary to Proposition 3.7.

Corollary 3.8. *(Comparison - allocation to risky asset $i \in \{1, ..., N_a^r\}$: $CD(\delta)$ and $QD(\beta=\delta)$) Suppose that Assumption 3.1, Assumption 3.2 and wealth dynamics (3.11)-(3.12) are applicable. For any risky asset $i \in \{1, ..., N_a^r\}$, the following comparison results hold.*

At time $t = t_0$, if the benchmark allocation to risky asset $i \in \{1, ..., N_a^r\}$ satisfies $\hat{\varrho}_i(t_0, w_0) \geq 0$, we have

$$\varrho^*_{qd,i}(t_0;\beta=\delta) \;>\; \varrho^*_{cd,i}(t_0;\delta). \tag{3.34}$$

At time $t = T$, we have

$$E^{t_0,w_0}_{\boldsymbol{\varrho}^*_{qd}}\left[W^*_{qd}(T;\beta=\delta)\cdot\varrho^*_{qd,i}(T;\beta=\delta)\right] \;<\; E^{t_0,w_0}_{\boldsymbol{\varrho}^*_{cd}}\left[W^*_{cd}(T;\delta)\cdot\varrho^*_{cd,i}(T;\delta)\right]. \tag{3.35}$$

Proof. See Appendix B.3. □

In the numerical results of Section 5, we demonstrate that even when the assumptions of this section (e.g. Assumption 3.1, Assumption 3.2 and wealth dynamics (3.11)-(3.12)) no longer hold, the conclusions regarding the relative risky asset allocation profiles given in Proposition 3.7 and Corollary 3.8 remain qualitatively applicable, with important implications for the out-of-sample benchmark outperformance of the strategies.

3.5 The limits of benchmark outperformance

As discussed in the Introduction, the benchmark strategies $\hat{\boldsymbol{\varrho}}$ chosen in practice by large pension funds are typically simple constant proportion investment strategies. However, it is worth investigating the feasibility of outperforming a benchmark strategy that is already "better" in some sense than a constant proportion strategy. This would naturally require the specification of the preferred alternative benchmark strategy, as well as the sense in which it can be considered "better" than a standard constant proportion benchmark strategy.

For purposes of concreteness, in this section we consider the QD- and CD-optimal investment strategies themselves as benchmark strategies to be outperformed. While outperforming a constant proportion benchmark using the QD- and CD-optimal strategies is not too challenging (see Section 5), the subsequent results show that outperforming benchmark strategies of increasing sophistication could require taking on more risk. Specifically, as shown in the numerical results of Section 5, taking on more risk in the sense of increasing the allocation to the risky asset basket unsurprisingly also increases the likelihood of ultimately underperforming the benchmark.

To gain the necessary intuition, we continue working under the stylized assumptions of this section (we emphasize that these assumptions are relaxed in Section 4 and Section 5), and introduce the following definition.

Definition 3.1. *(m-compounded optimal investment strategies) Suppose that a benchmark strategy $\hat{\boldsymbol{\varrho}}$ satisfying Assumption 3.2 is given. Fix values of $\beta > 0$ and $\delta > 0$. Under Assumption 3.1 and wealth dynamics (3.11)-(3.12), the $QD(\beta)$-optimal strategy $\boldsymbol{\varrho}^*_{qd}$ and $CD(\delta)$-optimal strategy $\boldsymbol{\varrho}^*_{cd}$ are therefore available as a result of Propositions 3.3 and 3.5, respectively. Let $\boldsymbol{\varrho}^{[0]*}_{qd} := \boldsymbol{\varrho}^*_{qd}$ and $\boldsymbol{\varrho}^{[0]*}_{cd} := \boldsymbol{\varrho}^*_{cd}$ for $m \equiv 0$. For general $m \in \mathbb{N}$, define the m-compounded $QD(\beta)$-optimal strategy $\boldsymbol{\varrho}^{[m]*}_{qd}$ as the QD-optimal strategy as per Proposition 3.3 using the $(m-1)$-compounded $QD(\beta)$-optimal strategy $\boldsymbol{\varrho}^{[m-1]*}_{qd}$ as its benchmark strategy to be outperformed. Similarly, we define the m-compounded $CD(\delta)$-optimal strategy $\boldsymbol{\varrho}^{[m]*}_{cd}$ as the CD-optimal strategy as per Proposition 3.5 using the $(m-1)$-compounded $CD(\delta)$-optimal strategy $\boldsymbol{\varrho}^{[m-1]*}_{cd}$ as its benchmark strategy to be outperformed.*

In other words, Definition 3.1 posits the following stylized situation: from a given benchmark strategy $\hat{\boldsymbol{\varrho}}$ (which may be a constant proportion strategy, but this is not a requirement), the corresponding QD- and CD-optimal investment strategies are constructed, giving $\boldsymbol{\varrho}^{[m=0]*}_{qd} = \boldsymbol{\varrho}^*_{qd}$ and $\boldsymbol{\varrho}^{[m=0]*}_{cd} = \boldsymbol{\varrho}^*_{cd}$. These QD- and CD-optimal strategies are then in turn used as the "benchmark strategies" to be substituted into the expressions for the QD- and CD-optimal controls ((3.13) and (3.18), respectively), which gives the $(m=1)$-compounded optimal investment strategies $\boldsymbol{\varrho}^{[m=1]*}_{qd}$ and $\boldsymbol{\varrho}^{[m=1]*}_{cd}$. Under the stylized assumptions of this section, nothing prevents the indefinite continuation of this recursive substitution, so that we therefore arrive at the m-compounded optimal investment strategies $\boldsymbol{\varrho}^{[m]*}_{qd}$ and $\boldsymbol{\varrho}^{[m]*}_{cd}$ for arbitrary $m \in \mathbb{N}$ as per Definition 3.1. The following proposition provides the closed-form expressions for the m-compounded investment strategies.

Proposition 3.9. *(m-compounded QD- and CD-optimal strategies) We assume that Assumption 3.1, Assumption 3.2 and wealth dynamics (3.11)-(3.12) are applicable. The optimal fraction of the investor's wealth to be invested in risky asset $i \in \{1, ..., N_a^r\}$ according to the m-compounded QD-optimal investment strategy for any $m \in \mathbb{N}$ is given by the i^{th} component of the vector $\boldsymbol{\varrho}^{[m]*}_{qd}$, where*

$$
\begin{aligned}
& W^{[m]*}_{qd}\left(t^-;\beta\right) \cdot \boldsymbol{\varrho}^{[m]*}_{qd}\left(t, \boldsymbol{X}^{[m]*}_{qd}\left(t^-;\beta\right);\beta\right) \\
= \; & \left[\left(\frac{e^{(m+1)\beta T}-1}{e^{\beta T}-1}\right) \cdot h_{qd}\left(t;\beta,q\right) - \left(W^{[m]*}_{qd}\left(t^-;\beta\right) - e^{(m+1)\beta T}\hat{W}\left(t^-\right)\right)\right] \cdot \left(\boldsymbol{\Sigma}+\boldsymbol{\Lambda}\right)^{-1}\tilde{\boldsymbol{\mu}} \\
& + e^{(m+1)\beta T}\hat{W}\left(t^-\right) \cdot \hat{\boldsymbol{\varrho}}\left(t, \hat{W}\left(t^-\right)\right).
\end{aligned}
\tag{3.36}
$$

The optimal fraction of the investor's wealth to be invested in risky asset $i \in \{1, ..., N_a^r\}$ according to the m-compounded CD-optimal investment strategy for any $m \in \mathbb{N}$ is given by the i^{th} component of the vector $\boldsymbol{\varrho}^{[m]*}_{cd}$, where

$$W_{cd}^{[m]*}\left(t^{-};\delta\right)\cdot\boldsymbol{\varrho}_{cd}^{[m]*}\left(t,\boldsymbol{X}_{cd}^{[m]*}\left(t^{-};\delta\right);\delta\right)$$
$$=\left[\left(\frac{\left(g_{cd}\left(t;\delta\right)\right)^{m+1}-1}{g_{cd}\left(t;\delta\right)-1}\right)\cdot h_{cd}\left(t;\delta,q\right)-\left(W_{cd}^{[m]*}\left(t^{-};\delta\right)-\left(g_{cd}\left(t;\delta\right)\right)^{m+1}\hat{W}\left(t^{-}\right)\right)\right]\cdot\left(\boldsymbol{\Sigma}+\boldsymbol{\Lambda}\right)^{-1}\tilde{\boldsymbol{\mu}}$$
$$+\left(g_{cd}\left(t;\delta\right)\right)^{m+1}\hat{W}\left(t^{-}\right)\cdot\hat{\boldsymbol{\varrho}}\left(t,\hat{W}\left(t^{-}\right)\right). \tag{3.37}$$

In (3.36) *and* (3.37), $W_{qd}^{[m]*}(t;\beta)$ *and* $W_{cd}^{[m]*}(t;\delta)$ *and denote the investor's wealth processes (3.11) under the m-compounded QD- and CD-optimal controls, respectively, while* $\boldsymbol{X}_{qd}^{[m]*}(t;\delta)=\left(W_{qd}^{[m]*}(t;\delta),\hat{W}(t),\hat{\boldsymbol{\varrho}}\left(t,\hat{W}(t)\right)\right)$ *and* $\boldsymbol{X}_{cd}^{[m]*}(t;\delta)=\left(W_{cd}^{[m]*}(t;\delta),\hat{W}(t),\hat{\boldsymbol{\varrho}}\left(t,\hat{W}(t)\right)\right)$.

Proof. See Appendix B.4. □

Recall from Proposition 3.3 that the QD-optimal strategy (i.e. $\boldsymbol{\varrho}_{qd}^{[m=0]*}=\boldsymbol{\varrho}_{qd}^{*}$) can be interpreted as pursuing the implicit wealth target $e^{\beta T}\hat{W}(t)$ at time t, while the CD-optimal strategy ($\boldsymbol{\varrho}_{cd}^{[m=0]*}=\boldsymbol{\varrho}_{cd}^{*}$) can be interpreted as setting an implicit wealth target of $g_{cd}(t;\delta)\hat{W}(t)$ at time t. Therefore, setting contributions to zero for simplicity, Proposition 3.9 shows that the m-compounded QD-optimal strategy $\boldsymbol{\varrho}_{qd}^{[m]*}$ and m-compounded CD-optimal investment strategy $\boldsymbol{\varrho}_{cd}^{[m]*}$ could be interpreted as simply pursuing significantly more aggressive implicit wealth targets at time t, namely $\left(e^{\beta T}\right)^{m+1}\hat{W}(t)$ and $\left(g_{cd}(t;\delta)\right)^{m+1}\hat{W}(t)$, respectively. This implies that we can obtain the necessary intuition regarding performance of QD- and CD-optimal strategies against increasingly sophisticated benchmark strategies by simply making the outperformance target more aggressive against for example a constant proportion benchmark strategy $\hat{\boldsymbol{\varrho}}$.

The forms of the m-compounded QD-optimal and CD-optimal strategies as per Proposition 3.3 imply that results analogous to Corollary 3.6, Proposition 3.7 and Corollary 3.8 can be derived for comparing the m-compounded strategies. However, it is potentially far more informative to compare the m-compounded QD-optimal and original QD-optimal strategy, and the m-compounded CD-optimal and original CD-optimal strategy, respectively, since this would clarify how the strategies behave when faced with more sophisticated benchmark strategies. To this end, similar to the risky asset basket definitions for the original strategies in (3.26), we define the total risky asset basket allocation for the m-compounded optimal strategies as

$$\mathcal{R}_{qd}^{[m]*}(t;\beta)=\sum_{k=1}^{N_a^r}\varrho_{qd,k}^{[m]*}(t;\beta),\qquad\text{and}\qquad\mathcal{R}_{cd}^{[m]*}(t;\delta)=\sum_{k=1}^{N_a^r}\varrho_{cd,k}^{[m]*}(t;\delta). \tag{3.38}$$

The following proposition (Proposition 3.10) compares selected aspects of the m-compounded and original optimal strategies that are amenable to closed-form analysis. Note that the comparisons of Proposition 3.10 require that the (original) benchmark strategy $\hat{\boldsymbol{\varrho}}$ is at least economically plausible, in the sense that a limit is placed on the total expected amount of short-selling of risky assets by the benchmark (see condition (B.1) in Appendix B.5 for an additional details and discussion).

Proposition 3.10. *(Optimal and m-compounded optimal strategies) Suppose that Assumption 3.1, Assumption 3.2 and wealth dynamics (3.11)-(3.12) are applicable. In addition, assume that the given benchmark strategy* $\hat{\mathcal{P}}$ *in (3.1) satisfies condition* (B.1). *Then the QD- and CD-optimal strategies as per Propositions 3.3 and 3.5 are outperformed in expectation by the corresponding m-compounded strategies as per Definition 3.1, in the sense that for any* $m\in\mathbb{N}$ *and* $t\in(t_0,T]$,

$$E_{\boldsymbol{\varrho}_{qd}^{[m]*}}^{t_0,w_0}\left[W_{qd}^{[m]*}(t;\beta)\right]>E_{\boldsymbol{\varrho}_{qd}^{*}}^{t_0,w_0}\left[W_{qd}^{*}(t;\beta)\right],\qquad\textit{and}\qquad E_{\boldsymbol{\varrho}_{cd}^{[m]*}}^{t_0,w_0}\left[W_{cd}^{[m]*}(t;\delta)\right]>E_{\boldsymbol{\varrho}_{cd}^{*}}^{t_0,w_0}\left[W_{cd}^{*}(t;\beta)\right]. \tag{3.39}$$

Furthermore, at time $t=t_0$, *if the total benchmark risky asset basket allocation satisfies* $\hat{\mathcal{R}}(t_0)=\sum_{k=1}^{N_a^r}\hat{\varrho}_k(t_0,w_0)\geq 0$, *the QD- and CD-optimal strategies allocate proportionally less wealth to the total risky asset basket than their corresponding m-compounded QD- and CD-optimal strategies. Specifically, for any* $m\in\mathbb{N}$, *we have*

$$\mathcal{R}_{qd}^{[m]*}(t_0;\beta)>\mathcal{R}_{qd}^{*}(t_0;\beta),\qquad\textit{and}\qquad\mathcal{R}_{cd}^{[m]*}(t_0;\delta)>\mathcal{R}_{cd}^{*}(t_0;\delta). \tag{3.40}$$

Proof. See Appendix B.5. □

An informal interpretation of Proposition 3.10 is that the m-compounded optimal strategies, while performing well in expectation (see (3.39)), does so by taking on more risk in the sense of increasing the allocation to the risky asset basket (see (3.40)). Note that while (3.40) is only proven at time $t = t_0$ in these closed-form results, numerical experiments (see Section 5) show that this is typical behavior for more aggressive benchmark outperformance targets also at times $t \in (t_0, T]$.

To conclude this section, we recall that Proposition 3.9 suggested that the outperformance against more sophisticated benchmark strategies can be assessed by considering more aggressive outperformance targets, while Proposition 3.10 shows that these more aggressive outperformance targets are achieved by taking on more risk. Within the framework of this section, taking on more risk is always feasible, since trading is allowed to continue in the event of insolvency and unlimited leverage is allowed. This is clearly not possible in more realistic circumstances, such as when leverage is restricted and trading in insolvency is ruled out. As a result, not only does this show the limitations and real trade-offs involved in benchmark outperformance in certain settings, but it also clearly illustrates the need to consider the numerical solutions of the outperformance problems when multiple investment constraints are considered, which we now discuss.

4 Numerical solutions

While the analytical results of Section 3 provide valuable intuition in more realistic investment settings when discrete rebalancing and multiple investment constraints are applicable (as confirmed by the results of Section 5), in such settings the solution techniques of Section 3 are typically no longer applicable, and therefore a numerical solution technique would be required.

In this section, we start by formulating a more realistic investment setting, then proceed to summarize the preferred neural network-based numerical solution approach to solve the QD and CD problems, which does not require assumptions regarding the parametric dynamics of the underlying assets, is entirely data-driven, and can handle discrete rebalancing and multiple investment constraints.

4.1 Discrete rebalancing with investment constraints

Instead of continuously rebalancing the portfolio in during the investment time horizon $[t_0 = 0, T]$, we assume a given set $\mathcal{T}$ of N_{rb} discrete rebalancing times,

$$\mathcal{T} \quad = \quad \{ t_n = n\Delta t | \, n = 0, ..., N_{rb} - 1 \} \, , \qquad \Delta t = T/N_{rb}, \tag{4.1}$$

where the assumption of equal spacing is used for notational convenience. At each rebalancing time $t_n \in \mathcal{T}$, a given amount of cash $q(t_n)$ is contributed to the portfolio. Note that the investor and benchmark strategies remain of the form (2.1), where $\mathcal{T}$ is now given by (4.1).

In this solution approach presented below, there is no requirement that the parametric dynamics of the N_a underlying assets (such as (3.5) and (3.10)) should be specified. Instead, the approach simply requires that at each time $t_{n+1} \in \mathcal{T} \cup T$, the return on each asset $i \in \{1, .., N_a\}$ over the time interval $[t_n, t_{n+1}]$, which is denoted by $R_i(t_n)$, is observabler, and a set of independent return sample paths over the investment horizon is given. As discussed below, for training and testing purposes (i.e. the solution of the problem and the out-of-sample testing of the resulting strategy), the returns $R_i(t_n)$ can be obtained by for example stationary block bootstrap resampling, which allows for the consideration of both serial correlation and cross-correlation of asset returns. Using the general formulation of Section 2, the investor and benchmark wealth dynamics are respectively given by

$$W\left(t_{n+1}^{-}\right) \quad = \quad \left[W\left(t_{n}^{-}\right) + q\left(t_{n}\right)\right] \cdot \sum_{i=1}^{N_a} p_i\left(t_n, \boldsymbol{X}\left(t_{n}^{-}\right)\right) \cdot \left[1 + R_i\left(t_n\right)\right], \tag{4.2}$$

$$\hat{W}\left(t_{n+1}^{-}\right) \quad = \quad \left[\hat{W}\left(t_{n}^{-}\right) + q\left(t_{n}\right)\right] \cdot \sum_{i=1}^{N_a} \hat{p}_i\left(t_n, \hat{\boldsymbol{X}}\left(t_{n}^{-}\right)\right) \cdot \left[1 + R_i\left(t_n\right)\right], \tag{4.3}$$

where $n = 0, ..., N_{rb} - 1$ and $W\left(t_0^{-}\right) = \hat{W}\left(t_0^{-}\right) := w_0 > 0$. The minimal form of the information incorporated by the investor's strategy is $\boldsymbol{X}(t_n) = \left(W(t_n), \hat{W}(t_n)\right)$, although this can be augmented with additional market information without difficulty (Van Staden et al. (2023)).

