Multi-period Mean CVAR Asset Allocation: Is it Advantageous to be Time Consistent?

Peter Forsyth[1]

[1]Cheriton School of Computer Science
University of Waterloo

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Motivation

Long Term Investor saving for retirement

- Investor has DC (Defined Contribution) pension plan
- Invests for 30+ years, yearly contributions
- Rebalances infrequently (i.e. once a year)
- Desires to end up with a target wealth level, used to fund retirement
  - $\rightarrow$ What is the optimal dynamic allocation to bonds and stocks?

What objective function should we use?

- Traditionally, various utility functions (i.e. power law)
  - $\rightarrow$ Difficult for end users to interpret
  - $\rightarrow$ Objective function maximizes *utils* not target wealth
- Much recent work on multi-period mean-variance objective functions

Previous Work: Multi-period Mean Variance

Suppose we want to find the (dynamic) rebalancing strategy (the control) which solves

$$\min Var[W_T]$$
$$\quad such\ that\ E[W_T] = \text{ specified}$$

$$W_T = \text{ terminal wealth}$$
$$Var = \text{ variance}$$
$$E[\cdot] = \text{ Expectation}$$

Pre-commitment solution (multi-period MV optimal)

- Optimal multi-period MV control: minimize (Zhou and Li, 2000)

$$E\left[\left(\min(W_T - W^*, 0)\right)^2\right]$$

$\rightarrow$ varying W^* traces out efficient frontier

- But this solution is not *time consistent*
- Suppose we compute the pre-commitment solution at $t = 0$
  - Determine feedback (closed loop) controls as a function of state variables
- Recompute strategy at some later time $t > 0$
  - Strategy (as a function of state) may not agree with $t = 0$ strategy

  $\Rightarrow$ Investor has incentive to deviate from the pre-commitment policy computed at $t = 0$

Time Consistent Solution

Add constraint to MV objective function

- Force time consistency (Basak and Chabakauri, 2010; Bjork and Murgoci, 2010; Wang and Forsyth, 2011).

But, note result from (Bjork and Murgoci, 2010), which I paraphrase

Theorem 1

Given the optimal control from the time consistent MV problem[1]*, this same control is optimal for an alternative objective function, which is unconstrained and time consistent.*

In other words:

⇒ Forcing time consistency changes the objective function.

[1] The result is more general, and applies to *non-standard* problems

Pre-commitment $\Rightarrow$ induced time consistent strategy

Pre-commitment MV solution at time zero is found by minimizing

$$E\left[\left(\min(W_T - W^*, 0)^2\right)^2\right] \qquad (1)$$

If, $\forall t > 0$, we fix W^*, then

- $\rightarrow$ The pre-commitment control computed at time zero is the time consistent[2] control for objective function (1)
- $\rightarrow$ Termed *time consistent mean-variance* **induced** *utility function*, (Strub, Li, Cui; 2019)

[2]Proof: eqn (1) can be optimized using dynamic programming.

Summary: MV optimization, pre-commitment vs. time consistent

Forcing time consistency

⇒ Equivalent to unconstrained alternative objective function

Pre-commitment policy

⇒ Equivalent at time zero to alternative *induced* objective function

⇒ This induced objective function has time consistent controls

Both approaches give rise to alternative objective functions

⇒ Both controls are time consistent

⇒ Investor has no incentive to deviate from control computed at time zero

⇒ Neither strategy can be dismissed out of hand

This talk

- Study both approaches for **mean-CVAR** asset allocation

Mean-CVAR Objective Function

CVAR_α is the mean of the worst α fraction of outcomes
$\hookrightarrow$ A larger value is better

$$W_T \equiv \text{ terminal wealth at time } T \quad ; \quad g(W_T) \equiv \text{ density of } W_T$$

$$\text{CVAR}_\alpha = \frac{\int_{-\infty}^{W_\alpha^*} W_T \; g(W_T) \; dW_T}{\alpha} \quad ; \quad \int_{-\infty}^{W_\alpha^*} g(W_T) \; dW_T = \alpha$$

- $g(W_T)$ is the density of final wealth, not losses

Plan:

