Multi-period Mean CVAR Asset Allocation: Is it Advantageous to be Time Consistent?

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Motivation

Long Term Investor saving for retirement

- Investor has DC (Defined Contribution) pension plan
- Invests for 30+ years, yearly contributions
- Rebalances infrequently (i.e. once a year)
- Desires to end up with a target wealth level, used to fund retirement
  → What is the optimal dynamic allocation to bonds and stocks?

What objective function should we use?

- Traditionally, various utility functions (i.e. power law)
  → Difficult for end users to interpret
  → Objective function maximizes *utils* not target wealth
- Much recent work on multi-period mean-variance objective functions
Suppose we want to find the (dynamic) rebalancing strategy (the control) which solves

$$\min \ Var[W_T]$$

such that $E[W_T] = \text{specified}$

$W_T = \text{terminal wealth}$

$Var = \text{variance}$

$E[\cdot] = \text{Expectation}$
Pre-commitment solution (multi-period MV optimal)

- Optimal multi-period MV control: minimize (Zhou and Li, 2000)

\[
E \left[ \left( \min(W_T - W^*, 0) \right)^2 \right] \rightarrow \text{varying } W^* \text{ traces out efficient frontier}
\]

- But this solution is not *time consistent*
- Suppose we compute the pre-commitment solution at \( t = 0 \)
  - Determine feedback (closed loop) controls as a function of state variables
- Recompute strategy at some later time \( t > 0 \)
  - Strategy (as a function of state) may not agree with \( t = 0 \) strategy
  \( \Rightarrow \) Investor has incentive to deviate from the pre-commitment policy computed at \( t = 0 \)
Add constraint to MV objective function

- Force time consistency (Basak and Chabakauri, 2010; Bjork and Murgoci, 2010; Wang and Forsyth, 2011).

But, note result from (Bjork and Murgoci, 2010), which I paraphrase

**Theorem 1**
Given the optimal control from the time consistent MV problem\(^1\), this same control is optimal for an alternative objective function, which is unconstrained and time consistent.

In other words:

⇒ Forcing time consistency changes the objective function.

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\(^1\)The result is more general, and applies to *non-standard* problems
Pre-commitment $\Rightarrow$ induced time consistent strategy

Pre-commitment MV solution at time zero is found by minimizing

$$E \left[ \left( \min(W_T - W^*, 0)^2 \right)^2 \right]$$

(1)

If, $\forall t > 0$, we fix $W^*$, then

$\rightarrow$ The pre-commitment control computed at time zero is the time consistent$^2$ control for objective function (1)

$\rightarrow$ Termed *time consistent mean-variance induced utility function*, (Strub, Li, Cui; 2019)

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$^2$Proof: eqn (1) can be optimized using dynamic programming.
Summary: MV optimization, pre-commitment vs. time consistent

Forcing time consistency

⇒ Equivalent to unconstrained alternative objective function

Pre-commitment policy

⇒ Equivalent at time zero to alternative induced objective function

⇒ This induced objective function has time consistent controls

Both approaches give rise to alternative objective functions

⇒ Both controls are time consistent

⇒ Investor has no incentive to deviate from control computed at time zero

⇒ Neither strategy can be dismissed out of hand

This talk

• Study both approaches for mean-CVAR asset allocation
Mean-CVAR Objective Function

\( \text{CVAR}_\alpha \) is the mean of the worst \( \alpha \) fraction of outcomes
\( \leftrightarrow \) A larger value is better

\[
W_T \equiv \text{terminal wealth at time } T ; \quad g(W_T) \equiv \text{density of } W_T
\]

\[
\text{CVAR}_\alpha = \frac{\int_{-\infty}^{W^*_\alpha} W_T g(W_T) \, dW_T}{\alpha} ; \quad \int_{-\infty}^{W^*_\alpha} g(W_T) \, dW_T = \alpha
\]

- \( g(W_T) \) is the density of final wealth, not losses

Plan:

- Consider mean-CVAR objective function with scalarization parameter \( \kappa > 0 \)

\[
\max \left( \text{CVAR}_\alpha + \kappa E[W_T] \right)
\]

- Compare pre-commitment and time consistent mean-CVAR strategies
Two Asset Portfolio: stock index and bond index

\[ S_t \equiv \text{amount invested in stock index} \]
\[ dS_t = \left( \text{Single factor jump diffusion} \right) \]
\[ \text{jump size } \rightarrow \text{ double exponential} \]

\[ B_t \equiv \text{amount invested in risk free bond;} \]
\[ dB_t = rB_t \, dt \quad ; \quad r = \text{risk free rate} \]

\[ W_t \equiv S_t + B_t = \text{total wealth} \]

\[ W_0 \equiv \text{Initial wealth} \]