As is typical in the case of many active funds, we assume the investor has investment constraints of no short selling and no leverage allowed, resulting in sets of admissibility for the investor strategy $\mathcal{P}$ given by

$$\mathcal{A} = \{\mathcal{P} = \{\boldsymbol{p}(t_n, \boldsymbol{X}(t_n)) : t_n \in \mathcal{T}\} | \boldsymbol{p}(t_n, \boldsymbol{X}(t_n)) \in \mathcal{Z}, \quad \forall t_n \in \mathcal{T}\}, \tag{4.4}$$

$$\text{where} \quad \mathcal{Z} = \left\{ (y_1, ..., y_{N_a}) \in \mathbb{R}^{N_a} : \sum_{i=1}^{N_a} y_i = 1 \text{ and } y_i \geq 0 \text{ for all } i = 1, ..., N_a \right\}. \tag{4.5}$$

The investor's wealth remains non-negative given (4.2), (4.4)-(4.5) and $w_0 > 0$.

Solving investment problems (2.2) and (2.4) (note that we now focus on the discrete-time formulation of the QD problem) subject to these constraints requires a numerical solution technique, which we now discuss.

4.2 Neural network solution approach

To solve problems of the form (2.2) and (2.4) numerically, we use an existing neural network-based approach that does not rely on dynamic programming (Ni et al. (2022); Van Staden et al. (2023)). This approach, which is briefly summarized in this section, offers some clear advantages over competing approaches to solve similar problems, such as the class of Reinforcement Learning (RL) algorithms (see for example Dixon et al. (2020); Gao et al. (2020); Lucarelli and Borrotti (2020); Park et al. (2020)):

(i) The investment strategy is approximated directly using a neural network (NN), and we do not require dynamic programming (DP) based techniques such as RL to solve the benchmark outperformance problems. In particular, the problem of error amplification of the high-dimensional conditional expectation functions over value iterations associated with DP-based techniques (see for example Li et al. (2020); Tsang and Wong (2020); Wang and Foster (2020)) is avoided entirely. In addition, it can be shown under some conditions that problems of the form (2.2) and (2.4) have optimal controls that are relatively low dimensional relative to the objective functional (Van Staden et al. (2023)). Therefore, the direct approximation of the control can be considered a more efficient numerical solution approach. In somewhat different settings, the approach of solving for the control directly without the use of DP techniques has also been suggested in Buehler et al. (2019); Han and Weinan (2016); Reppen et al. (2022).

(ii) As discussed below, the rebalancing time t_n serves as an input (or feature) for the NN, which ensures that the number of parameters of the NN does not scale with the number of rebalancing events. In addition, use of t_n as a feature guarantees the smooth behavior of the control with respect to time in the limit as $\Delta t \to 0$, which is a practical requirement of a reasonable investment policy (see Van Staden et al. (2023)). These benefits place our approach in contrast to the approaches of for example Buehler et al. (2019); Han and Weinan (2016); Huré et al. (2021); Tsang and Wong (2020).

A detailed description of the NN-based numerical solution approach can be found in the literature (Van Staden et al. (2023)), while some algorithm implementation details specifically for the QD and CD problems are given in Appendix E. We therefore only briefly highlight some key aspects of the approach in this section.

The numerical solution of problems (2.2) and (2.4) requires the solution of the feedback control $(t_n, \boldsymbol{X}(t_n)) \to \mathcal{P}(t_n, \boldsymbol{X}(t_n)) := \boldsymbol{p}(t_n, \boldsymbol{X}(t_n)) \in \mathcal{Z}, \forall t_n \in \mathcal{T}$. We approximate the control function $\boldsymbol{p}(t, \boldsymbol{X})$ by a NN $\boldsymbol{F}(t, \boldsymbol{X}(t); \boldsymbol{\theta}) \equiv \boldsymbol{F}(\cdot, \boldsymbol{\theta})$, where $\boldsymbol{\theta} \in \mathbb{R}^{\eta_\theta}$ is the set of NN parameters (i.e. the NN weights and biases), in other words

$$\boldsymbol{p}(t, \boldsymbol{X}(t)) \simeq \boldsymbol{F}(t, \boldsymbol{X}(t); \boldsymbol{\theta}) \equiv \boldsymbol{F}(\cdot, \boldsymbol{\theta}). \tag{4.6}$$

In terms of the structure of the NN $\boldsymbol{F}$, we use a fully-connected feed-forward NN with at least 3 inputs (or features), namely $(t_n, \boldsymbol{X}(t_n)) = \left(t_n, W(t_n), \hat{W}(t_n)\right)$, while additional trading signals can be incorporated as additional features if required. The number of output nodes correspond to the number of assets, while a softmax activation function in the output layer guarantees outputs in the set $\mathcal{Z} \subset \mathbb{R}^{N_a}$. Given any particular input $(t_n, \boldsymbol{X}(t_n))$, the NN therefore automatically generates the asset allocation $\boldsymbol{p}(t_n, \boldsymbol{X}(t_n)) \in \mathcal{Z}$ as per (4.6), so that problems (2.2) and (2.4) now can be solved respectively as the *unconstrained* optimization problems

$$\inf_{\boldsymbol{\theta} \in \mathbb{R}^{\eta_\theta}} E^{t_0, w_0}_{\boldsymbol{F}(\cdot;\boldsymbol{\theta})} \left[\left(W(T; \boldsymbol{\theta}) - e^{\beta T} \hat{W}(T) \right)^2 \right], \qquad \inf_{\boldsymbol{\theta} \in \mathbb{R}^{\eta_\theta}} E^{t_0, w_0}_{\boldsymbol{F}(\cdot;\boldsymbol{\theta})} \left[\sum_{n=0}^{N_{rb}} \left(W(t_n^-; \boldsymbol{\theta}) - e^{\delta t_n} \hat{W}(t_n^-) \right)^2 \right]. \tag{4.7}$$

The expectations in (4.7) are approximated by using a finite set of samples from the set $Y = \left\{ Y^{(j)} : j = 1, ..., N_d \right\}$, where each $Y^{(j)}$ represents a time series of *joint* asset return observations $R_i, i \in \{1, .., N_a\}$, observed at each

$t_n \in \mathcal{T}$. Conventionally, Y is referred to as the "training" data set for the NN (Goodfellow et al. (2016)), and we discuss its construction in more detail below. For a given NN parameter vector $\boldsymbol{\theta} \in \mathbb{R}^{\eta_\theta}$ and returns path $Y^{(j)} \in Y$, dynamics (4.2)-(4.3), control (4.6) gives the corresponding wealth outcomes $W^{(j)}(t_n;\boldsymbol{\theta})$ and $\hat{W}^{(j)}(t_n)$ for $t_n \in \mathcal{T}$, the following approximations to (4.7) are solved

$$\min_{\boldsymbol{\theta}\in\mathbb{R}^{\eta_\theta}} \left\{ \frac{1}{N_d} \sum_{j=1}^{N_d} \left(W^{(j)}(T;\boldsymbol{\theta}) - e^{\beta T} \hat{W}^{(j)}(T) \right)^2 \right\}, \quad \min_{\boldsymbol{\theta}\in\mathbb{R}^{\eta_\theta}} \left\{ \frac{1}{N_d} \sum_{j=1}^{N_d} \sum_{n=0}^{N_{rb}} \left(W^{(j)}(t_n^-;\boldsymbol{\theta}) - e^{\delta t_n} \hat{W}^{(j)}(t_n^-) \right)^2 \right\}. \tag{4.8}$$

Solving (4.8) using stochastic gradient descent, we obtain the optimal parameter vectors $\boldsymbol{\theta}_k^*, k \in \{qd, cd\}$. For further details regarding hyperparameters and ground truth solutions, please refer to Appendix E.

Using $\boldsymbol{\theta}_k^*, k \in \{qd, cd\}$, the resulting optimal strategies $\boldsymbol{p}_k^*(\cdot, \boldsymbol{X}(\cdot)) \simeq \boldsymbol{F}(\cdot, \boldsymbol{\theta}_k^*), k \in \{qd, cd\}$ are implemented on a testing data set Y^{test} (which is similar in structure to Y but typically contains different data or data-generating assumptions) to assess the out-of-sample performance of the respective strategies.

As for the construction of the training and testing data sets (Y and Y^{test}), while the NN solution methodology as outlined above is agnostic as to the construction technique underlying these data sets, it is clearly of great practical significance for solving and assessing the performance of the strategies. For purposes of confirming whether the numerical approach has been implemented correctly, it is possible to assume parametric dynamics for the underlying assets and generate Y by sampling $R_i(t_n)$ using Monte Carlo simulation, so that the resulting numerical solutions under certain conditions can be compared to the corresponding analytical solutions (see Appendix E). However, in more practical applications as well as in the results of Section 5.3, practitioners may prefer to use historical data directly without imposing any parametric assumptions, so that some augmentation technique is necessarily required due to the sparsity of historical financial data for long-term investments.

In this paper, for illustrative purposes we use stationary block bootstrap resampling (Politis and Romano (1994)) to generate Y and Y^{test} from different historical time periods. We emphasize that the use of block bootstrap resampling is popular with practitioners (Cavaglia et al. (2022); Cogneau and Zakalmouline (2013); Dichtl et al. (2016); Scott and Cavaglia (2017); Simonian and Martirosyan (2022)), as well as academics (Anarkulova et al., 2022), and is designed for weakly stationary time series with serial dependence. While bootstrap sampling methods have been proposed for resampling non-stationary time series (Politis (2003), Politis et al. (1999)), this is not explored further in the results of Section 5.

5 Illustrative investment results

In this section, the results associated with the QD- and CD-optimal investment strategies are illustrated first using closed-form solutions (Section 3) under stylized assumptions, and then using numerical solutions (Section 4) associated with the more realistic setting of Subsection 4.1.

To ensure that the examples remain relevant in practice, we assume that the investor constructs portfolios to outperform standard constant proportion benchmarks based on a broad stock market index and Treasury bills and bonds, similar to the benchmarks used by active portfolio managers for government pension plans (Canadian Pension Plan (2022); Government Pension Fund Global (2022)). In constructing strategies to outperform these benchmarks, we consider cases of modest as well as more aggressive benchmark outperformance targets compared to what is typically seen in practice, since more aggressive outperformance targets could be used as a proxy for assessing the optimal strategy behavior against benchmarks of increasing sophistication (see Subsection 3.5). We make the assumption that the investor may not necessarily be limited to investing in the same underlying assets as the benchmark, but is also able to invest in some widely-recognized equity factors (Ang (2014)).

5.1 Investment scenarios

The key investment scenario assumptions used for illustrative purposes are summarized in Table 5.1. For closed-form solutions, continuous rebalancing is approximated using 3600 time steps during the time horizon of 10 years. The time horizon is chosen to reflect the concerns of an investor with medium to long-term benchmark outperformance requirements.

Table 5.2 provides a summary of the underlying assets and the constant proportion benchmarks considered, while more detailed definitions of the assets and associated data sources can be found in Appendix C. Note that investor portfolio P0 will be constructed to outperform benchmark BM0 in order to illustrate the closed-form

Table 5.1: Key investment scenario assumptions.

Parameter	Closed-form solutions (no constraints)	Numerical solutions (realistic constraints)	
Investment constraints	None	No short-selling, no leverage allowed	
T	10 years	10 years	
w_0	120	120	
Rebalancing frequency	Continuous	Annual	Quarterly
N_{rb} (# rebalancing events)	3600	10	40
Contributions	$q = 12$ (rate per year)	$q(t_n) = 12, \forall n$ (annual contribution)	$q(t_n) = 3, \forall n$ (quarterly contribution)

solutions of Section 3. In this case, the broad equity market index ("Market") plays the role of the single "risky asset basket" in the terminology of Subsection 3.4. In contrast, investor portfolio P1 will be constructed to outperform benchmark BM1 to illustrate the numerical solutions subject to no short-selling and no leverage investment constraints, as outlined in Section 4 .

Table 5.2: Portfolios "Px", $x \in \{0, 1\}$ constructed by the investor using assets indicated by "✓", to outperform benchmarks "BMx", $x \in \{0, 1\}$ with asset holdings as a percentage of wealth $\hat{p}_i$ as indicated. Definitions and data sources of historical time series are provided in Appendix C.

Assets		Investor portfolios		Benchmarks	
Label	Asset description	P0	P1	BM0	BM1
T30	30-day Treasury bill	✓	✓	30%	15%
B10	10-year Treasury bond		✓		15%
Market	Market portfolio (broad equity market index)	✓	✓	70%	70%
Size	Portfolio of small stocks		✓		
Value	Portfolio of value stocks		✓		
	Number of candidate assets (N_a):	2	5	2	3

As discussed in Subsection 3.4, we will compare investment results on the basis of equal expectations of terminal wealth. In particular, the parameters $\delta^{\mathcal{E}}$ and $\beta^{\mathcal{E}}$ are selected for the $CD\left(\delta = \delta^{\mathcal{E}}\right)$ and $QD\left(\beta = \beta^{\mathcal{E}}\right)$ problems, respectively, such that

$$E_{\mathcal{P}_{cd}^*}^{t_0,w_0}\left[W_{cd}^*\left(T; \delta = \delta^{\mathcal{E}}\right)\right] \equiv E_{\mathcal{P}_{qd}^*}^{t_0,w_0}\left[W_{qd}^*\left(T; \beta = \beta^{\mathcal{E}}\right)\right] \equiv \mathcal{E}. \tag{5.1}$$

The rationale for comparing the results on the basis of equal expectation is discussed in Subsection 3.4, while the additional numerical comparison results on the basis of equal parameters (i.e. setting $\delta = \beta$ for both problems resulting in different expectations of terminal wealth) reported in Appendix B.6 demonstrate that the main qualitative conclusions of this section are not affected by changing the basis of comparison.

Table 5.3 summarizes the data sets used for the illustration of results, as well as the target value of the expectation (5.1) chosen for illustrative purposes. As shown, the historical data periods for data sets DS0, DS0b, DS1, DS1b and DS3 are chosen specifically to incorporate periods of high inflation such as 1963-1985, since this data might be more relevant to current market conditions than more recent data (e.g. data of the last 30 years) associated with atypically low and declining real interest rates. Note that sets DS0b and DS1b use the same assumptions as data sets DS0 and DS1, respectively, except that the benchmark outperformance targets implied by the chosen values of $\mathcal{E}$ for DS0b and DS1b are significantly more aggressive. In particular, the target $\mathcal{E}$ is in (5.1) is chosen to be some multiple e^{kT} of the benchmark expected value $E_{\hat{\mathcal{P}}}^{t_0,w_0}\left[\hat{W}(T)\right]$, where k is between 1% and 2% to reflect typical practitioner benchmark outperformance targets, whereas DS0b and DS1b use values of $k \simeq 2.4\%$. We also include data sets DS2 and DS2b, not only for illustrating the effect the rebalancing frequency on the results, but also to demonstrate the robustness of conclusions when using only the most recent data following the popularization of equity factors Size and Value by Fama and French (1992).

As for data set construction, note that DS0 and DS0b are simulated using specified dynamics for the underlying assets of investor portfolio P0 (Table 5.2): the Kou (2002) model is used for the "risky asset" (Market), while the "risk free" asset (T30) evolves according to (3.5). The model calibrations and resulting parameters are discussed in Appendix C. For all other data sets, we make no parametric model assumptions regarding the distribution or dynamics of underlying asset returns, and instead use the historical data directly by implementing stationary block bootstrap resampling for the construction of data sets (see Politis and Romano (1994) and the discussion in Section 4). Note that for all data sets, including in the case of estimation of model parameters for DS0, the historical returns time series was inflation-adjusted prior to the construction of the data sets (see Appendix C).

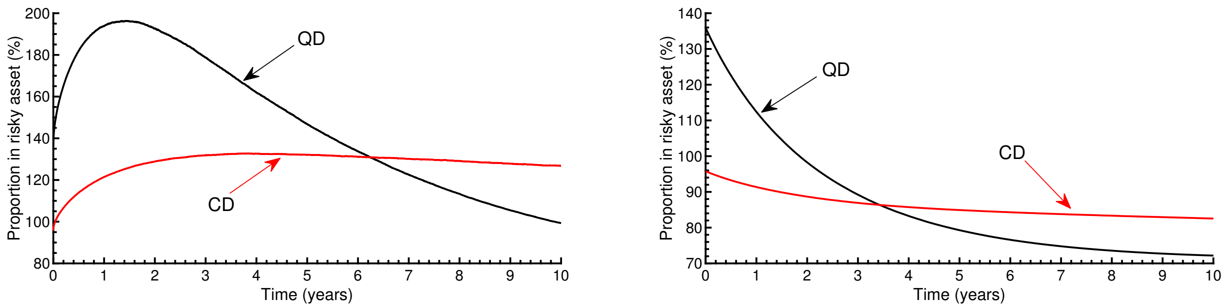
Table 5.3: Data sets, abbreviated as "DSx", $x \in \{0, 1, 2, 2b, 3\}$ used for the illustration of results. "SBBR" refers to stationary block bootstrap resampling with expected blocksize ("Exp. blksize") in months as indicated. The training and testing data sets consists of $N_d = 10^6$ and $N_d^{test} = 5 \times 10^5$ joint paths of asset price returns, respectively.