- Consider mean-CVAR objective function with scalarization parameter $\kappa > 0$

$$\max \Big(\text{CVAR}_\alpha + \kappa E[W_T] \Big)$$

- Compare pre-commitment and time consistent mean-CVAR strategies

Two Asset Portfolio: stock index and bond index

$$S_t \equiv \textit{amount} \text{ invested in stock index}$$

$$dS_t = \Big(\text{Single factor jump diffusion} \Big)$$

$$\textit{jump size} \rightarrow \textit{double exponential}$$

$$B_t \equiv \textit{amount} \text{ invested in risk free bond;}$$

$$dB_t = rB_t \, dt \quad ; \quad r = \text{ risk free rate}$$

$$W_t \equiv S_t + B_t = \text{ total wealth}$$

$$W_0 \equiv \text{Initial wealth}$$

Discrete Rebalancing times:

$$\mathcal{T} \equiv \{t_0 = 0 < t_1 < \cdots < t_M = T\}. \qquad (2)$$

Rebalancing

At rebalancing times t_i, let $t_i^+ \equiv \lim_{\epsilon \to 0^+} t_i + \epsilon \ \ ; \ \ t_i^- \equiv \lim_{\epsilon \to 0^+} t_i - \epsilon$

Inject cash q_i

$$W(t_i^+) = W(t_i^-) + q_i$$

Determine optimal fraction in stocks $p_i(\cdot)$

$$\begin{aligned} p_i(\cdot) &= p(W(t_i^+), t_i) \\ & S(t_i^+) = p_i(W_i^+) W_i^+ \ ; \ B(t_i^+) = (1 - p_i(W_i^+)) W_i^+ \end{aligned}$$

Admissible controls $\mathcal{P}$

$$\begin{aligned} \mathcal{P} &= \{ p_i(\cdot) \in \mathcal{Z} \ : \ i = 0, \ldots, M-1 \} \\ & \mathcal{Z} = [0,1] \ ; \ \ \text{no leverage, no shorting} \end{aligned}$$

Tail of controls at t_n

$$\mathcal{P}_n = \{ p_n(\cdot), \ldots, p_{M-1}(\cdot) \} \ .$$

Alternate definition of CVAR

Given an expectation under control $E_{\mathcal{P}}[\cdot]$ (Rockafeller and Uryasev, 2000)

$$\begin{aligned} \text{CVAR}_\alpha &= \max_{W^*} E_{\mathcal{P}}\left(W^* + \frac{1}{\alpha}\left[(W_T - W^*)^-\right]\right) \\ & (W_T - W^*)^- \equiv \min(W_T - W^*, 0) \ . \end{aligned}$$

Mean-CVAR problem (Miller and Yang, 2017)

$$\underbrace{\max_{\mathcal{P}} \left\{ \overbrace{\max_{W^*} E_{\mathcal{P}}\left(W^* + \frac{1}{\alpha}\left[(W_T - W^*)^-\right]\right)}^{CVAR\ \ under\ \ control\ \ \mathcal{P}} + \kappa \overbrace{E_{\mathcal{P}}\Big(W_T \Big)}^{Wealth} \right\}}_{maximize\ \ over\ \ \mathcal{P}} .$$

$(TCMC_{t_n}(\kappa))$: Time Consistent Mean CVAR

Defined via value function $J(s, b, t)$

$$
\begin{aligned}
&(TCMC_{t_n}(\kappa)) : \\
&\qquad J\left(s, b, t_n^-\right) = \max_{\mathcal{P}_n} \max_{W^*} E_{\mathcal{P}_n}^{(W_n^+, t_n^+)}\left[W^* + \frac{1}{\alpha}(W_T - W^*)^- + \kappa W_T\right] \\
&\qquad\qquad W_n^+ = s + b + q_n
\end{aligned}
$$

$$
\begin{aligned}
\text{s.t.}\mathcal{P}_n = \left\{p_n(\cdot), \mathcal{P}_{n+1}^*\right\} = \left\{p_n(\cdot), p_{n+1}^*(\cdot), \ldots, p_{M-1}^*(\cdot)\right\} \qquad (3) \\
\text{where } \mathcal{P}_{n+1}^* \text{ is optimal for problem } (TCMC_{t_{n+1}}(\kappa))
\end{aligned}
$$

Intuition:

- Time consistent constraint in (3)
  - → Optimize control today, knowing that future controls are optimal for future problems

Embed problem ($TCMC_{t_n}(\kappa)$) in 3-d space

Define auxiliary function $V(s, b, W^*, t)$

$$V\left(s, b, W^*, t_n^-\right) = E_{\mathcal{P}_n}^{(W_n^+, W^*, t_n^+)}\left[W^* + \frac{1}{\alpha}(W_T - W^*)^- + \kappa W_T\right]$$
$$W_n^+ = s + b + q_n$$