Discrete Rebalancing times:

\[ \mathcal{T} \equiv \{ t_0 = 0 < t_1 < \cdots < t_M = T \}. \] (2)
Rebalancing

At rebalancing times $t_i$, let $t_i^+ \equiv \lim_{\epsilon \to 0^+} t_i + \epsilon$ ; $t_i^- \equiv \lim_{\epsilon \to 0^+} t_i - \epsilon$

Inject cash $q_i$

\[ \mathcal{W}(t_i^+) = \mathcal{W}(t_i^-) + q_i \]

Determine optimal fraction in stocks $p_i(\cdot)$

\[ p_i(\cdot) = p(\mathcal{W}(t_i^+), t_i) \]

\[ S(t_i^+) = p_i(\mathcal{W}_i^+) \mathcal{W}_i^+ ; \quad B(t_i^+) = (1 - p_i(\mathcal{W}_i^+)) \mathcal{W}_i^+ \]

Admissible controls $\mathcal{P}$

\[ \mathcal{P} = \{ p_i(\cdot) \in \mathcal{Z} : i = 0, \ldots, M - 1 \} \]

\[ \mathcal{Z} = [0, 1] ; \quad \text{no leverage, no shorting} \]

Tail of controls at $t_n$

\[ \mathcal{P}_n = \{ p_n(\cdot), \ldots, p_{M-1}(\cdot) \} \]
Alternate definition of CVAR

Given an expectation under control $E_P[\cdot]$ (Rockafeller and Uryasev, 2000)

$$CVAR_\alpha = \max_{W^*} E_P \left( W^* + \frac{1}{\alpha} [(W_T - W^*)^-] \right)$$

$$(W_T - W^*)^- \equiv \min(W_T - W^*, 0).$$

Mean-CVAR problem (Miller and Yang, 2017)

$$\max_{\mathcal{P}} \left\{ \max_{W^*} E_{\mathcal{P}} \left( W^* + \frac{1}{\alpha} [(W_T - W^*)^-] \right) + \kappa E_{\mathcal{P}} (W_T) \right\}.$$
**\(T \text{CMC}_{t_n}(\kappa)\): Time Consistent Mean CVAR**

Defined via value function \(J(s, b, t)\)

\[
(T \text{CMC}_{t_n}(\kappa)) : \\
J(s, b, t_n^-) = \max_{\mathcal{P}_n} \max_{W^*} E_{\mathcal{P}_n}^{(W_n^+, t_n^+)} \left[ W^* + \frac{1}{\alpha} (W_T - W^*)^- + \kappa W_T \right] \\
W_n^+ = s + b + q_n
\]

\[s.t. \mathcal{P}_n = \{ p_n(\cdot), \mathcal{P}_{n+1}^* \} = \{ p_n(\cdot), p_{n+1}^*(\cdot), \ldots, p_{M-1}^*(\cdot) \} \tag{3}\]

where \(\mathcal{P}_{n+1}^*\) is optimal for problem \((T \text{CMC}_{t_{n+1}}(\kappa))\)

**Intuition:**

- Time consistent constraint in (3)
  - Optimize control today, knowing that future controls are optimal for future problems
Embed problem \( (TCMC_{t_n}(\kappa)) \) in 3-d space

Define auxiliary function \( V(s, b, W^*, t) \)

\[
V(s, b, W^*, t_n^-) = E_{P_n}^{(W_n^+, W^*, t_n^+)} \left[ W^* + \frac{1}{\alpha} (W_T - W^*)^- + \kappa W_T \right]
\]

\[
W_n^+ = s + b + q_n
\]

Dynamic programming solution for optimal control

\[
p_n(w) = \arg \max_{p' \in \mathbb{Z}} \left\{ \max_{W^*} V(w, p', w(1 - p'), W^*, t_n^+) \right\}.
\]

But we advance the solution (backwards) for all values of \( W^* \)

\[
V(s, b, W^*, t_n^-) = V(w^+ p_n(w^+), w^+ (1 - p_n(w^+)), W^*, t_n^+ )
\]

\[
w^+ = s + b + q_n.
\]
Expanded state space formulation II

For $t \in (t_{n-1}, t_n)$ (i.e. between rebalancing dates)

- Solve 2-d PIDE, with $W^*$ regarded as a parameter

**Intuition**

- Optimal $W^*$ depends on state, time and future controls
  - → Solve for all possible values of $W^*$, additional state variable
- Now have a true 3-d problem
  - → Coupling for different $W^*$ values occurs through optimal controls at each rebalancing date
- Recover original value function

$$J(s, b, t_n^-) = \max_{W^*} V(s, b, W^*, t_n^-),$$
(\text{PCMC}_{t_0} (\kappa)) : Pre-commitment Mean-CVAR