Data set label		DS0	DS0b	DS1	DS1b	DS2	DS2b	DS3
Rebal. frequency		Continuous	Continuous	Annual	Annual	Annual	Quarterly	Annual
Data set construction		Model simulation	Model simulation	SBBR	SBBR	SBBR	SBBR	SBBR
Benchmark		BM0	BM0	BM1	BM1	BM1	BM1	BM1
Investor portfolio		P0	P0	P1	P1	P1	P1	P1
Training data set Y	Data period	1963:07 - 2020:12	1963:07 - 2020:12	1963:07 - 2009:12	1963:07 - 2009:12	1995:01 - 2009:12	1995:01 - 2009:12	1963:07 - 1995:12
	Exp. blksize (months)	N/a	N/a	6	6	3	3	6
	$E_{\hat{\mathcal{P}}}^{t_0,w_0}\left[\hat{W}(T)\right]$	367	367	362	362	384	379	367
	$\mathcal{E} = E_{\mathcal{P}_k^*}^{t_0,w_0}\left[W_k^*(T)\right]$	405	465	400	460	420	420	405
Testing data set Y^{test}	Data period	N/a	N/a	2010:01 - 2020:12	2010:01 - 2020:12	2010:01 - 2020:12	2010:01 - 2020:12	1996:01 - 2020:12
	Exp. blksize (months)	N/a	N/a	3	3	3	3	3

5.2 Illustration of closed-form solutions

The closed-form solutions of Section 3 are illustrated using 2 assets, since a risky asset basket (here simply referred to as the "risky asset" given by "Market" in Table 5.2) and a risk-free asset (T30 in Table 5.2) are sufficient to illustrate the key aspects of the strategies - see Subsection 3.4. As a result, portfolio P0 is constructed to outperform benchmark BM0 (Table 5.2), with parameters based on the Kou (2002) model and data set DS0 - see Table 5.3 and Appendix C.

Figure 5.1 compares the 95th and 50th percentiles of the proportion of wealth invested in the risky asset according to the closed-form CD- and QD-optimal strategies, which illustrate the results of Proposition 3.7 and Corollary 3.8. In particular, we observe that the CD strategy does not take similarly extreme positions as the QD strategy early in the investment time horizon.



(a) Proportion of wealth in risky asset: 95th percentiles

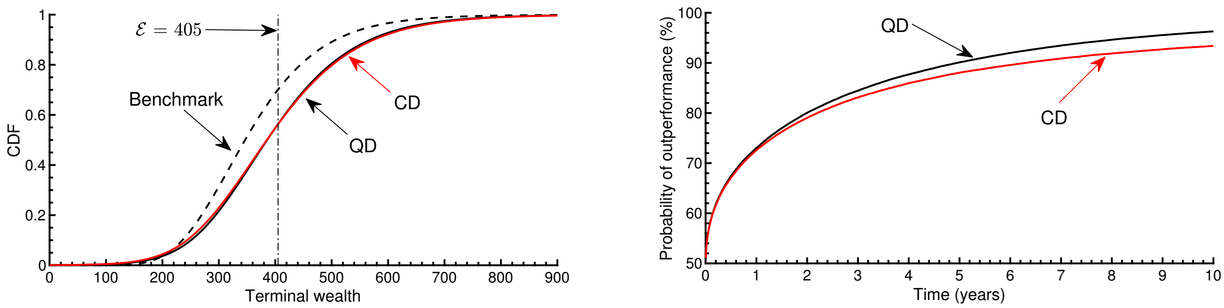
(b) Proportion of wealth in risky asset: 50th percentiles

Figure 5.1: Closed-form solutions, no constraints, investor portfolio P0, benchmark BM0, data set DS0: Selected percentiles of the optimal proportion of wealth in the risky asset according to each strategy.

Figure 5.2(a) shows that despite the fundamental differences in investment strategies illustrated in Figure 5.1, the terminal wealth distributions associated with the closed-form investment strategies are nearly identical. While surprising, it is not uncommon for significantly different strategies to nevertheless yield very similar final wealth distributions - see for example Dang and Forsyth (2016) where such strategies are described as "non-unique" strategies.

However, Figure 5.2(b) illustrates that the probability that the investor would report outperforming the benchmark during the investment time horizon is slightly higher for the QD strategy than for the CD strategy, although the overall levels of outperformance in Figure 5.2(b) are unrealistically high due to the stylized assumptions used in deriving the closed-form solutions. Note that there is no contradiction in obtaining nearly identical wealth distributions (Figure 5.2(a)) together with differences in benchmark outperformance (Figure 5.2(b)), since the latter offers pathwise comparisons relative to the benchmark while the former presents terminal wealth distributions only. Mathematically similar marginal (wealth) distributions may be associated with different joint distributions, and Figure 5.2(b) illustrates one key aspect of the joint distribution of $W_j^*(t), \hat{W}(t), j \in \{cd, qd\}$.

For assessing the implications of setting a significantly more aggressive benchmark outperformance target,



(a) CDFs of terminal wealth **(b)** Probability of benchmark outperformance

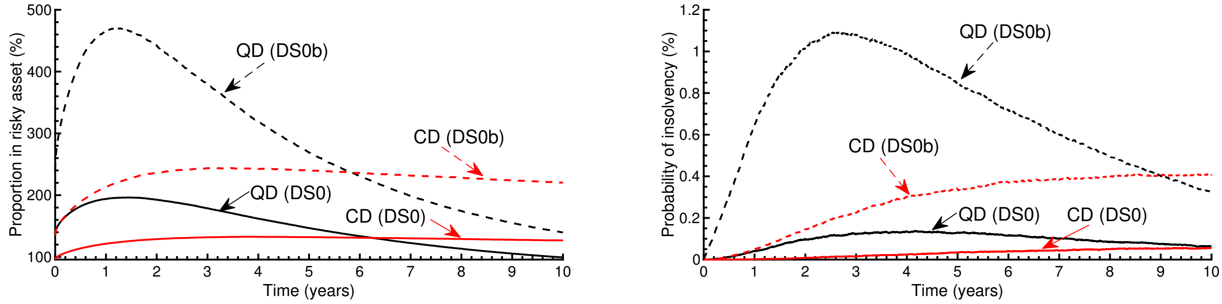
Figure 5.2: Closed-form solutions, no constraints, investor portfolio P0, benchmark BM0, data set DS0: (a) Simulated CDFs of the benchmark terminal wealth $\hat{W}(T)$, and investor's terminal wealth $W_j^*(T)$,$j \in \{cd, qd\}$, where $W_j^*(T)$ has expected value $\mathcal{E} = 405$, regardless of strategy. (b) Probability of benchmark outperformance over time, $t \rightarrow P_{\mathcal{P}_{qd}^*}^{t_0,w_0}\left[W_j^*(t) > \hat{W}(t)\right]$, $j \in \{cd, qd\}$.

Figure 5.3 compares, for data sets DS0 and DS0b, the 95th percentiles of the proportion of wealth in the risky asset basket as well as the probability of trading in insolvency over time. While the relative behavior of the QD- and CD-optimal strategies in terms of the risky asset basket allocation in the case of DS0b remains qualitatively similar to the case of DS0 (Figure 5.3(a)), it is clear that the more aggressive target for DS0b results in the completely unrealistic behavior of the closed-form investment strategies, where for example the QD-optimal strategy borrows to invest more than four times the total wealth in the risky asset basket around years 1 and 2. This is clearly only plausible if unrestricted leverage and trading in insolvency is possible (which is allowed under the stylized assumptions of Section 3), with Figure 5.3(b) confirming that the QD strategy relies (under these stylized assumptions) more on trading in insolvency than the CD-optimal strategy for most of the investment time horizon. As a result, Figure 5.3 clearly illustrates the importance of assessing the behavior of the optimal strategies under more realistic investment constraints (Subsection 5.3 below).

However, valuable intuition is gained from the closed-form solutions. In particular, we observe that while the terminal wealth distributions are nearly identical (Figure 5.2(a)), the risky asset basket allocation of the CD-optimal strategy has less variation across time (Figure 5.1), with more aggressive outperformance targets resulting in larger allocations to the risky asset basket (Figure 5.3(a)). While these observations remain applicable in the case of numerical solutions under more realistic assumptions, we will see that the benchmark outperformance results (Figure 5.2(b)) no longer hold qualitatively out-of-sample when investment constraints are applied, with the CD-optimal strategy gaining the advantage.

5.3 Illustration of numerical solutions

We now illustrate the investment results using the optimal strategies obtained numerically in the case of discrete rebalancing, multiple assets, and investment constraints. These results are obtained using the numerical solution approach discussed in Section 4. We emphasize that we make no parametric model assumptions regarding the



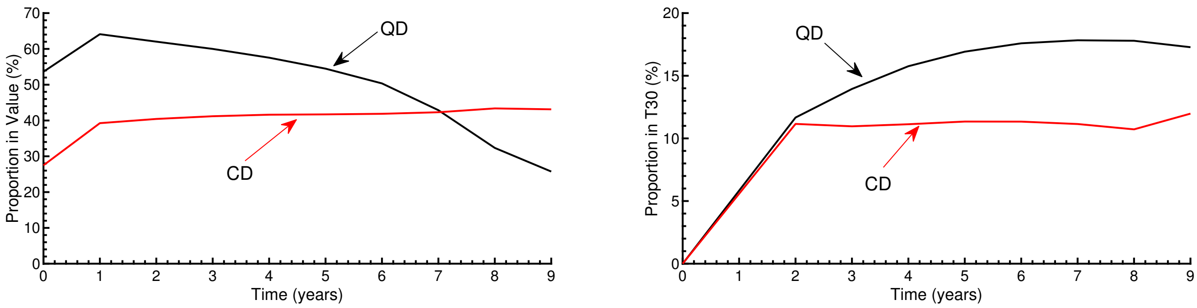
(a) Proportion of wealth in risky asset: 95th percentiles

(b) Probability of trading in insolvency

Figure 5.3: Closed-form solutions, no constraints, investor portfolio P0, benchmark BM0, data sets DS0 and DS0b: Selected percentiles of the optimal proportion of wealth in the risky asset according to each strategy.

distribution or dynamics of underlying asset returns. The results are only illustrated for data set DS1 in Table 5.3, with key out-of-sample results associated with the other data sets in Table 5.3 provided in Appendix D. Note that we continue comparing investment strategies on the basis of equal expectations, (5.3), where the same expected value of terminal wealth is obtained on the *training* data set of the neural network. Additional results provided in Appendix B.6 show that comparing results on the basis of equal parameters ($\delta = \beta$) results in qualitatively similar conclusions.

Figure 5.4 illustrates that in the case of discrete rebalancing and investment constraints, the qualitative conclusions from the closed-form solutions still hold (see Figure (5.1)). In particular, since Value and T30 represents the assets with the highest and lowest standard deviation of returns of the assets in Table 5.2, the CD-optimal strategy takes less extreme positions in these assets at key points during the investment time horizon.



(a) Proportion of wealth in Value: 95th percentile

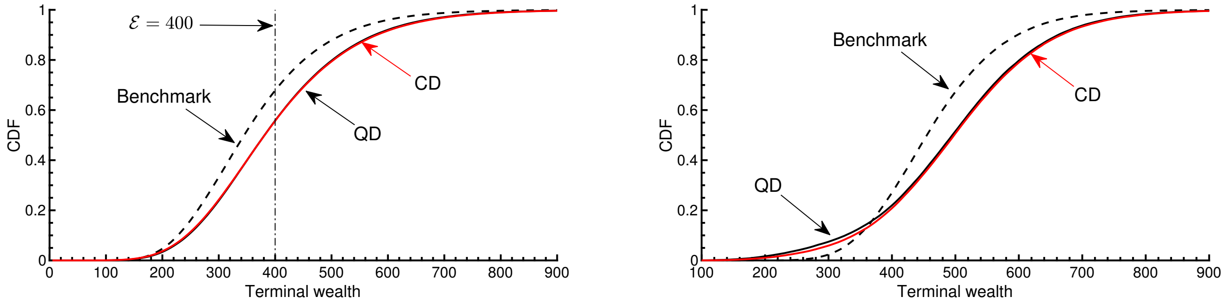
(b) Proportion of wealth in T30: 95th percentile

Figure 5.4: Numerical solutions, with constraints, investor portfolio P1, benchmark BM1, data set DS1: 95th percentiles of the proportion of wealth invested in Value and T30 over time. Note that the final rebalancing event is at $t = T - \Delta t = 9$ years. Other assets are shown in Figure B.3.

Figure 5.5 shows that in the case of discrete rebalancing and investment constraints, the terminal wealth distributions remain almost identical, both in-sample (training data) and out-of-sample (testing data), despite the fact that the underlying investment strategies exhibit the differences illustrated in Figure 5.4 (see Figure B.3 for other assets). As in the case of the closed-form solutions (see Figure 5.1), we can view the resulting CD- and QD-optimal investment strategies as "non-unique" (Dang and Forsyth (2016)) since they generate nearly identical terminal wealth distributions.

While Figure 5.5 only shows results associated with DS1, the results for other data sets in Table 5.3 are similar and illustrated in Appendix D.

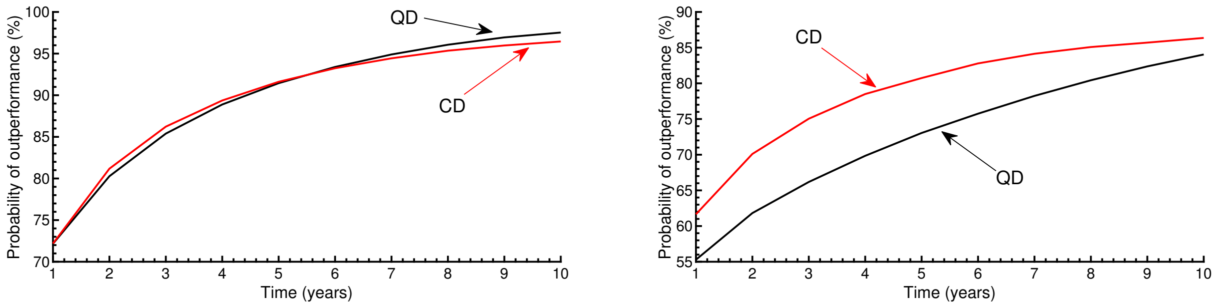
Considering the probability of benchmark outperformance over time, Figure 5.6(a) shows that the in-sample (training data set) results for the CD- and QD-optimal investment strategies are very similar. However, Figure 5.6(b) shows that *out-of-sample* (i.e. for the testing data set), the CD-optimal strategy consistently achieves a higher probability of benchmark outperformance during the investment time horizon than the QD-optimal strategy, with some "convergence" closer to maturity. While the results in Figure 5.6 are only shown for data



(a) CDFs of terminal wealth: Training data (DS1) **(b)** CDFs of terminal wealth: Testing data (DS1)

Figure 5.5: Numerical solutions, with constraints, investor portfolio P1, benchmark BM1, data set DS1: CDFs of terminal benchmark wealth $\hat{W}(T)$ and terminal investor wealth $W_k^*(T)$, $k \in \{qd, cd\}$, where the investor terminal wealth has the same expected value $\mathcal{E} = 400$ on the training data set.

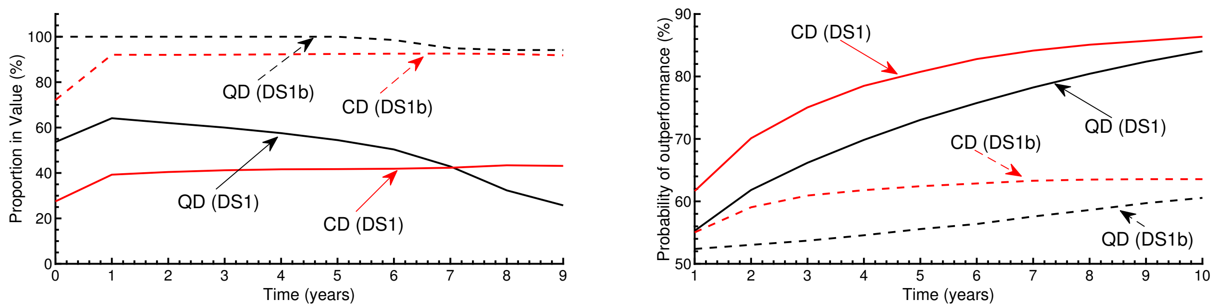
set DS1, the results in Appendix D indicate that the CD-optimal strategy also delivers qualitatively similar out-of-sample results to those of Figure 5.6(b) in the case of the other data sets in Table 5.3.



(a) Probability of outperformance: Training data (DS1) **(b)** Probability of outperformance: Testing data (DS1)

Figure 5.6: Numerical solutions, with constraints, investor portfolio P1, benchmark BM1, data set DS1: Probability of benchmark outperformance over time, $t \to P_{\mathcal{P}_{qd}^*}^{t_0,w_0}\left[W_j^*(t) > \hat{W}(t)\right]$, $j \in \{cd, qd\}$.

Figure 5.7 illustrates the implications of using a more aggressive benchmark outperformance target for the allocation to Value as well as for the probability of outperformance, with Figure D.4 in Appendix D showing the effect on the wealth distributions of more aggressive outperformance targets. In contrast to Figure 5.3(a),



(a) Proportion of wealth in Value: 95th percentile **(b)** Probability of outperformance: Testing data

Figure 5.7: Numerical solutions, with constraints, investor portfolio P1, benchmark BM1, data sets DS1 and DS1b: 95th percentile of the proportion of wealth invested in Value over time, as well as the probability of benchmark outperformance on the testing data sets. Note that the final rebalancing event is at $t = T - \Delta t = 9$ years.

the allocation to any asset now cannot exceed 100% of the total wealth since no short selling or leverage is allowed. Figure 5.7(a) shows that the allocation to Value (the asset index with the highest empirical return but also highest standard deviation of returns) is exactly 100% for the QD-optimal strategy and around 90% for the CD-optimal strategy when using a significant proportion of the investment time horizon a significantly more aggressive benchmark outperformance target in the case of DS1b. This lack of diversification significantly reduces the probability of benchmark outperformance, as the comparative results for DS1 and DS1b in Figure

5.7(b) show, while increasing both the upside and downside wealth outcomes relative to the benchmark (Figure D.4).

However, it is important to maintain a broader perspective with regards to practical applicability when considering the aggressive outperformance target of DS1b, which has been provided for illustrative purposes only. Considering instead the case of the fairly modest outperformance target used for DS1, we observe that in the case of the out-of-sample results illustrated in Figure 5.6(b), the CD strategy has a ∼85% probability of outperforming the benchmark. The median Internal Rate of Return (IRR) for the CD strategy in this case is 9.39% while the median IRR for the benchmark is 8.22%, which gives a median outperformance of 116 bps in the out-of-sample testing data. As a point of reference, the CPP outperformance for the last 5 years was about 80bps (see CPP 2021 annual report(Canadian Pension Plan, 2021)). This illustrates that the proposed optimal strategies do not require overly aggressive outperformance targets in order to yield excellent outperformance results.