Dynamic programming solution for optimal control

$$p_n(w) = \arg\max_{p' \in \mathcal{Z}} \left\{ \max_{W^*} V(w\ p', w\ (1-p'), W^*, t_n^+) \right\} .$$

But we advance the solution (backwards) **for all values of** W^*

$$V(s, b, W^*, t_n^-) = V\left(\ w^+\ p_n(w^+), w^+\ (1 - p_n(w^+)\ ), W^*, t_n^+\ \right)$$
$$w^+ = s + b + q_n\ .$$

Expanded state space formulation II

For $t \in (t_{n-1}, t_n)$ (i.e. between rebalancing dates)

- Solve 2-d PIDE, with W^* regarded as a parameter

Intuition

- Optimal W^* depends on state, time and future controls
  - $\rightarrow$ Solve for all possible values of W^*, additional state variable
- Now have a true 3-d problem
  - $\rightarrow$ Coupling for different W^* values occurs through optimal controls at each rebalancing date
- Recover original value function

$$J\left(s, b, t_n^-\right) = \max_{W^*} V(s, b, W^*, t_n^-) \ ,$$

$(PCMC_{t_0}(\kappa))$: Pre-commitment Mean-CVAR

Defined via value function $\hat{J}(s, b, t_0)$

$$(PCMC_{t_0}(\kappa)) :$$

$$\hat{J}\left(0, W_0, t_0^-\right) = \max_{\mathcal{P}_0} \max_{W^*} E_{\mathcal{P}_0}^{(W_0^+, t_0^+)}\left[W^* + \frac{1}{\alpha}(W_T - W^*)^- + \kappa W_T\right]$$

$$W_0^+ = W_0 + q_0 \;;\; W_0 = \text{ initial wealth}$$

Compared with time-consistent formulation

- **No time consistent constraint**
- **Optimality at** $t = t_0$

Re-formulate: interchange $\max_{\mathcal{P}_0} \max_{W^*} E[\cdot]$

$$\hat{J}\left(0, W_0, t_0^-\right) = \max_{W^*} \max_{\mathcal{P}_0} E_{\mathcal{P}_0}^{(W_0^+, t_0^+)}\left[W^* + \frac{1}{\alpha}(W_T - W^*)^- + \kappa W_T\right]$$

Expanded State Space Formulation (Miller and Yang, 2017)

Define auxiliary function $\hat{V}(s, b, W^*, t)$

$$\hat{V}(s, b, W^*, t_n^-) \quad = \quad E_{\mathcal{P}_n}^{(W_n^+, W^*, t_n^+)}\left[W^* + \frac{1}{\alpha}(W_T - W^*)^- + \kappa W_T\right]$$
$$W_n^+ = s + b + q_n$$

Dynamic programming solution for control:

$$\hat{p}_n(w, W^*) \quad = \mathop{\arg\max}_{p' \in \mathcal{Z}} \left\{ \hat{V}(w\ p', w\ (1 - p'), W^*, t_i^+) \right\} .$$

Advance solution backwards (**fixed** W^*)

$$\hat{V}(s, b, W^*, t_n^-) \quad = \quad \hat{V}\big(\ w^+\ \hat{p}_n(w^+, W^*), w^+\ (1 - \hat{p}_n(w^+, W^*)\ ), W^*, t_n^+\ \big)$$
$$w^+ = s + b + q_n$$

Pre-commitment formulation II

Remark 1 (Contrast with time-consistent case)

No coupling of the solution for different W^* values from the optimal control.

As usual: for $t \in (t_{n-1}, t_n)$, solve 2-d PIDE

Original pre-commitment value function is recovered via

$$\hat{J}\left(0, W_0, t_0^-\right) = \max_{W'} \hat{V}(0, W_0, W', t_0^-) \tag{4}$$

Formulation requires

- Inner HJB equation solve (W^* is fixed for $t \in [0, T]$)
- Outer optimize (4) over W^* at $t = t_0^-$

Equivalent/Induced Time Consistent Problem

Let

$$W^*(t_0) \;=\; \arg\max_{W'} \hat{V}(s = 0, b = W_0, W', t = t_0)$$

Proposition 1 (Equivalent/Induced Time Consistent Problem)

The pre-commitment mean-CVAR strategy $\mathcal{P}^*$ determined by solving $\hat{J}(0, W_0, t_0)$ is the time consistent strategy for the equivalent problem TCEQ (with fixed $W^*(t_0)$), with value function $\tilde{J}(s, b, t)$ defined by[3]

$$(TCEQ_{t_n}(\kappa\alpha)):$$

$$\tilde{J}\left(s, b, t_n^-\right) = \max_{\mathcal{P}_n \in \mathcal{A}} E_{\mathcal{P}_n}^{W_n^+, t_n^+}\left[(W_T - \overbrace{W^*(t_0)}^{constant})^- + (\kappa\alpha) W_T \right]$$

$$W_n^+ = s + b + q_n$$

[3]Proof: W^* is constant, and multiply PCMC objective by $\alpha > 0$.