Defined via value function \( \hat{J}(s, b, t_0) \)

\[
(\text{PCMC}_{t_0} (\kappa)) : \\
\hat{J}(0, W_0, t_0^-) = \max_{\mathcal{P}_0} \max_{W^*} E^{(W_0^+, t_0^+)}_{\mathcal{P}_0} \left[ W^* + \frac{1}{\alpha} (W_T - W^*)^- + \kappa W_T \right] \\
W_0^+ = W_0 + q_0 \; ; \; W_0 = \text{initial wealth}
\]

Compared with time-consistent formulation

- No time consistent constraint
- Optimality at \( t = t_0 \)

Re-formulate: interchange \( \max_{\mathcal{P}_0} \max_{W^*} E[\cdot] \)

\[
\hat{J}(0, W_0, t_0^-) = \max_{W^*} \max_{\mathcal{P}_0} E^{(W_0^+, t_0^+)}_{\mathcal{P}_0} \left[ W^* + \frac{1}{\alpha} (W_T - W^*)^- + \kappa W_T \right]
\]
Expanded State Space Formulation (Miller and Yang, 2017)

Define auxiliary function $\hat{V}(s, b, W^*, t)$

$$
\hat{V}(s, b, W^*, t_n^-) = E_{P_n}^{(W_n^+, W^*, t_n^+)} \left[ W^* + \frac{1}{\alpha} (W_T - W^*)^- + \kappa W_T \right] \\
W_n^+ = s + b + q_n
$$

Dynamic programming solution for control:

$$
\hat{p}_n(w, W^*) = \arg \max_{p' \in \mathcal{Z}} \left\{ \hat{V}(w p', w (1 - p'), W^*, t_i^+) \right\}.
$$

Advance solution backwards (fixed $W^*$)

$$
\hat{V}(s, b, W^*, t_n^-) = \hat{V} \left( w^+ \hat{p}_n(w^+, W^*), w^+ (1 - \hat{p}_n(w^+, W^*)), W^*, t_n^+ \right) \\
w^+ = s + b + q_n
$$
Pre-commitment formulation II

Remark 1 (Contrast with time-consistent case)

No coupling of the solution for different $W^*$ values from the optimal control.

As usual: for $t \in (t_{n-1}, t_n)$, solve 2-d PIDE

Original pre-commitment value function is recovered via

$$\hat{J}(0, W_0, t_0^-) = \max_{W'} \hat{V}(0, W_0, W', t_0^-)$$  \hspace{1cm} (4)

Formulation requires

- Inner HJB equation solve ($W^*$ is fixed for $t \in [0, T]$)
- Outer optimize (4) over $W^*$ at $t = t_0^-$
Equivalent/Induced Time Consistent Problem

Let

$$ W^*(t_0) = \arg \max_{W'} \hat{V}(s = 0, b = W_0, W', t = t_0) $$

Proposition 1 (Equivalent/Induced Time Consistent Problem)

The pre-commitment mean-CVAR strategy $P^*$ determined by solving $\hat{J}(0, W_0, t_0)$ is the time consistent strategy for the equivalent problem $TCEQ$ (with fixed $W^*(t_0)$), with value function $\tilde{J}(s, b, t)$ defined by

$$ (TCEQ_{t_n} (\kappa \alpha)) : $$

$$ \tilde{J}(s, b, t_n^-) = \max_{P_n \in A} E^{W_n^+, t_n^+}_{P_n} \left[ (W_T - \overbrace{W^*(t_0)}^{constant})^- + (\kappa \alpha)W_T \right] $$

$$ W_n^+ = s + b + q_n $$

---

3Proof: $W^*$ is constant, and multiply PCMC objective by $\alpha > 0$. 
Intuition: TCEQ

Induced alternative objective function ($TCEQ_{t_n} (\kappa \alpha)$):

- Solve pre-commitment problem at time zero
  - Determine target shortfall $W^*(t_0)$ (i.e. VAR at level $\alpha$)

- With this fixed $W^*(t_0)$
  - Solve $\forall t$, problem $(TCEQ_{t_n} (\kappa \alpha))$
  - This strategy is time consistent

Intuitively appealing

- If you have a billion dollars
  - You don’t worry as long as you have 50 million left

- If you have only a million dollars
  - You probably get worried if your wealth < 500,000

- Contrast with time consistent Mean-CVAR
  - Disaster level of wealth always relative to current wealth

- Fixed target based strategies popular with actuaries (Vigna, 2017)
Numerical Methods