In addition, for the portfolio manager with frequent reporting requirements, the CD-optimal strategy offers some clear advantages compared to the QD-optimal strategy. In particular, the CD-optimal strategy is associated with less extreme positions in the riskiest asset early in the investment time horizon (Figure 5.4(a)) while delivering a higher probability of benchmark outperformance during the investment time horizon in out-of-sample testing (Figure 5.6(b)). At the same time, this is achieved without adversely impacting the terminal wealth distribution of the CD-optimal strategy relative to that of the QD-optimal strategy (Figure 5.5).

6 Conclusion

In this paper, we proposed a novel objective function (the CD objective) for constructing dynamic optimal investment strategies that directly target a favorable tracking difference relative to the benchmark at multiple points in time during the investment time horizon.

After presenting closed-form results (derived under stylized assumptions) to gain intuition regarding the behavior of the CD-optimal investment strategies, we discussed the numerical solutions of portfolio optimization problems in the case of discrete rebalancing and multiple investment constraints.

Our results demonstrate that in comparison to targeting a favorable tracking difference only at maturity via the QD objective, the CD-optimal strategies: (i) deliver very similar terminal wealth distributions both in-sample and out-of-sample as the QD strategies, while (ii) requiring less extreme positions in the riskiest assets early in the investment time horizon.

The fact that CD-optimal strategy has a nearly identical terminal wealth distribution as the QD-optimal strategy, while its positions in underlying assets imply an improved risk profile across time, illustrates that it is insufficient to evaluate risk in a dynamic strategy based on the statistics (or even the entire distribution) of the terminal wealth alone. Risk assessment in the strategy itself is relevant, since our numerical results show that while the QD-optimal strategy achieves slightly better results in the probability of benchmark outperformance in training data, the CD-optimal strategy outperforms the QD-optimal strategy in testing data.

Our theoretical analysis and empirical investigations illustrate that the proposed CD objective function may be attractive for active portfolio managers expected to deliver a favorable tracking difference relative to a benchmark while having frequent reporting requirements to stakeholders.

We leave a comparison of the CD-optimal investment strategies to other benchmark outperformance strategies in the literature (for example, strategies maximizing the Information Ratio relative to the benchmark) for future work.

7 Acknowledgements

Forsyth's work was supported by the Natural Sciences and Engineering Research Council of Canada (NSERC) grant RGPIN-2017-03760. Li's work was supported by he Natural Sciences and Engineering Research Council of Canada (NSERC) grant RGPIN-2020-04331.

References

Al-Aradi, A. and S. Jaimungal (2018). Outperformance and tracking: dynamic asset allocation for active and passive portfolio management. *Applied Mathematical Finance* 25(3), 268–294.

Al-Aradi, A. and S. Jaimungal (2021). Active and passive portfolio management with latent factors. *Quantitative Finance* .

Alekseev, A. G. and M. V. Sokolov (2016). Benchmark-based evaluation of portfolio performance: a characterization. *Annals of Finance* 12, 409–440.

Anarkulova, A., S. Cederburg, and M. S. O'Doherty (2022). Stocks for the long run? Evidence from a broad sample of developed markets. *Journal of Financial Economics* 143:1, 409–433.

Ang, A. (2014). *Asset management - A systematic approach to factor investing.* Oxford University Press.

Applebaum, D. (2004). *Lévy processes and stochastic calculus.* Cambridge University Press.

Basak, S., A. Shapiro, and L. Tepla (2006). Risk management with benchmarking. *Management Science* 52(4), 542–557.

Bernard, C. and S. Vanduffel (2014). Mean-variance optimal portfolios in the presence of a benchmark with applications to fraud detection. *European Journal of Operational Research* 234, 469–480.

Bjork, T. (2009). *Arbitrage Theory in Continuous Time.* Oxford University Press.

Bo, L., H. Liao, and X. Yu (2021). Optimal tracking portfolio with a ratcheting capital benchmark. *SIAM Journal on Control and Optimization* 59(3), 2346–2380.

Boyde, E. (2021). Etfs: why tracking difference usually matters more than tracking error. *Financial Times* https://www.ft.com/content/44b08a6c-55d7-4841-8e8e-aad6451a4cc3.

Browne, S. (1999a). Beating a moving target: optimal portfolio strategies for outperforming a stochastic benchmark. *Finance and Stochastics* 3(275-294).

Browne, S. (1999b). Reaching goals by a deadline: Digital options and continuous-time active portfolio management. *Advances in Applied Probability* 31(2), 551–577.

Browne, S. (2000). Risk-constrained dynamic active portfolio management. *Management Science* 46(9), 1188–1199.

Buehler, H., L. Gonon, J. Teichmann, and B. Wood (2019). Deep hedging. *Quantitative Finance* 19(8), 1271–1291.

Canadian Pension Plan (2021). Cpp investments annual report. `https://www.cppinvestments.com/the-fund/our-performance/financial-results`.

Canadian Pension Plan (2022). Our investment strategy. `www.cppinvestments.com/the-fund/how-we-invest/our-investment-strategy`.

Cavaglia, S., L. Scott, K. Blay, and S. Hixon (2022). Multi-asset class factor premia: A strategic asset allocation perspective. *The Journal of Portfolio Management* 48:9, 14–32.

Charteris, A. and K. McCullough (2020). Tracking error vs tracking difference: Does it matter? *Investment Analysts Journal* 49(3), 269–287.

Cogneau, P. and V. Zakalmouline (2013). Block bootstrap methods and the choice of stocks for the long run. *Quantitative Finance* 13, 1443–1457.

Dang, D. and P. Forsyth (2016). Better than pre-commitment mean-variance portfolio allocation strategies: A semi-self-financing Hamilton–Jacobi–Bellman equation approach. *European Journal of Operational Research* 250(3), 827–841.

Davis, M. and S. Lleo (2008). Risk-sensitive benchmarked asset management. *Quantitative Finance* 8(4), 415–426.

Dichtl, H., W. Drobetz, and M. Wambach (2016). Testing rebalancing strategies for stock-bond portfolos across different asset allocations. *Applied Economics* 48, 772–788.

Dixon, M. F., I. Halperin, and P. Bilokon (2020). *Machine learning in finance.* Springer International Publishing.

ESMA (2014). Guidelines on ETFs and other UCITS issues: Guidelines for competent authorities and UCITS management companies. *European Securities and Markets Authority* ESMA/2014/937EN.

ETF.com (2021). Understanding tracking difference and tracking error. *ETF.com Education Center Article* https://www.etf.com/etf-education-center/etf-basics/understanding-tracking-difference-and-tracking-error.

Fama, E. and K. French (2015). A five-factor asset pricing model. *Journal of Financial Economics* 116(1), 1–22.

Fama, E. F. and K. R. French (1992). The cross-section of expected stock returns. *Journal of Finance* 47, 427–465.

Forsyth, P. and K. Vetzal (2017). Dynamic mean variance asset allocation: Tests for robustness. *International Journal of Financial Engineering* 4:2. 1750021 (electronic).

Forsyth, P. and K. Vetzal (2019). Optimal asset allocation for retirement saving: Deterministic vs. time consistent adaptive strategies. *Applied Mathematical Finance* 26(1), 1–37.

Gaivoronski, A. A., S. Krylov, and N. van der Wijst (2005). Optimal portfolio selection and dynamic benchmark tracking. *European Journal of Operational Research* 163(1), 115–131.

Gao, Z., Y. Gao, Y. Hu, Z. Jiang, and J. Su (2020). Application of deep q-network in portfolio management. In *2020 5th IEEE International Conference on Big Data Analytics (ICBDA)*, pp. 268–275.

Gerrard, R., M. Hiabu, I. Kyriakou, and J. P. Nielsen (2019). Communication and personal selection of pension saver's financial risk. *European Journal of Operational Research* 274(3), 1102–1111.

Gerrard, R., I. Kyriakou, J. P. Nielsen, and P. Vodička (2022). On optimal constrained investment strategies for long-term savers in stochastic environments and probability hedging. *European Journal of Operational Research* 307(2), 948–962.

Goetzmann, W., J. Ingersoll, M. Spiegel, and I. Welch (2002). Sharpening Sharpe ratios. *NBER Working Paper 9116* .

Goetzmann, W., J. Ingersoll, M. Spiegel, and I. Welch (2007). Manipulation and manipulation-proof performance measures. *Review of Financial Economics* 20, 1503–1546.

Goodfellow, I., Y. Bengio, and A. Courville (2016). *Deep learning.* MIT press.

Government Pension Fund Global (2022). Investment strategy. `www.nbim.no/en/the-fund/how-we-invest/investment-strategy/`.

Granziol, D., X. Wan, and S. Roberts (2020). Gadam: Combining adaptivity with iterate averaging gives greater generalisation. *Working paper* .

Han, J. and E. Weinan (2016). Deep learning approximation for stochastic control problems. *NIPS Deep Reinforcement Learning Workshop* .

Hougan, M. (2015). Tracking difference, the perfect ETF metric. *ETF.com* https://www.etf.com/sections/blog/tracking-difference-perfect-etf-metric.

Huré, C., H. Pham, A. Bachouch, and N. Langrené (2021). Deep neural networks algorithms for stochastic control problems on finite horizon: Convergence analysis. *SIAM Journal on Numerical Analysis* 59(1).

Kashyap, A. K., N. Kovrijnykh, J. Li, and A. Pavlova (2021). The benchmark inclusion subsidy. *Journal of Financial Economics* Forthcoming.

Korn, R. and C. Lindberg (2014). Portfolio optimization for an investor with a benchmark. *Decisions in Economics and Finance* 37, 373–384.

Kou, S. G. (2002). A jump-diffusion model for option pricing. *Management Science* 48(8), 1086–1101.

Kou, S. G. (2007). Jump-diffusion models for asset pricing in financial engineering. In J. Birge and V. Linetsky, eds., *Handbooks in OR and MS, Vol. 15*, chapter 2, pp. 73–116. Elsevier B.V., New York.

Lehalle, C.-A. and G. Simon (2021). Portfolio selection with active strategies: how long only constraints shape convictions. *Journal of Asset Management* 22(2), 443–463.

Li, Z., K. H. Tsang, and H. Y. Wong (2020). Lasso-based simulation for high-dimensional multi-period portfolio optimization. *IMA Journal of Management Mathematics* 31(3), 257–280.

Lim, A. and B. Wong (2010). A benchmark approach to optimal asset allocation for insurers and pension funds. *Insurance: Mathematics and Economics* 46(2), 317–327.

Lu, W.-M., J. S. Liu, Q. L. Kweh, and C.-W. Wang (2016). Exploring the benchmarks of the taiwanese investment trust corporations: Management and investment efficiency perspectives. *European Journal of Operational Research* 248(2), 607–618.

Lucarelli, G. and M. Borrotti (2020). A deep q-learning portfolio management framework for the cryptocurrency market. *Neural Computing and Applications* 32(23), 17229–17244.

Merton, R. (1976). Option pricing when underlying stock returns are discontinuous. *Journal of Financial Economics* 3, 125–144.

Ni, C., Y. Li, P. Forsyth, and R. Carroll (2022). Optimal asset allocation for outperforming a stochastic benchmark target. *Quantitative Finance* .

Nicolosi, M., F. Angelini, and S. Herzel (2018). Portfolio management with benchmark related incentives under mean reverting processes. *Annals of Operations Research* 266, 373–394.

Oderda, G. (2015). Stochastic portfolio theory optimization and the origin of rule based investing. *Quantitative Finance* 15(8), 1259–1266.

Oksendal, B. and A. Sulem (2019). *Applied Stochastic Control of Jump Diffusions.* Springer, 3rd edition.

Park, H., M. K. Sim, and D. G. Choi (2020). An intelligent financial portfolio trading strategy using deep q-learning. *Expert Systems with Applications* 158.

Pastant, M.-S. (2018). Tracking difference - a must when comparing etfs. *BNPParibas* https://investors-corner.bnpparibas-am.com/investing/tracking-difference-comparing-etfs.

Politis, D. and J. Romano (1994). The stationary bootstrap. *Journal of the American Statistical Association* 89, 1303–1313.

Politis, D., J. Romano, and M. Wolf (1999). *Subsampling.* Springer-Verlag. Chapter 4: Subsampling for nonstationary time series.

Politis, D. N. (2003). The impact of bootstrap methods on time series analysis. *Statistical Science* 18:2, 219–230.

Reppen, A. M., H. M. Soner, and V. Tissot-Daguette (2022). Deep stochastic optimization in finance. arxiv.org/abs/2205.04604.

Scott, L. and S. Cavaglia (2017). A wealth management perspective on factor premia and the value of downside protection. *The Journal of Portfolio Management* 43:3, 1–9.

Simonian, J. and A. Martirosyan (2022). Sharpe parity redux. *The Journal of Portfolio Management* 48:9, 183–193.

Tepla, L. (2001). Optimal investment with minimum performance constraints. *Journal of Economic Dynamics and Control* 25, 1629–1645.

Tsang, K. H. and H. Y. Wong (2020). Deep-learning solution to portfolio selection with serially dependent returns. *SIAM Journal on Financial Mathematics* 11(2), 593–619.

Van Staden, P. M., D. Dang, and P. Forsyth (2021). On the distribution of terminal wealth under dynamic mean-variance optimal investment strategies. *SIAM Journal on Financial Mathematics* 12(2), 566–603.

Van Staden, P. M., P. A. Forsyth, and Y. Li (2023). Beating a benchmark: dynamic programming may not be the right numerical approach. *SIAM Journal on Financial Mathematics* 14(2), 407–451.

Vanguard (2014). Understanding tracking difference and tracking error. *Vanguard White Paper* https://www.vanguard.com.hk/documents/understanding-td-and-te-en.pdf.

Wander, B. H. (2000). The volatility of relative performance as a risk measure. *The Journal of Investing* 9:2 (summer), 39–44.

Wang, R. and D. P. Foster (2020). What are the statistical limits of offline RL with linear function approximation? *Working paper* .

Yao, D. D., S. Zhang, and X. Y. Zhou (2006). Tracking a financial benchmark using a few assets. *Operations Research* 54(2), 232–246.

Zhang, Q. and Y. Gao (2017). Portfolio selection based on a benchmark process with dynamic value-at-risk constraints. *Journal of Computational and Applied Mathematics* 313, 440–447.

Zhao, Y. (2007). A dynamic model of active portfolio management with benchmark orientation. *Journal of Banking and Finance* 31, 3336–3356.

Zhou, X. and D. Li (2000). Continuous time mean variance portfolio selection: a stochastic LQ framework. *Applied Mathematics and Optimization* 42, 19–33.

Appendix A: Proofs of key results

In this appendix, proofs of the key results of Section 3 are presented.

A.1: Proof of Theorem 3.4

Fix $(t, w, \hat{w}) \in [t_0, T] \times \mathbb{R}^2$, $\delta > 0$ as well as the investor strategy $\boldsymbol{\varrho}(t) = \boldsymbol{\varrho}(t, \boldsymbol{X}(t))$ and benchmark strategy $\hat{\boldsymbol{\varrho}}(t) = \hat{\boldsymbol{\varrho}}\left(t, \hat{W}(t)\right)$, where we omit dependence of the controls on $\boldsymbol{X}(t)$ and $\hat{W}(t)$ for notational simplicity.

Proceeding informally, suppose that the objective functional of problem (3.4),

$$J(t, w, \hat{w}; \boldsymbol{\varrho}) \quad = \quad E_{\boldsymbol{\varrho}}^{t,w,\hat{w}}\left[\int_t^T \left(W(s) - e^{\delta s}\hat{W}(s)\right)^2 ds\right], \tag{A.1}$$

is sufficiently smooth. For $t \in [0, T)$ and $h > 0$ such that $t + h \leq T$, the tower property gives

$$E_{\boldsymbol{\varrho}}^{t,w,\hat{w}}\left[\int_t^{t+h} dJ\left(s, W(s), \hat{W}(s); \boldsymbol{\varrho}\right)\right] \quad = \quad -E_{\boldsymbol{\varrho}}^{t,w,\hat{w}}\left[\int_t^{t+h} \left(W(s) - e^{\delta s}\hat{W}(s)\right)^2 ds\right]. \tag{A.2}$$

Applying Itô's lemma for jump processes (see for example Oksendal and Sulem (2019)), and taking expectations, we also have

$$\begin{aligned}
& E_{\boldsymbol{\varrho}}^{t,w,\hat{w}}\left[\int_t^{t+h} dJ\left(s, W(s), \hat{W}(s); \boldsymbol{\varrho}\right)\right] \\
= \quad & E_{\boldsymbol{\varrho}}^{t,w,\hat{w}}\left[\int_t^{t+h}\left(\frac{\partial J}{\partial t} + \frac{\partial J}{\partial w}\cdot\left\{W(s)\cdot\left[r + \boldsymbol{\alpha}^\top\boldsymbol{\varrho}(s)\right] + q\right\} + \frac{\partial J}{\partial \hat{w}}\cdot\left\{\hat{W}(s)\cdot\left[r + \boldsymbol{\alpha}^\top\hat{\boldsymbol{\varrho}}(s)\right] + q\right\}\right)\cdot ds\right] \\
& + E_{\boldsymbol{\varrho}}^{t,w,\hat{w}}\left[\int_t^{t+h}\frac{1}{2}\left(\frac{\partial^2 J}{\partial \hat{w}^2}\cdot\hat{W}^2(s)\left(\hat{\boldsymbol{\varrho}}(s)\right)^\top\boldsymbol{\Sigma}\hat{\boldsymbol{\varrho}}(s) + \frac{\partial^2 J}{\partial w^2}\cdot W^2(s)\left(\boldsymbol{\varrho}(s)\right)^\top\boldsymbol{\Sigma}\boldsymbol{\varrho}(s)\right)\cdot ds\right] \\
& + E_{\boldsymbol{\varrho}}^{t,w,\hat{w}}\left[\int_t^{t+h}\frac{\partial^2 J}{\partial w \partial \hat{w}}\cdot W(s)\hat{W}(s)\left(\boldsymbol{\varrho}(s)\right)^\top\boldsymbol{\Sigma}\hat{\boldsymbol{\varrho}}(s)\cdot ds\right] \\
& + E_{\boldsymbol{\varrho}}^{t,w,\hat{w}}\left[\sum_{i=1}^{N_a^r}\lambda_i\int_t^{t+h}\left[\int_0^\infty \phi\left(s, W(s^-), \hat{W}(s^-), \xi_i\right) f_{\xi_i}(\xi_i)\, d\xi_i - J\left(W(s^-), \hat{W}(s^-), s\right)\right] ds\right]
\end{aligned} \tag{A.3}$$

where

$$\phi\left(s, W\left(s^{-}\right), \hat{W}\left(s^{-}\right), \xi_i\right) = J\left(s, W\left(s^{-}\right) + \varrho_i\left(s^{-}\right) W\left(s^{-}\right)(\xi_i - 1), \hat{W}\left(s^{-}\right) + \hat{\varrho}_i\left(s^{-}\right) \hat{W}\left(s^{-}\right)(\xi_i - 1)\right). \tag{A.4}$$

Setting (A.2) and (A.3) equal, we proceed informally by dividing by h, taking limits as $h \downarrow 0$, interchanging the limit and expectation, and using the dynamic programming principle to establish (3.15).