Intuition: TCEQ

Induced alternative objective function ($TCEQ_{t_n}(\kappa\alpha)$):

- Solve pre-commitment problem at time zero
  - $\rightarrow$ Determine target shortfall $W^*(t_0)$ (i.e. VAR at level α)
- With this fixed $W^*(t_0)$
  - $\rightarrow$ Solve $\forall t$, problem ($TCEQ_{t_n}(\kappa\alpha)$)
  - $\rightarrow$ This strategy is time consistent

Intuitively appealing

- If you have a billion dollars
  - $\rightarrow$ You don't worry as long as you have 50 million left
- If you have only a million dollars
  - $\rightarrow$ You probably get worried if your wealth $< 500,000$
- Contrast with time consistent Mean-CVAR
  - $\rightarrow$ Disaster level of wealth always relative to current wealth
- Fixed target based strategies popular with actuaries (Vigna, 2017)

Numerical Methods

Time consistent Mean-CVAR

- Discretize in (s, b, W^*) directions (3-d)
  - Solve PIDE between rebalancing times using ϵ-Monotone Fourier method (Forsyth, Labahn; 2019)
  - Discretize equity fraction, solve optimization problems at rebalancing times by exhaustive search and linear interpolation

Pre-commitment mean-CVAR

- Discretize in (s, b) directions (2-d), W^* is a fixed parameter
  - Solve PIDE: as above
  - Solve optimization problems: as above
- Use 1-d optimization method for outer optimization over W^*
  - Each evaluation of objective function requires HJB equation solve

Numerical Example

Stock Index

- Fit to CRSP US cap-weighted stock index *1926:1-2017:12* (**real**, i.e. inflation adjusted)

Bond Index

- Average one month **real** T-bill return, *1926:1-2017:12*, ($r = .00464$)

Investment Parameters	
Expiry time T	30 years
Initial wealth	0
Rebalancing frequency	yearly
Cash injection $\{q_i\}_{i=0,\ldots,29}$	20,000

Default Strategy: rebalance to constant weight $p = 0.4$[4]

$E[W_T]$	CVAR (5%)	$Median[W_T]$
1162	598	1084

Units: thousands of dollars

Choose κ (scalarization parameter) for pre-commitment and time consistent Mean-CVAR

- $Median[W_T]$ matches median for $p = 0.4$ strategy (approximately)

[4] A typical glide path strategy: $p = .8, t = 0$; $p = 0.0, t = T$; time average $p \simeq .4$. Glide path and constant weight with same time average p, $\rightarrow$ same distribution of W_T (Forsyth and Vetzal, 2019).

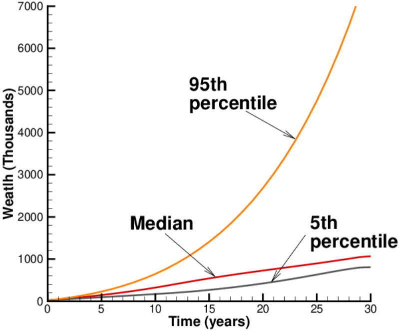
Compare CVAR, same $Median[W_T]$

Strategy	CVAR (5%)
Pre-commitment	682
Constant weight ($p = 0.4$)	598
Time consistent	530