Using the preceding results merely as a guide to the intuition as to the form of (3.15), the formal proof of (3.15) proceeds by using a suitably smooth test function instead of the objective functional - see for example Applebaum (2004); Oksendal and Sulem (2019).

A.2: Proof of Proposition 3.5

In the definitions of the functions D and F in (3.20) and (3.21), respectively, we have emphasized the dependence on the parameters δ and q for the purposes of the subsequent analysis. However, for the purposes of this proof, we will simply use the notation $D(t) \coloneqq D(t;\delta)$ and $F(t) \coloneqq F(t;\delta,q)$. As a result of Assumption 3.2, we take $\hat{\boldsymbol{\varrho}}$ as given, so that the quadratic source term $\left(w - e^{\delta t}\hat{w}\right)^2$ in (3.15) suggests an ansatz for the value function V_{cd} in Theorem 3.4 of the form

$$V_{cd}\left(t, w, \hat{w}, \hat{\boldsymbol{\varrho}}\right) = A(t) w^2 + \hat{A}(t)\hat{w}^2 + D(t) w\hat{w} + F(t) w + \hat{F}(t)\hat{w} + C(t), \tag{A.5}$$

where $A, \hat{A}, D, F, \hat{F}$ and C are unknown functions of time. If (A.5) is correct, then the pointwise supremum in (3.15) is attained by $\boldsymbol{\varrho}^*_{cd}$ satisfying the relationship

$$\left[w \cdot \frac{\partial^2 V_{cd}}{\partial w^2}\right] \cdot \boldsymbol{\varrho}^*_{cd} = -\left[\frac{\partial V_{cd}}{\partial w} \cdot \left(\boldsymbol{\Sigma} + \boldsymbol{\Lambda}\right)^{-1} \tilde{\boldsymbol{\mu}} + \hat{w} \cdot \frac{\partial^2 V_{cd}}{\partial w \partial \hat{w}} \cdot \hat{\boldsymbol{\varrho}}\right]. \tag{A.6}$$

Since (A.5) implies that the relevant partial derivatives are of the form

$$\frac{\partial V_{cd}}{\partial w} = 2A(t) w + F(t) + D(t)\hat{w}, \qquad \frac{\partial^2 V_{cd}}{\partial w^2} = 2A(t), \qquad \text{and} \qquad \frac{\partial^2 V_{cd}}{\partial w \partial \hat{w}} = D(t), \tag{A.7}$$

respectively, substitution into (A.6) results in the optimal control $\boldsymbol{\varrho}^*_{cd}$ of the form (3.18), where h_{cd} and g_{cd} are given by (3.19). It now only remains to determine the functions A, D and F. Substituting (A.5) and (A.6) into the PIDE (3.15)-(3.16), we obtain the following set of ordinary differential equations (ODEs) for the functions A, D and F on $t \in [t_0, T]$,

$$\frac{d}{dt}A(t) = -1 - (2r - \eta) A(t), \qquad A(T) = 0, \tag{A.8}$$

$$\frac{d}{dt}D(t) = -(2r - \eta) D(t) + 2e^{\delta t}, \qquad D(T) = 0, \tag{A.9}$$

$$\frac{d}{dt}F(t) = -(r - \eta) F(t) - 2qA(t) - qD(t), \qquad F(T) = 0, \tag{A.10}$$

where η is given by (3.9). Solving the ODEs (A.8), (A.9) and (A.10) then results in the functions A, D and F reported in (3.20) and (3.21), respectively.

A.3: Properties of g_{cd}

The following lemma analyzes the properties of g_{cd} in (3.19) .

Lemma A.1. *(Properties of g_{cd}) The function $g_{cd}(t;\delta) = -\frac{1}{2}D(t;\delta)/A(t)$ in (3.19) has the following properties for $t \in [t_0 = 0, T]$ and $\delta > 0$:*

(i) For a fixed $t \in [t_0 = 0, T]$, the function $\delta \to g_{cd}(t;\delta)$ is strictly increasing on $\delta \in (0, \infty)$.

(ii) For a fixed $\delta > 0$, the function $t \to g_{cd}(t;\delta)$ is strictly increasing on $t \in [t_0, T]$.

(iii) By continuity,

$$g_{cd}(T;\delta) = e^{\delta T}. \tag{A.11}$$

(iv) $g_{cd}(t;\delta)$ *admits the following bounds:*

$$e^{\delta t} \;<\; g_{cd}(t;\delta) \;<\; e^{\delta T}, \qquad \forall t \in [t_0, T). \tag{A.12}$$

Proof. The definition (3.19) can be used to obtain the following alternative forms of g_{cd},

$$g_{cd}(t;\delta) \;=\; e^{\delta t} \cdot \left(\frac{e^{(2r-\eta+\delta)(T-t)} - 1}{(2r-\eta+\delta)(T-t)} \right) \cdot \left(\frac{(2r-\eta)(T-t)}{e^{(2r-\eta)(T-t)} - 1} \right) \tag{A.13}$$

$$= \; e^{\delta(T+t)} \cdot \left(\frac{\int_t^T e^{(\eta-2r-\delta)u} du}{\int_t^T e^{(\eta-2r)u} du} \right). \tag{A.14}$$

To prove property (i) of Lemma A.1, it is sufficient to note that since the following auxiliary function is non-negative and strictly increasing,

$$\phi_{cd}(y) := \frac{(e^y - 1)}{y}, \qquad \forall y \in \mathbb{R}, \tag{A.15}$$

we can use (A.13) to show that for a fixed $t \in [t_0, T]$, the function $\delta \to g_{cd}(t;\delta)$ is the product of two non-negative, strictly increasing functions of $\delta \in (0, \infty)$. Property (iii) follows from taking limits as $t \uparrow T$ in (A.13). Next, we observe that since $\delta > 0$ and $e^{-\delta(u-t)} < 1 < e^{\delta(T-u)}$ for $u \in (t, T)$, the monotonicity of (Riemann) integrals imply that

$$0 \;<\; \int_t^T e^{(\eta-2r)u} e^{-\delta(u-t)} du \;<\; \int_t^T e^{(\eta-2r)u} du \;<\; \int_t^T e^{(\eta-2r)u} e^{\delta(T-u)} du, \qquad \forall t \in [t_0, T). \tag{A.16}$$

Re-arranging (A.16) and using the alternative form (A.14) of g_{cd}, we obtain the bounds (A.12) reported in property (iv). Finally, to prove property (ii), we start by observing that we can use (A.14) to obtain

$$\frac{d}{dt} g_{cd}(t;\delta) \;=\; \delta \cdot g_{cd}(t;\delta) - \frac{(\eta - 2r)}{e^{(\eta-2r)(T-t)} - 1} \left[e^{\delta T} - g_{cd}(t;\delta) \right]. \tag{A.17}$$

Taking limits in (A.17) as $t \to T$, we use (A.11) to obtain $\lim_{t \uparrow T} \left[\frac{d}{dt} g_{cd}(t;\delta) \right] = \frac{1}{2} \delta e^{\delta T} > 0$, and therefore we only need to show that $\frac{d}{dt} g_{cd}(t;\delta) > 0$ if $t < T$. In the case where $\eta - 2r > \delta\,(> 0)$, this follows in a straightforward fashion from the expression (A.17), the bounds (A.12) and the properties of the function (A.15). To show that we also have $\frac{d}{dt} g_{cd}(t;\delta) > 0$ for $t < T$ in the case where $\eta - 2r \leq \delta$, we note that (A.14) can be used to show that

$$\frac{d}{dt} g_{cd}(t;\delta) > 0 \quad \Longleftrightarrow \quad \delta(T-t) > \frac{(\eta - 2r - \delta)(T-t)}{\left[e^{(\eta-2r-\delta)(T-t)} - 1 \right]} - \frac{(\eta - 2r)(T-t)}{\left[e^{(\eta-2r)(T-t)} - 1 \right]}, \quad \forall t < T, \quad \delta > 0. \tag{A.18}$$

Since we are now only concerned with the case where $\eta - 2r \leq \delta$ in (A.18), the inequality in (A.18) suggests we consider the properties of the auxiliary function

$$\varphi_{cd}(x, y) \;=\; y - \frac{(x-y)}{\left[e^{(x-y)} - 1 \right]} + \frac{x}{\left[e^{x} - 1 \right]}, \qquad \forall x \leq y, \text{ and } y > 0. \tag{A.19}$$

Taking limits in (A.19), and noting that $x > 0$ in a sufficiently small neighborhood of $y > 0$, it follows that $\lim_{x \uparrow y} \varphi_{cd}(x, y) > 0$. In the case of the strict inequality $x < y$, the properties of ϕ_{cd} in (A.15) can again be used to show $\varphi_{cd}(x, y) > 0$. In summary, we therefore have $\varphi_{cd}(x, y) > 0$, $\forall x \leq y$ and $y > 0$, and thus by (A.18) implying $\frac{d}{dt} g_{cd}(t;\delta) > 0$, $\forall t < T$ and $\eta - 2r \leq \delta$, completing the proof of property (ii). □

A.4: Properties of h_{cd}

The following lemma analyzes the properties of h_{cd} in (3.19) .

Lemma A.2. *(Properties of h_{cd}) The function $h_{cd}(t;\delta,q) = -\frac{1}{2} F(t;\delta,q) / A(t)$ in (3.19) has the following properties for $t \in [t_0 = 0, T]$, $\delta > 0$ and $q \geq 0$:*

(i) *For fixed values of $t \in [t_0 = 0, T]$ and $q > 0$, the function $\delta \to h_{cd}(t;\delta,q)$ is strictly increasing on $\delta \in (0, \infty)$. If $q = 0$, $h_{cd}(t;\delta,q) \equiv 0$.*

(ii) For fixed values of $t \in [t_0 = 0, T]$ and $\delta > 0$, the function $q \rightarrow h_{cd}(t; \delta, q)$ is strictly increasing on $q \in [0, \infty)$.

(iii) By continuity,

$$h_{cd}(T; \delta, q) \quad = \quad 0. \tag{A.20}$$

(iv) $h_{cd}(t; \delta, q)$ admits the following bounds:

$$0 \; \leq \; h_{cd}(t; \delta, q) \; \leq \; h_{qd}(t; \beta = \delta, q), \qquad \forall t \in [t_0 = 0, T]. \tag{A.21}$$

Proof. Using the function $A(t)$ in (3.20), it can be shown that h_{cd} can be written in terms of the function g_{cd} in (3.19) as follows,

$$h_{cd}(t; \delta, q) \quad = \quad q \cdot \int_t^T [g_{cd}(u; \delta) - 1] \cdot \left[\frac{A(u) e^{(r-\eta)u}}{A(t) e^{(r-\eta)t}} \right] du. \tag{A.22}$$

Property (i) of Lemma A.2 therefore immediately follows from the corresponding property (i) of $g_{cd}(t; \delta)$ reported in Lemma A.1. Next, we observe that $A(t) \geq 0$ for all $t \geq T$, while the bounds (A.12) imply that $g_{cd}(t; \delta) > 1$ for all $t \in [t_0, T]$ and all $\delta > 0$. Therefore, since neither $g_{cd}(t; \delta)$ nor $A(t)$ depends on the rate of contribution $q \geq 0$, property (ii) also follows from (A.22). Property (iii) is obvious from taking the limit as $t \uparrow T$ in (A.22). Considering property (iv), we start by observing that

$$\frac{A(u) e^{(r-\eta)u}}{A(t) e^{(r-\eta)t}} = \left[\frac{A(u) e^{(2r-\eta)u}}{A(t) e^{(2r-\eta)t}} \right] \cdot e^{-r(u-t)} = \left[\frac{e^{(2r-\eta)T} - e^{(2r-\eta)u}}{e^{(2r-\eta)T} - e^{(2r-\eta)t}} \right] \cdot e^{-r(u-t)}, \qquad \forall u \in [t, T]. \tag{A.23}$$

Combining the expression (A.23) with (A.12) and (A.11), we observe that regardless of the sign of $(2r - \eta)$, we have

$$0 \; \leq \frac{A(u) e^{(2r-\eta)u}}{A(t) e^{(2r-\eta)t}} \; \leq \; 1 \; \leq \; \frac{e^{\delta T} - 1}{[g_{cd}(u; \delta) - 1]}, \qquad \forall u \in [t, T]. \tag{A.24}$$

Multiplying (A.24) by $q \cdot [g_{cd}(u; \delta) - 1] \cdot e^{-r(u-t)} \geq 0$, and subsequently integrating $u \in [t, T]$, the monotonicity of integrals together with (3.14), (A.22) and (A.23) yields the desired bounds (A.21) reported in property (iv). □

Note that Lemma A.2 does not report the behavior of the function $t \rightarrow h_{cd}(t; \delta, q)$ for fixed values of q and δ, since it can be shown (using results (A.22) and (A.8)) that

$$\frac{d}{dt} h_{cd}(t; \delta, q) \quad = \quad \left(r + \frac{1}{A(t)} \right) \cdot h_{cd}(t; \delta, q) - q \cdot [g_{cd}(t; \delta) - 1]. \tag{A.25}$$

The first term of (A.25) is typically non-negative (for example it is guaranteed if $r > 0$) by (A.21), while the second term of (A.25) is non-positive by (A.12). Numerical experiments show that $t \rightarrow h_{cd}(t; \delta, q), t \in [t_0 = 0, T]$ can therefore be increasing or decreasing on different sub-intervals of $[t_0, T]$ depending on the exact combinations of parameters. However, the properties of $h_{cd}(t; \delta, q)$ reported in Lemma A.2 are sufficient to analyze the implications of using the CD-optimal control.

Appendix B: Additional results - comparison of investment strategies

This appendix complements the discussion and results of Subsection 3.4 and Section 5.

B.1: Comparison of expectations and parameters

We show that under the assumptions of Section 3 (Assumption 3.1, Assumption 3.2 and wealth dynamics (3.11)-(3.12)), the claims (3.30) and (3.31) hold.

Naturally, some information regarding the benchmark strategy as feedback control $\hat{\boldsymbol{\varrho}}\left(t, \hat{W}(t)\right) = \left(\hat{\varrho}_k\left(t, \hat{W}(t)\right) : k = 1, ...\right.$ is required, specifically that it has to be at least somewhat economically reasonable. To make this concrete, the following two propositions place a very weak requirement on the benchmark strategy, namely that $\hat{\boldsymbol{\varrho}}$ satisfies

$$E_{\hat{\boldsymbol{\varrho}}}^{t_0,w_0}\left[\hat{W}(t)\cdot\tilde{\boldsymbol{\mu}}^{\top}\hat{\boldsymbol{\varrho}}\left(t,\hat{W}(t)\right)\right]=\sum_{i=1}^{N_a^r}(\mu_i-r)\cdot E_{\hat{\boldsymbol{\varrho}}}^{t_0,w_0}\left[\hat{W}(t)\cdot\hat{\varrho}_i\left(t,\hat{W}(t)\right)\right]\geq 0,\quad \forall t\in[t_0,T]. \tag{B.1}$$

Condition (B.1) can be interpreted as a "weighted no short-selling in expectation" restriction, since it is for example satisfied if $E_{\hat{\boldsymbol{\varrho}}}^{t_0,w_0}\left[\hat{W}(t)\cdot\hat{\varrho}_i\left(t,\hat{W}(t)\right)\right]\geq 0$ for all $i\in\{1,...,N_a^r\}$, i.e. if there is no risky asset for which the benchmark's expected investment is negative (this follows since $(\mu_i-r)>0$ by assumption). Condition (B.1) is clearly reasonable for most benchmark strategies used in practice, where trading (let alone short-selling) would typically be restricted when $\hat{W}(t)<0$. Furthermore, considering the application of (B.1) in the proofs of Proposition B.1 and Proposition B.2, it is clear that (B.1) can be refined substantially when more is known about the benchmark strategy, for example in the case where the benchmark is a deterministic function of time (e.g. "glide path" strategies) or a constant proportion investment strategy (see Forsyth and Vetzal (2019)). However, for our current purposes, (B.1) is convenient due to its relative generality.

We start by verifying the relationship (3.31) under the assumption of equal parameters, $\delta=\beta$.

Proposition B.1. *(Comparison of wealth expectations, $CD(\delta)$ and $QD(\beta=\delta)$) Suppose that Assumption 3.1, Assumption 3.2 and wealth dynamics (3.11)-(3.12) are applicable. In addition, assume that the given benchmark strategy $\hat{\mathcal{P}}$ in (3.1) satisfies the condition (B.1).*

Let $E_{\boldsymbol{\varrho}_{qd}^*}^{t_0,w_0}\left[W_{qd}^*(t;\beta=\delta)\right]$ denote the expectation of the $QD(\beta=\delta)$-optimal wealth under control (3.13) with parameter value $\beta=\delta$, where δ is the value used to obtain $E_{\boldsymbol{\varrho}_{cd}^*}^{t_0,w_0}\left[W_{cd}^*(t;\delta)\right]$ under control (3.18). Then

$$E_{\boldsymbol{\varrho}_{cd}^*}^{t_0,w_0}\left[W_{cd}^*(t;\delta)\right] \quad < \quad E_{\boldsymbol{\varrho}_{qd}^*}^{t_0,w_0}\left[W_{qd}^*(t;\beta=\delta)\right],\qquad \forall t\in(t_0,T]. \tag{B.2}$$

Proof. For any benchmark strategy satisfying Assumption 3.2 and wealth dynamics (3.12), let $\hat{K}(t)$ and $\hat{\chi}(t)$ denote the functions

$$\hat{K}(t)=E_{\hat{\boldsymbol{\varrho}}}^{t_0,w_0}\left[\hat{W}(t)\right],\qquad \hat{\chi}(t)=E_{\hat{\boldsymbol{\varrho}}}^{t_0,w_0}\left[\hat{W}(t)\cdot\tilde{\boldsymbol{\mu}}^{\top}\hat{\boldsymbol{\varrho}}\left(t,\hat{W}(t)\right)\right],\qquad t\in[t_0,T], \tag{B.3}$$

where the wealth dynamics (3.12) imply that $\hat{K}(t)$ can be written in terms of $\hat{\chi}(t)$ as

$$\hat{K}(t) \quad = \quad w_0e^{rt}+\int_0^t\left[\hat{\chi}(u)+q\right]e^{r(t-u)}du. \tag{B.4}$$

For benchmark strategies also satisfying condition (B.1), which by (B.3) means that we are given $\hat{\chi}(t)\geq 0$, then by (B.4) we also have $\hat{K}(t)>0$. As a result, with η given by (3.9), we have

$$\eta\cdot\hat{K}(t)+\hat{\chi}(t) \quad > \quad 0,\qquad \forall t\in[t_0,T]. \tag{B.5}$$

Now consider the investor strategies. Substituting the CD-optimal control (3.18) into the investor wealth dynamics (3.11), we take expectations and use the definitions (B.3) to obtain

$$\begin{aligned}E_{\boldsymbol{\varrho}_{cd}^*}^{t_0,w_0}\left[W_{cd}^*(t;\delta)\right] \quad &= \quad w_0e^{(r-\eta)t}+q\int_0^t e^{(r-\eta)(t-u)}du+\eta\cdot\int_0^t h_{cd}(u;\delta,q)\cdot e^{(r-\eta)(t-u)}du\\ &\quad +\int_0^t g_{cd}(u;\delta)\cdot\left[\eta\cdot\hat{K}(u)+\hat{\chi}(u)\right]e^{(r-\eta)(t-u)}du,\end{aligned} \tag{B.6}$$

where h_{cd} and g_{cd} are given by (3.19). (B.6). Note that if more is known about the benchmark strategy, closed-form expressions for $\hat{K}(t)$ and $\hat{\chi}(t)$ might allow further simplification of (B.6).

Similarly, substituting the QD-optimal control (3.13) into the investor wealth dynamics (3.11) and taking expectations yields

$$\begin{aligned}E_{\boldsymbol{\varrho}_{qd}^*}^{t_0,w_0}\left[W_{qd}^*(t;\beta)\right] \quad &= \quad w_0e^{(r-\eta)t}+q\int_0^t e^{(r-\eta)(t-u)}du+\eta\cdot\int_0^t h_{qd}(u;\beta,q)\cdot e^{(r-\eta)(t-u)}du\\ &\quad +\int_0^t e^{\beta T}\cdot\left[\eta\cdot\hat{K}(u)+\hat{\chi}(u)\right]e^{(r-\eta)(t-u)}du,\end{aligned} \tag{B.7}$$

where h_{qd} is given by (3.14). Setting $\beta \equiv \delta$ in (B.7), the difference in expectations (B.7) and (B.6) is given by

$$
\begin{aligned}
& E^{t_0,w_0}_{\boldsymbol{\varrho}^*_{qd}}\left[W^*_{qd}\left(t;\beta=\delta\right)\right] - E^{t_0,w_0}_{\boldsymbol{\varrho}^*_{cd}}\left[W^*_{cd}\left(t;\delta\right)\right] \\
= \;& \eta \cdot \int_0^t \left[h_{qd}\left(u;\beta=\delta,q\right) - h_{cd}\left(u;\delta,q\right)\right] e^{(r-\eta)(t-u)} du \\
& + \int_0^t \left[e^{\delta T} - g_{cd}\left(u;\delta\right)\right] \cdot \left[\eta \cdot \hat{K}\left(u\right) + \hat{\chi}\left(u\right)\right] e^{(r-\eta)(t-u)} du. \qquad \text{(B.8)}
\end{aligned}
$$

From Lemma A.1, we know that $e^{\delta T} > g_{cd}\left(t;\delta\right), \forall t < T$ (see (A.12)), while Lemma A.2 shows that $h_{qd}\left(t;\beta=\delta,q\right) \geq h_{cd}\left(t;\delta,q\right)$, $\forall t \leq T$ (see (A.21)). Combining these results with (B.5), expression (B.8) implies that (B.2) holds. □

The following proposition verifies the claim that if we insist on achieving equal expectations of terminal wealth (3.29), the parameters satisfy (3.30).

Proposition B.2. *(Comparison of parameter values $\delta^{\mathcal{E}}$ and $\beta^{\mathcal{E}}$, equal expectations $\mathcal{E}$). Suppose that Assumption 3.1, Assumption 3.2 and wealth dynamics (3.11)-(3.12) are applicable. In addition, assume that the given benchmark strategy $\hat{\mathcal{P}}$ in (3.1) satisfies the condition (B.1).*

If the investor chooses parameter values $\delta^{\mathcal{E}}, \beta^{\mathcal{E}} > 0$ such that the resulting CD$\left(\delta=\delta^{\mathcal{E}}\right)$-optimal and QD$\left(\beta=\beta^{\mathcal{E}}\right)$-optimal controls both result in the same expected value of terminal wealth $\mathcal{E}$,

$$
E^{t_0,w_0}_{\boldsymbol{\varrho}^*_{cd}}\left[W^*_{cd}\left(T;\delta=\delta^{\mathcal{E}}\right)\right] \equiv E^{t_0,w_0}_{\boldsymbol{\varrho}^*_{qd}}\left[W^*_{qd}\left(T;\beta=\beta^{\mathcal{E}}\right)\right] \equiv \mathcal{E}, \qquad \text{(B.9)}
$$

then

$$
\delta^{\mathcal{E}} > \beta^{\mathcal{E}}. \qquad \text{(B.10)}
$$

Proof. Since the benchmark strategy satisfies Assumption 3.2, wealth dynamics (3.12) and condition (B.1), we know that (B.3), (B.4) and (B.5) hold. Considering the QD-optimal strategy, suppose that the investor chooses the parameter value $\beta = \beta^{\mathcal{E}} > 0$ for the QD problem such that $E^{t_0,w_0}_{\boldsymbol{\varrho}^*_{qd}}\left[W^*_{qd}\left(T;\beta^{\mathcal{E}}\right)\right] \equiv \mathcal{E}$. By (B.2), we therefore have

$$
E^{t_0,w_0}_{\boldsymbol{\varrho}^*_{cd}}\left[W^*_{cd}\left(T;\delta=\beta^{\mathcal{E}}\right)\right] < E^{t_0,w_0}_{\boldsymbol{\varrho}^*_{qd}}\left[W^*_{qd}\left(T;\beta^{\mathcal{E}}\right)\right] \equiv \mathcal{E}. \qquad \text{(B.11)}
$$

Considering the CD-optimal strategy, the definition of the value of $\delta^{\mathcal{E}}$ in (B.9) together with (B.11) therefore implies that

$$
E^{t_0,w_0}_{\boldsymbol{\varrho}^*_{cd}}\left[W^*_{cd}\left(T;\delta=\beta^{\mathcal{E}}\right)\right] < E^{t_0,w_0}_{\boldsymbol{\varrho}^*_{cd}}\left[W^*_{cd}\left(T;\delta=\delta^{\mathcal{E}}\right)\right] \equiv \mathcal{E}. \qquad \text{(B.12)}
$$

By Lemma A.1, we know that for any t, the function $\delta \rightarrow g_{cd}\left(t;\delta\right)$ is strictly increasing in $\delta \in (0,\infty)$. Similarly, by Lemma A.2 we know that if $q > 0$, the function $\delta \rightarrow h_{cd}\left(t;\delta,q\right)$ is also strictly increasing in δ, otherwise it is identically zero. Therefore, setting $t = T$ in (B.6), we conclude that the function $\delta \rightarrow E^{t_0,w_0}_{\boldsymbol{\varrho}^*_{cd}}\left[W^*_{cd}\left(T;\delta\right)\right]$ is strictly increasing on $\delta \in (0,\infty)$. This observation, together with (B.12), implies that we must have $\delta^{\mathcal{E}} > \beta^{\mathcal{E}}$, thereby proving (B.10). □

B.2: Proof of Proposition 3.7

For any $t \in [t_0 = 0, T]$, recalling the definition of $\hat{K}\left(t\right)$ in (B.3), define the functions

$$
K^*_{qd}\left(t;\beta\right) = E^{t_0,w_0}_{\boldsymbol{\varrho}^*_{qd}}\left[W^*_{qd}\left(t;\beta\right)\right], \qquad K^*_{cd}\left(t;\delta\right) = E^{t_0,w_0}_{\boldsymbol{\varrho}^*_{cd}}\left[W^*_{cd}\left(t;\delta\right)\right], \qquad \text{(B.13)}
$$

as well as

$$
F\left(t\right) = E^{t_0,w_0}_{\boldsymbol{\varrho}^*_{qd}}\left[W^*_{qd}\left(t;\beta=\delta\right) \cdot \mathcal{R}^*_{qd}\left(t;\beta=\delta\right)\right] - E^{t_0,w_0}_{\boldsymbol{\varrho}^*_{cd}}\left[W^*_{cd}\left(t;\delta\right) \cdot \mathcal{R}^*_{cd}\left(t;\delta\right)\right]. \qquad \text{(B.14)}
$$

Using the expressions for the optimal controls (3.13) and (3.18), and setting $\beta = \delta$ in the QD-optimal control

(3.13), $F(t)$ is given by

$$
\begin{aligned}
F(t) &= \left[h_{qd}(t;\delta,q) - h_{cd}(t;\delta,q)\right] \cdot \sum_{k=1}^{N_a^r} \left[(\boldsymbol{\Sigma}+\boldsymbol{\Lambda})^{-1}\tilde{\boldsymbol{\mu}}\right]_k \\
&\quad + \left[K_{cd}^{*}(t;\delta) - K_{qd}^{*}(t;\delta)\right] \cdot \sum_{k=1}^{N_a^r} \left[(\boldsymbol{\Sigma}+\boldsymbol{\Lambda})^{-1}\tilde{\boldsymbol{\mu}}\right]_k \\
&\quad + \left[e^{\delta T} - g_{cd}(t;\delta)\right] \cdot \left(E_{\hat{\boldsymbol{\varrho}}}^{t_0,w_0}\left[\hat{W}(t)\cdot\hat{\mathcal{R}}(t)\right] + \hat{K}(t) \cdot \sum_{k=1}^{N_a^r} \left[(\boldsymbol{\Sigma}+\boldsymbol{\Lambda})^{-1}\tilde{\boldsymbol{\mu}}\right]_k \right). \qquad \text{(B.15)}
\end{aligned}
$$

Setting $t=t_0$, we have $K_{qd}^{*}(t_0;\delta) = K_{cd}^{*}(t_0;\delta) = \hat{K}(t_0) = w_0$, so (B.15) simplifies to

$$
\begin{aligned}
F(t_0) &= w_0 \cdot \left[\mathcal{R}_{qd}^{*}(t_0;\delta) - \mathcal{R}_{cd}^{*}(t_0;\delta)\right] \qquad \text{(by definition (B.14))}, \qquad \text{(B.16)} \\
&= \left[h_{qd}(t_0;\delta,q) - h_{cd}(t_0;\delta,q)\right] \cdot \sum_{k=1}^{N_a^r} \left[(\boldsymbol{\Sigma}+\boldsymbol{\Lambda})^{-1}\tilde{\boldsymbol{\mu}}\right]_k \\
&\quad + \left[e^{\delta T} - g_{cd}(t_0;\delta)\right] \cdot w_0 \left(\hat{\mathcal{R}}(t_0) + \sum_{k=1}^{N_a^r} \left[(\boldsymbol{\Sigma}+\boldsymbol{\Lambda})^{-1}\tilde{\boldsymbol{\mu}}\right]_k \right). \qquad \text{(B.17)}
\end{aligned}
$$

By Lemma A.2 (see (A.21)), we have $\left[h_{qd}(t_0;\delta,q) - h_{cd}(t_0;\delta,q)\right] \geq 0$. Furthermore, by Lemma A.1 (see (A.12)), we have the strict inequality $\left[e^{\delta T} - g_{cd}(t_0;\delta)\right] > 0$. Given the additional assumption of $\hat{\mathcal{R}}(t_0) \geq 0$ in Proposition 3.7, and since $\sum_{k=1}^{N_a^r}\left[(\boldsymbol{\Sigma}+\boldsymbol{\Lambda})^{-1}\tilde{\boldsymbol{\mu}}\right]_k > 0$ and $w_0 > 0$, we therefore have the strict inequality $F(t_0) > 0$. Using (B.16), we have therefore confirmed that $\left[\mathcal{R}_{qd}^{*}(t_0;\delta) - \mathcal{R}_{cd}^{*}(t_0;\delta)\right] > 0$, which is the claim (3.32) of Proposition 3.7.

Setting $t=T$ in (B.15), we have

$$
\begin{aligned}
F(T) &= E_{\boldsymbol{\varrho}_{qd}^{*}}^{t_0,w_0}\left[W_{qd}^{*}(T;\delta)\cdot\mathcal{R}_{qd}^{*}(T;\delta)\right] - E_{\boldsymbol{\varrho}_{cd}^{*}}^{t_0,w_0}\left[W_{cd}^{*}(T;\delta)\cdot\mathcal{R}_{cd}^{*}(T;\delta)\right] \qquad \text{(by definition (B.14))}, \qquad \text{(B.18)} \\
&= \left[h_{qd}(T;\delta,q) - h_{cd}(T;\delta,q)\right] \cdot \sum_{k=1}^{N_a^r} \left[(\boldsymbol{\Sigma}+\boldsymbol{\Lambda})^{-1}\tilde{\boldsymbol{\mu}}\right]_k \\
&\quad + \left[K_{cd}^{*}(T;\delta) - K_{qd}^{*}(T;\delta)\right] \cdot \sum_{k=1}^{N_a^r} \left[(\boldsymbol{\Sigma}+\boldsymbol{\Lambda})^{-1}\tilde{\boldsymbol{\mu}}\right]_k \\
&\quad + \left[e^{\delta T} - g_{cd}(T;\delta)\right] \cdot \left(E_{\hat{\boldsymbol{\varrho}}}^{t_0,w_0}\left[\hat{W}(T)\cdot\hat{\mathcal{R}}(T)\right] + \hat{K}(T) \cdot \sum_{k=1}^{N_a^r} \left[(\boldsymbol{\Sigma}+\boldsymbol{\Lambda})^{-1}\tilde{\boldsymbol{\mu}}\right]_k \right). \qquad \text{(B.19)}
\end{aligned}
$$

By Lemma A.1, $\left[e^{\delta T} - g_{cd}(T;\delta)\right] = 0$, the final term of (B.19) vanishes, and thus no assumptions (other than Assumption 3.2) regarding the benchmark strategy is required. In addition, the first term of (B.19) vanishes as well, since $h_{cd}(T;\delta,q) = h_{qd}(T;\delta,q) = 0$ by Lemma A.2. By Proposition B.1 and definitions (B.13), we have $K_{cd}^{*}(T;\delta) < K_{qd}^{*}(T;\beta=\delta)$, and since $\sum_{k=1}^{N_a^r}\left[(\boldsymbol{\Sigma}+\boldsymbol{\Lambda})^{-1}\tilde{\boldsymbol{\mu}}\right]_k > 0$, we therefore have $F(T) < 0$. Rearranging (B.18), we therefore obtain result (3.33) of Proposition 3.7.

B.3: Proof of Corollary 3.8

Recalling the definitions in (3.26), as well as the definition of $F(t)$ in (B.14), by linearity we have

$$
F(t) = \sum_{i=1}^{N_a^r} F_i(t), \qquad \text{(B.20)}
$$

where $F_i(t), i \in \{1, ..., N_a^r\}$ is defined as

$$
\begin{aligned}
F_i(t) &= E^{t_0,w_0}_{\boldsymbol{\varrho}^*_{qd}}\left[W^*_{qd}(t;\delta)\cdot\varrho^*_{qd,i}(t;\delta)\right] - E^{t_0,w_0}_{\boldsymbol{\varrho}^*_{cd}}\left[W^*_{cd}(t;\delta)\cdot\varrho^*_{cd,i}(t;\delta)\right] \\
&= \left[h_{qd}(t;\delta,q) - h_{cd}(t;\delta,q)\right]\cdot\left[(\boldsymbol{\Sigma}+\boldsymbol{\Lambda})^{-1}\tilde{\boldsymbol{\mu}}\right]_i \\
&\quad + \left[K^*_{cd}(t;\delta) - K^*_{qd}(t;\delta)\right]\cdot\left[(\boldsymbol{\Sigma}+\boldsymbol{\Lambda})^{-1}\tilde{\boldsymbol{\mu}}\right]_i \\
&\quad + \left[e^{\delta T} - g_{cd}(t;\delta)\right]\cdot\left(E^{t_0,w_0}_{\hat{\boldsymbol{\varrho}}}\left[\hat{W}(t)\cdot\hat{\varrho}_i(t,W(t))\right] + \hat{K}(t)\cdot\left[(\boldsymbol{\Sigma}+\boldsymbol{\Lambda})^{-1}\tilde{\boldsymbol{\mu}}\right]_i\right). \qquad \text{(B.21)}
\end{aligned}
$$

Comparing (B.21) with (B.15), it is therefore clear that the results reported in Corollary 3.8 follow from the results of Proposition 3.7.

B.4: Proof of Proposition 3.9

We prove Proposition 3.9 for the m-compounded CD-optimal strategy (3.37), since the proof of the m-compounded QD-optimal strategy (3.36) proceeds along similar lines. In order to lighten notation in the subsequent proof, we fix time t and parameter $\delta > 0$, and drop the arguments to write $\hat{\boldsymbol{\varrho}} := \hat{\boldsymbol{\varrho}}\left(t, \hat{W}(t^-)\right)$, $W^{[m]*}_{cd} := W^{[m]*}_{cd}(t^-;\delta)$, $\hat{W} := \hat{W}(t^-)$, $h_{cd} := h_{cd}(t;\delta,q)$ and $g_{cd} = g_{cd}(t;\delta)$. Similarly, the m-compounded CD-optimal strategy as per Definition 3.1 at the given time t with parameter $\delta > 0$ will simply be denoted by $\boldsymbol{\varrho}^{[m]*}_{cd} := \boldsymbol{\varrho}^{[m]*}_{cd}\left(t, \boldsymbol{X}^{[m]*}_{cd}(t^-;\delta);\delta\right)$ in this proof.