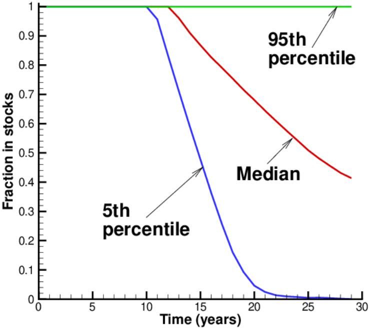
Units: thousands of dollars

More Details: pre-commitment

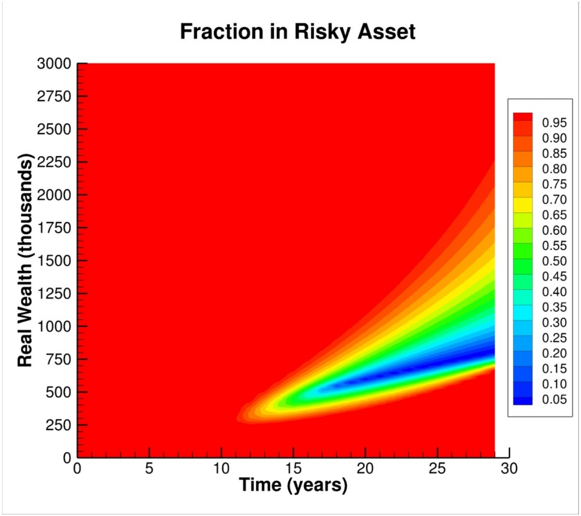
Percentiles: accumulated wealth



Percentiles: fraction in equities



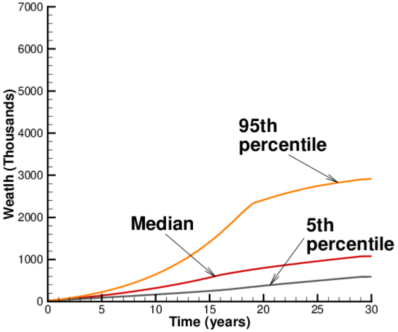
Control Heat Map: pre-commitment



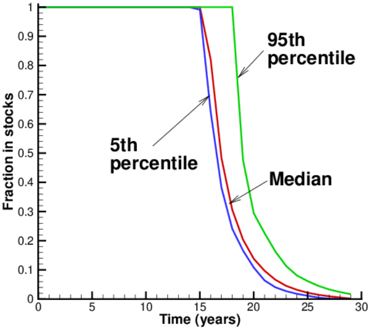
Red: all stock; Blue: all bond

More Details: time consistent

Percentiles: accumulated wealth



Percentiles: fraction in equities



Heat Map of controls: time consistent

- Mostly independent of wealth (except for small wealth values)
  - $\rightarrow$ Almost deterministic strategy
- Map:uninteresting, has (mostly) straight vertical lines

Time consistent constraint or induced time consistent objective?

Pre-commitment strategies

→ Investor has incentive to deviate from strategy computed at time zero

But, pre-commitment mean-CVAR strategy computed at time zero

→ Identical to time consistent target shortfall strategy, with fixed target

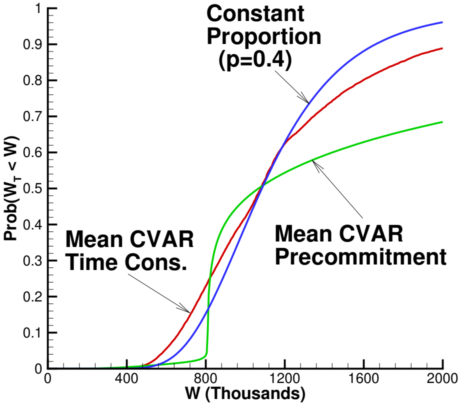
→ Investor has no incentive to deviate from this strategy, under this induced objective function

Time consistent strategies

- If we constrain a pre-commitment strategy to be time consistent
  - → Then this strategy is equivalent to an optimal strategy for an unconstrained alternative objective function

⇒ Both strategies: time consistent under alternative objective functions

Compare Strategies: Cumulative Distribution Functions



By design, all strategies have same $Median[W_T]$

- $\rightarrow$ Intersect at $Prob(\cdot) = 0.5$

Minimize left tail risk

- $\rightarrow$ Look for strategy which plots below other strategies in left tail

Time consistent mean-CVAR

- $\rightarrow$ Has worst tail risk of any strategy

Conclusions

- Adding time consistent constraint to mean-CVAR objective function
  - Equivalent to alternative, unconstrained objective function
  - $\rightarrow$ Under this alternative objective function, we no longer minimize tail risk
- Pre-commitment strategy at time zero
  - Equivalent to time consistent target shortfall strategy $\forall t > 0$
  - Minimizes tail risk w.r.t fixed target
  - Maximizes CVAR at time zero[5]
- It would appear that forcing time consistency in the mean-CVAR case is a bad idea!
- Consistent with poor performance of time consistent, MV case, wealth dependent risk aversion parameter
  - See (Wang, Forsyth; 2011), (Van Staden et al; 2018), (Bensoussan et al; 2019)

[5] Recall CVAR is the mean of the worst fraction of wealth outcomes $\rightarrow$ larger is better.