Let $m = 0$. By Definition 3.1, $\boldsymbol{\varrho}^{[m=0]*}_{cd}$ is simply the CD-optimal investment strategy (3.18). In particular, given the benchmark strategy $\hat{\boldsymbol{\varrho}}$, the assumptions of Proposition 3.9 imply that by Proposition 3.5, we simply have

$$
W^{[0]*}_{cd}\cdot\boldsymbol{\varrho}^{[0]*}_{cd} = \left[h_{cd} - \left(W^{[0]*}_{cd} - g_{cd}\hat{W}\right)\right]\cdot(\boldsymbol{\Sigma}+\boldsymbol{\Lambda})^{-1}\tilde{\boldsymbol{\mu}} + g_{cd}\hat{W}\cdot\hat{\boldsymbol{\varrho}}. \qquad \text{(B.22)}
$$

By Definition 3.1, $\boldsymbol{\varrho}^{[m=1]*}_{cd}$ is the CD-optimal investment strategy that uses strategy $\boldsymbol{\varrho}^{[m=0]*}_{cd}$ as its benchmark strategy to be outperformed. Applying Proposition 3.5 again, we therefore have

$$
W^{[1]*}_{cd}\cdot\boldsymbol{\varrho}^{[1]*}_{cd} = \left[h_{cd} - \left(W^{[1]*}_{cd} - g_{cd}W^{[0]*}_{cd}\right)\right]\cdot(\boldsymbol{\Sigma}+\boldsymbol{\Lambda})^{-1}\tilde{\boldsymbol{\mu}} + g_{cd}W^{[0]*}_{cd}\cdot\boldsymbol{\varrho}^{[0]*}_{cd}. \qquad \text{(B.23)}
$$

Substituting (B.22) into (B.23) and simplifying, we obtain

$$
\begin{aligned}
W^{[1]*}_{cd}\cdot\boldsymbol{\varrho}^{[1]*}_{cd} &= \left[h_{cd} + g_{cd}h_{cd} - W^{[1]*}_{cd} + (g_{cd})^2\hat{W}\right]\cdot(\boldsymbol{\Sigma}+\boldsymbol{\Lambda})^{-1}\tilde{\boldsymbol{\mu}} + (g_{cd})^2\hat{W}\cdot\hat{\boldsymbol{\varrho}} \\
&= \left[\left(\frac{(g_{cd})^2 - 1}{g_{cd} - 1}\right)h_{cd} - W^{[1]*}_{cd} + (g_{cd})^2\hat{W}\right]\cdot(\boldsymbol{\Sigma}+\boldsymbol{\Lambda})^{-1}\tilde{\boldsymbol{\mu}} + (g_{cd})^2\hat{W}\cdot\hat{\boldsymbol{\varrho}}, \qquad \text{(B.24)}
\end{aligned}
$$

which confirms that (3.37) holds for $m = 1$. Now assume that (3.37) holds for the $(m-1)$-compounded CD-optimal strategy $\boldsymbol{\varrho}^{[m-1]*}_{cd}$, which means that we assume

$$
W^{[m-1]*}_{cd}\cdot\boldsymbol{\varrho}^{[m-1]*}_{cd} = \left[\left(\frac{(g_{cd})^m - 1}{g_{cd} - 1}\right)\cdot h_{cd} - W^{[m-1]*}_{cd} + (g_{cd})^m\hat{W}\right]\cdot(\boldsymbol{\Sigma}+\boldsymbol{\Lambda})^{-1}\tilde{\boldsymbol{\mu}} + (g_{cd})^m\hat{W}\cdot\hat{\boldsymbol{\varrho}}. \qquad \text{(B.25)}
$$

By Definition 3.1, the m-compounded CD-optimal strategy $\boldsymbol{\varrho}^{[m]*}_{cd}$ uses $\boldsymbol{\varrho}^{[m-1]*}_{cd}$ as its benchmark strategy to be outperformed, and therefore applying Proposition 3.5 again we have

$$
W^{[m]*}_{cd}\cdot\boldsymbol{\varrho}^{[m]*}_{cd} = \left[h_{cd} - \left(W^{[m]*}_{cd} - g_{cd}W^{[m-1]*}_{cd}\right)\right]\cdot(\boldsymbol{\Sigma}+\boldsymbol{\Lambda})^{-1}\tilde{\boldsymbol{\mu}} + g_{cd}W^{[m-1]*}_{cd}\cdot\boldsymbol{\varrho}^{[m-1]*}_{cd}. \qquad \text{(B.26)}
$$

Substituting (B.25) into (B.26), we therefore obtain

$$
\begin{aligned}
W^{[m]*}_{cd}\cdot\boldsymbol{\varrho}^{[m]*}_{cd} &= \left[\left\{1 + g_{cd}\left(\frac{(g_{cd})^m - 1}{g_{cd} - 1}\right)\right\}\cdot h_{cd} - W^{[m]*}_{cd} + (g_{cd})^{m+1}\hat{W}\right]\cdot(\boldsymbol{\Sigma}+\boldsymbol{\Lambda})^{-1}\tilde{\boldsymbol{\mu}} \\
&\quad + (g_{cd})^{m+1}\hat{W}\cdot\hat{\boldsymbol{\varrho}}, \qquad \text{(B.27)}
\end{aligned}
$$

which confirms (3.37). By induction, (B.27) therefore holds for an arbitrary $m \in \mathbb{N}$. Since (3.36) can be established using similar arguments, this completes the proof of Proposition 3.9.

B.5: Proof of Proposition 3.10

Substituting the m-compounded CD-optimal control (3.37) into the investor wealth dynamics (3.11), we take expectations and use the definitions (B.3) to obtain

$$\begin{aligned} E^{t_0,w_0}_{\boldsymbol{\varrho}^{[m]*}_{cd}}\left[W^{[m]*}_{cd}(t;\delta)\right] &= w_0 e^{(r-\eta)t} + q\int_0^t e^{(r-\eta)(t-u)}du \\ &+\eta\cdot\int_0^t \left(\frac{(g_{cd}(u;\delta))^{m+1}-1}{g_{cd}(u;\delta)-1}\right)\cdot h_{cd}(u;\delta,q)\cdot e^{(r-\eta)(t-u)}du \qquad \text{(B.28)} \\ &+\int_0^t (g_{cd}(u;\delta))^{m+1}\cdot\left[\eta\cdot\hat{K}(u)+\hat{\chi}(u)\right]e^{(r-\eta)(t-u)}du. \qquad \text{(B.29)} \end{aligned}$$

As in the case of (B.6), we observe that if more is known about the benchmark strategy, closed-form expressions for $\hat{K}(t)$ and $\hat{\chi}(t)$ might allow further simplification of (B.29).

From Lemma A.1, we know that $g_{cd}(t;\delta) > e^{\delta t} > 1$ for $t\in(t_0,T]$ (see (A.12)), while Lemma A.2 shows that $h_{cd}(t;\delta,q)\geq 0$, $\forall t\leq T$ (see (A.21)). Combining (B.6) and (B.29) with (B.5) given condition (B.1), we obtain, for any $m\in\mathbb{N}$ and $t\in(t_0,T]$,

$$\begin{aligned} & E^{t_0,w_0}_{\boldsymbol{\varrho}^{[m]*}_{cd}}\left[W^{[m]*}_{cd}(t;\delta)\right] - E^{t_0,w_0}_{\boldsymbol{\varrho}^{*}_{cd}}\left[W^{*}_{cd}(t;\delta)\right] \\ =\ & \eta\cdot\int_0^t\left(\frac{(g_{cd}(u;\delta))^{m+1}-1}{g_{cd}(u;\delta)-1}-1\right)\cdot h_{cd}(u;\delta,q)\cdot e^{(r-\eta)(t-u)}du \\ & +\int_0^t\left[(g_{cd}(u;\delta))^{m+1}-g_{cd}(u;\delta)\right]\cdot\left[\eta\cdot\hat{K}(u)+\hat{\chi}(u)\right]e^{(r-\eta)(t-u)}du \\ >\ & 0. \end{aligned} \tag{B.30}$$

Using similar arguments, it follows that

$$\begin{aligned} & E^{t_0,w_0}_{\boldsymbol{\varrho}^{[m]*}_{qd}}\left[W^{[m]*}_{qd}(t;\beta)\right] - E^{t_0,w_0}_{\boldsymbol{\varrho}^{*}_{qd}}\left[W^{*}_{qd}(t;\beta)\right] \\ =\ & \eta\cdot\left(\frac{e^{(m+1)\beta T}-1}{e^{\beta T}-1}-1\right)\cdot\int_0^t h_{qd}(u;\beta,q)\cdot e^{(r-\eta)(t-u)}du \\ & +\left(e^{(m+1)\beta T}-e^{\beta T}\right)\cdot\int_0^t\left[\eta\cdot\hat{K}(u)+\hat{\chi}(u)\right]e^{(r-\eta)(t-u)}du \\ >\ & 0. \end{aligned} \tag{B.31}$$

From (B.30) and (B.31), the result (3.39) has been established.

To show (3.40) for the m-compounded CD-optimal and CD-optimal strategies, we first recall the definitions (B.3) and (B.13), and also define

$$K^{[m]*}_{cd}(t;\delta) = E^{t_0,w_0}_{\boldsymbol{\varrho}^{[m]*}_{cd}}\left[W^{[m]*}_{cd}(t;\delta)\right]. \tag{B.32}$$

Using similar steps as in the proof of Proposition 3.7, it can be shown that

$$\begin{aligned} & E^{t_0,w_0}_{\boldsymbol{\varrho}^{[m]*}_{cd}}\left[W^{[m]*}_{cd}(t;\delta)\cdot\mathcal{R}^{[m]*}_{cd}(t;\delta)\right] - E^{t_0,w_0}_{\boldsymbol{\varrho}^{*}_{cd}}\left[W^{*}_{cd}(t;\delta)\cdot\mathcal{R}^{*}_{cd}(t;\delta)\right] \\ =\ & \left(\frac{(g_{cd}(t;\delta))^{m+1}-1}{g_{cd}(t;\delta)-1}-1\right)\cdot h_{cd}(t;\delta,q)\cdot\sum_{k=1}^{N_a^r}\left[(\boldsymbol{\Sigma}+\boldsymbol{\Lambda})^{-1}\tilde{\boldsymbol{\mu}}\right]_k \\ +\ & \left[K^{*}_{cd}(t;\delta)-K^{[m]*}_{cd}(t;\delta)\right]\cdot\sum_{k=1}^{N_a^r}\left[(\boldsymbol{\Sigma}+\boldsymbol{\Lambda})^{-1}\tilde{\boldsymbol{\mu}}\right]_k \\ +\ & \left((g_{cd}(t;\delta))^{m+1}-g_{cd}(t;\delta)\right)\cdot\left(E^{t_0,w_0}_{\hat{\boldsymbol{\varrho}}}\left[\hat{W}(t)\cdot\hat{\mathcal{R}}(t)\right]+\hat{K}(t)\cdot\sum_{k=1}^{N_a^r}\left[(\boldsymbol{\Sigma}+\boldsymbol{\Lambda})^{-1}\tilde{\boldsymbol{\mu}}\right]_k\right). \end{aligned} \tag{B.33}$$

At time $t_0 = 0$, $K^*_{cd}(t_0;\delta) = K^{[m]*}_{cd}(t_0;\delta) = w_0$, so the second term of (B.33) vanishes. Note that the condition (B.1) ensures that $\hat{K}(t_0) > 0$ (see (B.4) and the associated discussion), while the assumption $\hat{\mathcal{R}}(t_0) \geq 0$ implies that $E^{t_0,w_0}_{\hat{\varrho}}\left[\hat{W}(t_0)\cdot\hat{\mathcal{R}}(t_0)\right] = w_0 \cdot \hat{\mathcal{R}}(t_0) \geq 0$. Considering in addition the properties of g_{cd} and h_{cd} as per Lemmas A.1 and A.2, from (B.33) we therefore have that

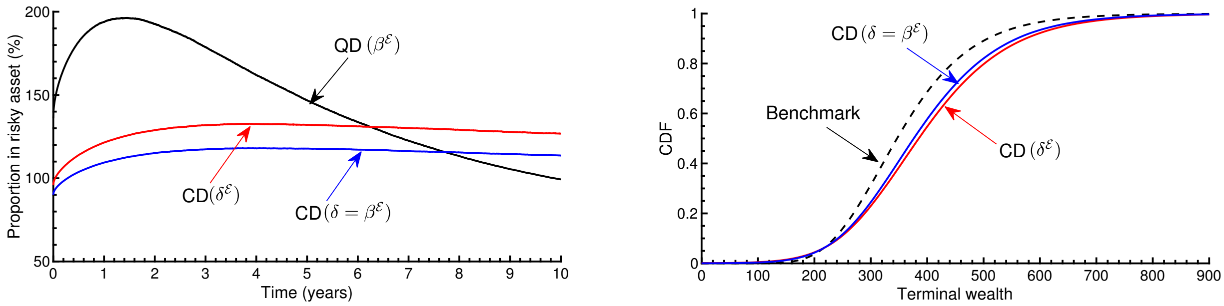
$$E^{t_0,w_0}_{\varrho^{[m]*}_{cd}}\left[W^{[m]*}_{cd}(t;\delta)\cdot\mathcal{R}^{[m]*}_{cd}(t;\delta)\right] - E^{t_0,w_0}_{\varrho^*_{cd}}\left[W^*_{cd}(t;\delta)\cdot\mathcal{R}^*_{cd}(t;\delta)\right] \quad > \quad 0, \tag{B.34}$$

so that (3.40) holds for the CD-optimal and m-compounded CD optimal strategies. The proof of (3.40) for the QD-optimal and m-compounded QD optimal strategies uses similar arguments, and is therefore omitted.

B.6: Numerical results: $CD\left(\delta^{\mathcal{E}}\right)$, $CD\left(\delta = \beta^{\mathcal{E}}\right)$ and $QD\left(\beta^{\mathcal{E}}\right)$

In Subsection 3.4, we noted that the closed-form comparison results were derived under the assumption of equal parameters (i.e. $CD(\delta)$ is compared to $QD(\beta = \delta)$), but that comparing results on the basis of equal expectations $\mathcal{E}$ of terminal wealth (i.e. comparing $CD\left(\delta^{\mathcal{E}}\right)$ with $QD\left(\beta^{\mathcal{E}}\right)$) can be more practical when comparing investment outcomes. In addition, we claimed in Subsection 3.4 that the difference $\left(\delta^{\mathcal{E}} - \beta^{\mathcal{E}}\right) > 0$ is typically sufficiently small in numerical experiments such that the results from assuming equal parameters $(\delta - \beta) \equiv 0$ for analytical purposes is sufficient to gain intuition into the relative behavior of the optimal strategies compared on the basis of equal expectations. In this appendix, we verify this claim by comparing the results for problems $CD\left(\delta^{\mathcal{E}}\right)$, $CD\left(\delta = \beta^{\mathcal{E}}\right)$ and $QD\left(\beta^{\mathcal{E}}\right)$.

In the case of closed-form solutions (no constraints), Figure B.1(a) can be compared with Figure 5.1(b), and Figure B.1(b) can be compared with Figure 5.2(a). Note that the qualitative conclusions regarding Figures B.1 and B.1 remain unchanged if we use $CD\left(\delta = \beta^{\mathcal{E}}\right)$ instead of $CD\left(\delta^{\mathcal{E}}\right)$ in the comparison with $QD\left(\beta^{\mathcal{E}}\right)$.



(a) Proportion of wealth in risky asset: 95th pctiles

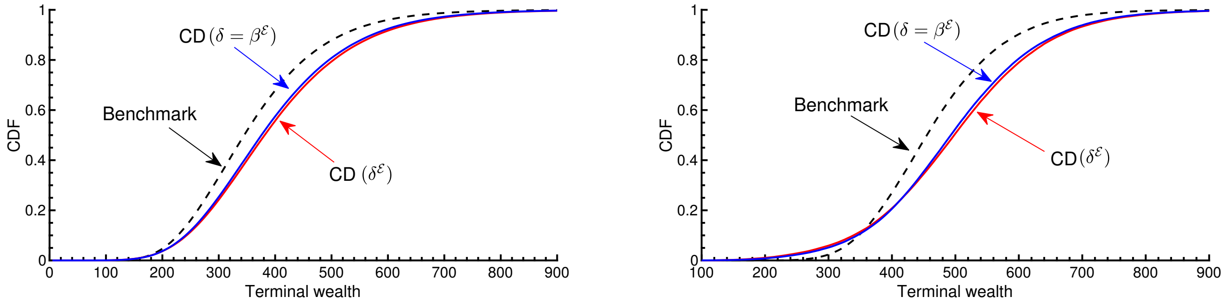
(b) CDFs of terminal wealth

Figure B.1: Analytical solutions, no constraints, investor portfolio P0, benchmark BM0, data set DS0: Effect of value of δ on problem CD(δ). CDFs of $\hat{W}(T)$, $W^*_{cd}\left(T;\delta = \beta^{\mathcal{E}}\right)$, and $W^*_{cd}\left(T;\delta^{\mathcal{E}}\right)$. In sub-figure (b), the CDF of $W^*_{qd}\left(T;\beta^{\mathcal{E}}\right)$ is not shown, since it is effectively indistinguishable from the CDF of $W^*_{cd}\left(T;\delta^{\mathcal{E}}\right)$; see Figure (5.2).

In the case of numerical solutions with constraints, Figure B.2 can be compared with Figure 5.5, and again qualitative conclusions are not affected, the CDF results of using $CD\left(\delta = \beta^{\mathcal{E}}\right)$ instead of $CD\left(\delta^{\mathcal{E}}\right)$ remain similar. Note that the CDFs of $QD\left(\beta^{\mathcal{E}}\right)$ are not shown in Figure B.2 because they are basically indistinguishable from the CDF results for $CD\left(\delta^{\mathcal{E}}\right)$.

For numerical solutions with constraints, Figure B.3 shows how the investment strategy is affected by using $CD\left(\delta = \beta^{\mathcal{E}}\right)$ instead of $CD\left(\delta^{\mathcal{E}}\right)$ in a comparison analysis with $QD\left(\beta^{\mathcal{E}}\right)$. The qualitative conclusions regarding Figure 5.4 remain unaffected.

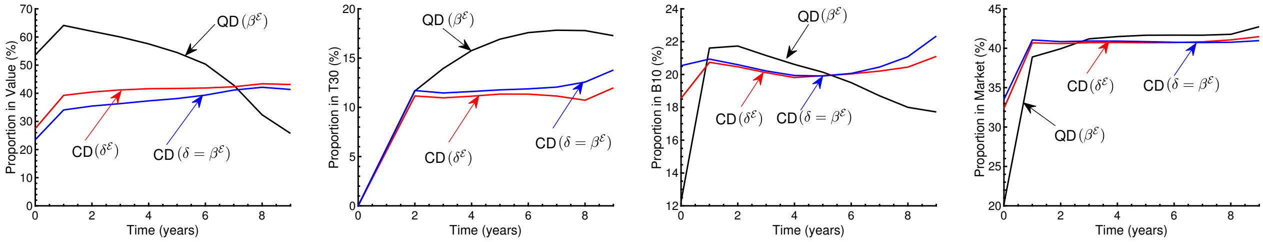
In the case of numerical solutions with constraints, Figure B.4 shows the same results as Figure 5.6, but with the results for $CD\left(\delta = \beta^{\mathcal{E}}\right)$ added. Again, we observe that the qualitative conclusions regarding Figure 5.6 are not affected.



(a) CDFs of terminal wealth: Training data (DS1)

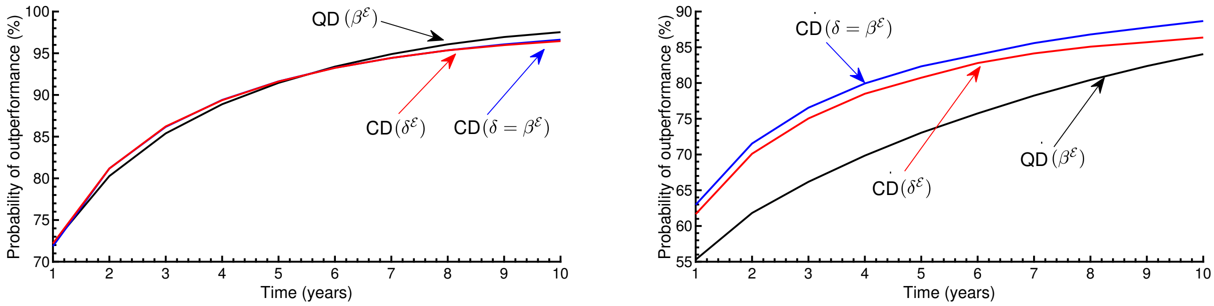
(b) CDFs of terminal wealth: Testing data (DS1)

Figure B.2: Numerical solutions, with constraints, investor portfolio P1, benchmark BM1, data set DS1: CDFs of $\hat{W}(T)$, $W_{cd}^{*}\left(T;\delta=\beta^{\mathcal{E}}\right)$, and $W_{cd}^{*}\left(T;\delta^{\mathcal{E}}\right)$. The CDFs of $W_{qd}^{*}\left(T;\beta^{\mathcal{E}}\right)$ on the training and testing data are not shown, since they are effectively indistinguishable from the corresponding CDFs of $W_{cd}^{*}\left(T;\delta^{\mathcal{E}}\right)$; see Figure (5.5).



(a) Value **(b)** T30 **(c)** B10 **(d)** Market

Figure B.3: Numerical solutions, with constraints, investor portfolio P1, benchmark BM1, data set DS1: 95th percentile of the proportion of wealth invested in each asset over time on the training data set (DS1). Zero investment in Size, thus it is omitted. Note the same scale on the y-axis, and that the last rebalancing event is at $t = T - \Delta t = 9$ years.



(a) Probability of outperformance: Training data (DS1)

(b) Probability of outperformance: Testing data (DS1)

Figure B.4: Numerical solutions, with constraints, investor portfolio P1, benchmark BM1, data set DS1: Probability of benchmark outperformance over time.

Appendix C: Source data

In this appendix, we provide details regarding the underlying data used to obtain the results presented in Section 5, as well as the supplementary results in Appendix B and Appendix D.

The historical returns data for the T-bills/bonds and the broad market index were obtained from the CRSP[2]. Historical returns data for the equity factors Size and Value (see Fama and French (2015, 1992)) were obtained from Kenneth French's data library[3] (KFDL). In more detail, the historical time series sourced for the underlying assets, with naming conventions as in Table 5.2, are as follows:

(i) T30 (30-day Treasury bill): CRSP, monthly returns for 30-day Treasury bill.

(ii) B10 (10-year Treasury bond): CRSP, monthly returns for 10-year Treasury bond.

[2] Calculations were based on data from the Historical Indexes 2020©, Center for Research in Security Prices (CRSP), The University of Chicago Booth School of Business. Wharton Research Data Services was used in preparing this article. This service and the data available thereon constitute valuable intellectual property and trade secrets of WRDS and/or its third party suppliers.

[3] See https://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html

(iii) Market (broad equity market index): CRSP, monthly returns, including dividends and distributions, for a capitalization-weighted index consisting of all domestic stocks trading on major US exchanges (the VWD index).

(iv) Size (Portfolio of small stocks): KFDL, "Portfolios Formed on Size", which consists of monthly returns on a capitalization-weighted index consisting of the firms (listed on major US exchanges) with market value of equity, or market capitalization, at or below the 30th percentile (i.e. smallest 30%) of market capitalization values of NYSE-listed firms.

(v) Value (Portfolio of value stocks): KFDL, "Portfolios Formed on Book-to-Market", which consists of monthly returns on a capitalization-weighted index of the firms (listed on major US exchanges) consisting of the firms (listed on major US exchanges) with book-to-market value of equity ratios at or above the 70th percentile (i.e. highest 30%) of book-to-market ratios of NYSE-listed firms.

All historical time series were obtained for the period from 1963:07 to 2020:12, and inflation-adjusted using inflation data from the US Bureau of Labor Statistics[4].

For the purposes of illustrating the closed-form solutions of Section 3 in Subsection 5.2, the (single) risky asset is assumed to correspond to the broad equity market index evolving according to the dynamics of the Kou (2002) model. As a result, $\log \xi$ is assumed to follow an asymmetric double-exponential distribution, with the PDF of ξ given by

$$f_{\xi}(\xi) = \nu\zeta_1\xi^{-\zeta_1-1}\mathbb{I}_{[\xi\geq 1]}(\xi) + (1-\nu)\,\zeta_2\xi^{\zeta_2-1}\mathbb{I}_{[0\leq\xi<1]}(\xi), \quad \upsilon \in [0,1] \text{ and } \zeta_1 > 1, \zeta_2 > 0, \tag{C.1}$$

where ν denotes the probability of an upward jump given that a jump occurs. Using the filtering technique for calibrating jump-diffusion processes (see Dang and Forsyth (2016); Forsyth and Vetzal (2017) for technical details), the resulting calibrated parameters are presented in Table C.1.

Table C.1: Calibrated, inflation-adjusted parameters for asset dynamics (3.5) and (3.10), with $f_{\xi}(\xi)$ given by (C.1). The calibration methodology of Dang and Forsyth (2016); Forsyth and Vetzal (2017) is used with a jump threshold parameter value of 3.

Parameter	r	μ	σ	λ	υ	ζ_1	ζ_2
Value	0.0074	0.0749	0.1392	0.2090	0.2500	7.7830	6.1074

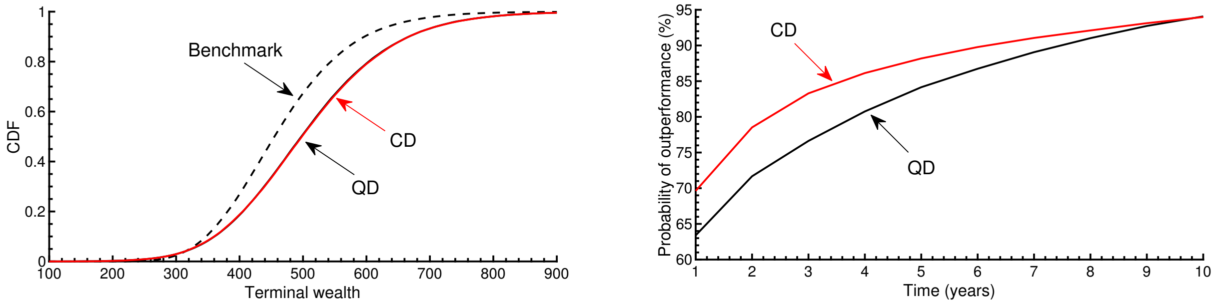
Appendix D: Additional numerical results

This appendix complements the numerical results of Section 5.1, which focused on the results associated with data set DS1 in Table 5.3. In this appendix, we report the key out-of-sample (testing) results associated with the other data sets in Table 5.3.

In summary, Figure D.1, Figure D.2 and Figure D.3 illustrate that the qualitative conclusions regarding the out-of-sample performance of the CD-optimal strategy relative to the QD-optimal strategy remain robust to rebalancing frequency assumptions and different data periods.

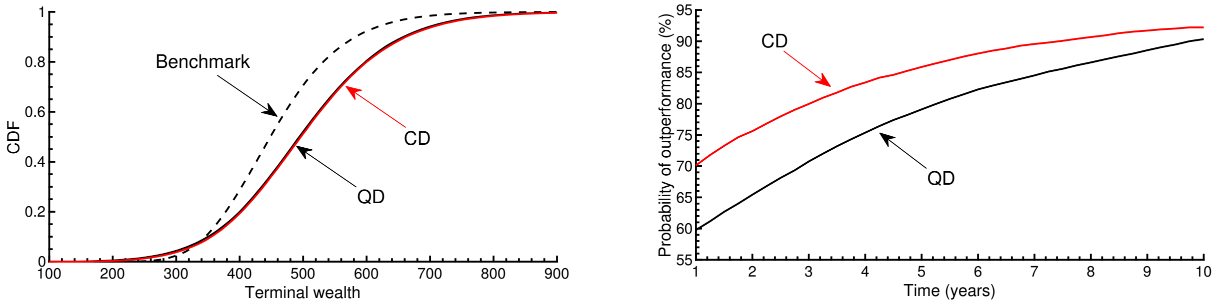
Figure D.4 compares the terminal wealth distributions obtained for DS1 and DS1b (see Table 5.3), and therefore shows effect on the terminal wealth of using a more aggressive benchmark outperformance target. We observe that a more aggressive outperformance target (DS1b) increases both the upside *and* downside wealth outcomes compared to a more modest outperformance target (DS1), since the resulting optimal wealth allocations are less diversified in the case of DS1b (see Figure 5.7(a)).

[4] The annual average CPI-U index, which is based on inflation data for urban consumers, were used - see `http://www.bls.gov.cpi`



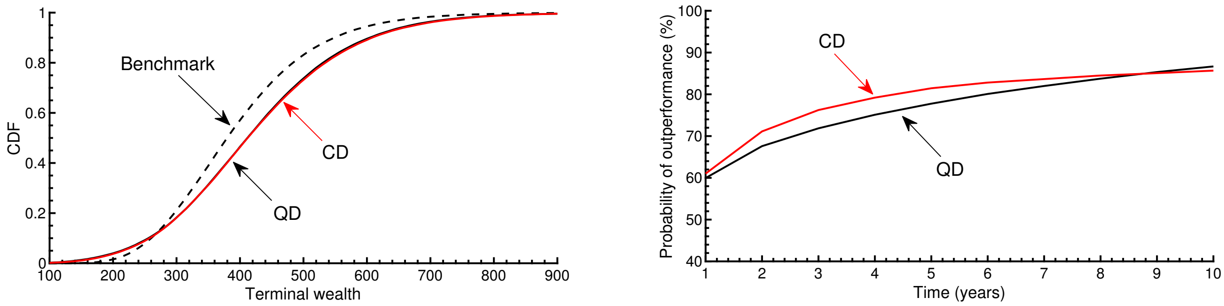
(a) CDFs of terminal wealth: Testing data (DS2) **(b)** Probability of outperformance: Testing data (DS2)

Figure D.1: Numerical solutions, with constraints, investor portfolio P1, benchmark BM1, data set DS2: Testing (out-of-sample) results. The CDFs of terminal wealth for CD and QD are indistinguishable.



(a) CDFs of terminal wealth: Testing data (DS2b) **(b)** Probability of outperformance: Testing data (DS2b)

Figure D.2: Numerical solutions, with constraints, investor portfolio P1, benchmark BM1, data set DS2b: Testing (out-of-sample) results.



(a) CDFs of terminal wealth: Testing data (DS3) **(b)** Probability of outperformance: Testing data (DS3)

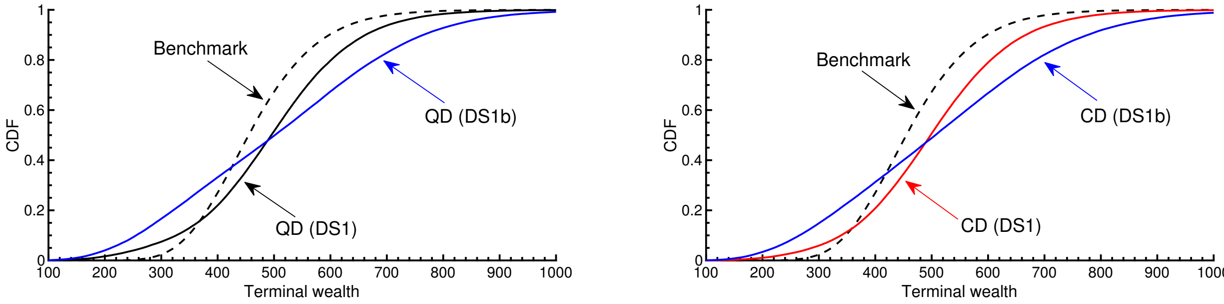
Figure D.3: Numerical solutions, with constraints, investor portfolio P1, benchmark BM1, data set DS3: Testing (out-of-sample) results.

Appendix E: NN approach: hyperparameters and ground truth

In this appendix, we summarize relevant implementation details of the numerical algorithm described in Section 4, and verify the numerical solutions using a ground truth analysis. Note that additional details regarding the algorithm can be found in Van Staden et al. (2023).

In the numerical results reported in Section 4, we implemented a fully-connected feedforward NN consisting of two hidden layers, each with $N_a + 2$ nodes. For training the NN for each problem, 64,000 stochastic gradient descent (SGD) steps were used based on the Gadam algorithm (Granziol et al. (2020)), each implementing a mini-batch size of 100 paths. For illustrative purposes, the minimal features were used (time, investor wealth, benchmark wealth). The adequacy of this configuration was verified using ground truth results (see below), as well as assessing the stability of results using repeated training on the same data set.

We now consider verifying the results of the implementation of the numerical algorithm using ground truth results. As discussed in Section 4, the proposed NN methodology automatically incorporates the investment constraints of no short-selling and no leverage. However, the closed-form solutions (Section 3) are based on



(a) QD: Testing data (DS1 and DS1b) **(b)** CD: Testing data (DS1 and DS1b)

Figure D.4: Numerical solutions, with constraints, investor portfolio P1, benchmark BM1, data set DS2: CDFs of terminal wealth for CD and QD on the Testing (out-of-sample) results of DS1 and DS1b, illustrating the effect of a more aggressive outperformance target.

Assumption 3.1, where no such constraints are applicable.

For the purposes of a ground truth analysis, the objective is to show the convergence of the numerical solutions (under suitable conditions) to the available closed-form solutions. Therefore, instead of changing the NN methodology to allow for trading in the event of bankruptcy (allowed under the stylized assumptions of Assumption 3.1), we observe as in Van Staden et al. (2023) that a relatively short time horizon (T =1 year) and modest outperformance target imply that the closed-form solutions typically do not require short-selling or leverage. In this case, the numerical solutions (with constraints) can approximate the closed-form solutions (no constraints) fairly accurately if the underlying data is identical. In terms of generating the underlying data, we use the parametric framework of Section 3 with parameters as in Table C.1. Analytical investment strategies are calculated based on these parameters, while the numerical approach uses 10^6 Monte Carlo simulations of these dynamics as training data for the neural network (see Section 4).

Table E.1 presents the resulting ground truth comparison results for investor portfolio P0, benchmark BM0 (Table 5.2), confirming that the numerical results do indeed correspond to the analytical results, as required. Note that contributions are set to zero in order to avoid discrete approximation errors when comparing a continuous contribution rate to discrete contribution amounts made at rebalancing times.

Table E.1: Ground truth comparison, investor portfolio P0, benchmark BM0: $w_0 = 100$, $q = q(t_n) = 0$, $T = 1$ year, $\mathcal{E} = 105.25$. Analytical solutions based on 360 rebalancing events approximating continuous rebalancing. Numerical results are based on only 36 discrete rebalancing events to ensure that computation times remain reasonable. The column "Ratio" gives $W_j^*(T)/\hat{W}(T), j \in \{qd, cd\}$.

Quantity	Analytical solutions: P0					Numerical solutions (using NN): P0				
	BM0	QD($\beta^{\mathcal{E}}$=0.054)		CD($\delta^{\mathcal{E}}$=0.072)		BM0	QD($\beta^{\mathcal{E}}$=0.054)		CD($\delta^{\mathcal{E}}$=0.072)	
	$\hat{W}(T)$	$W_{qd}^*(T)$	Ratio	$W_{cd}^*(T)$	Ratio	$\hat{W}(T)$	$W_{qd}^*(T)$	Ratio	$W_{cd}^*(T)$	Ratio
Mean	104.2	105.3	1.01	105.3	1.01	104.2	105.2	1.01	105.3	1.01
CExp 5%	85.6	80.2	0.93	80.3	0.93	85.6	80.2	0.93	80.3	0.93
5th pctile	90.7	87.4	0.96	87.4	0.96	90.7	87.2	0.96	87.3	0.96
Median	104.1	105.5	1.01	105.5	1.01	104.1	105.5	1.01	105.5	1.01
95th pctile	117.9	122.1	1.04	122.3	1.04	117.9	122.0	1.03	122.2	1.04