Time consistent Mean-CVAR

- Discretize in \((s, b, W^*)\) directions (3-d)
  - Solve PIDE between rebalancing times using \(\epsilon\)-Monotone Fourier method (Forsyth, Labahn; 2019)
  - Discretize equity fraction, solve optimization problems at rebalancing times by exhaustive search and linear interpolation

Pre-commitment mean-CVAR

- Discretize in \((s, b)\) directions (2-d), \(W^*\) is a fixed parameter
  - Solve PIDE: as above
  - Solve optimization problems: as above
- Use 1-d optimization method for outer optimization over \(W^*\)
  - Each evaluation of objective function requires HJB equation solve
Numerical Example

Stock Index
- Fit to CRSP US cap-weighted stock index 1926:1-2017:12 (real, i.e. inflation adjusted)

Bond Index
- Average one month real T-bill return, 1926:1-2017:12, \((r = .00464)\)

<table>
<thead>
<tr>
<th>Investment Parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td>Expiry time (T)</td>
</tr>
<tr>
<td>Initial wealth</td>
</tr>
<tr>
<td>Rebalancing frequency</td>
</tr>
<tr>
<td>Cash injection ({q_i}_{i=0,...,29})</td>
</tr>
</tbody>
</table>
Default Strategy: rebalance to constant weight $p = 0.4^4$

<table>
<thead>
<tr>
<th>$E[W_T]$</th>
<th>CVAR (5%)</th>
<th>Median $[W_T]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1162</td>
<td>598</td>
<td>1084</td>
</tr>
</tbody>
</table>

Units: thousands of dollars

Choose $\kappa$ (scalarization parameter) for pre-commitment and time consistent Mean-CVAR

- Median $[W_T]$ matches median for $p = 0.4$ strategy (approximately)

\[\text{A typical glide path strategy: } p = .8, t = 0; p = 0.0, t = T; \text{ time average } p \approx .4. \text{ Glide path and constant weight with same time average } p, \rightarrow \text{ same distribution of } W_T \text{ (Forsyth and Vetzal, 2019).} \]
Compare CVAR, same $Median[W_T]$ 

<table>
<thead>
<tr>
<th>Strategy</th>
<th>CVAR (5%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pre-commitment</td>
<td>682</td>
</tr>
<tr>
<td>Constant weight ($p = 0.4$)</td>
<td>598</td>
</tr>
<tr>
<td>Time consistent</td>
<td>530</td>
</tr>
</tbody>
</table>

Units: thousands of dollars
More Details: pre-commitment

Percentiles: accumulated wealth

Percentiles: fraction in equities
Control Heat Map: pre-commitment

Red: all stock; Blue: all bond
More Details: time consistent

Percentiles: accumulated wealth

Heat Map of controls: time consistent
- Mostly independent of wealth (except for small wealth values)
  → Almost deterministic strategy
- Map: uninteresting, has (mostly) straight vertical lines
Time consistent constraint or induced time consistent objective?

**Pre-commitment strategies**

→ Investor has incentive to deviate from strategy computed at time zero

But, pre-commitment mean-CVAR strategy computed at time zero

→ Identical to time consistent target shortfall strategy, with fixed target

→ Investor has no incentive to deviate from this strategy, under this induced objective function

**Time consistent strategies**

- If we constrain a pre-commitment strategy to be time consistent
  → Then this strategy is equivalent to an optimal strategy for an unconstrained alternative objective function

⇒ Both strategies: time consistent under alternative objective functions
Compare Strategies: Cumulative Distribution Functions

By design, all strategies have the same $\text{Median}[W_T]$

$\Rightarrow$ Intersect at $\text{Prob}(\cdot) = 0.5$

Minimize left tail risk

$\Rightarrow$ Look for strategy which plots below other strategies in left tail

Time consistent mean-CVAR

$\Rightarrow$ Has worst tail risk of any strategy
Conclusions

- Adding time consistent constraint to mean-CVAR objective function
  - Equivalent to alternative, unconstrained objective function
    \[ \Rightarrow \text{Under this alternative objective function, we no longer minimize tail risk} \]

- Pre-commitment strategy at time zero
  - Equivalent to time consistent target shortfall strategy \( \forall t > 0 \)
  - Minimizes tail risk w.r.t fixed target
  - Maximizes CVAR at time zero\(^5\)

- It would appear that forcing time consistency in the mean-CVAR case is a bad idea!

- Consistent with poor performance of time consistent, MV case, wealth dependent risk aversion parameter
  - See (Wang, Forsyth; 2011), (Van Staden et al; 2018), (Bensoussan et al; 2019)

\(^5\)Recall CVAR is the mean of the worst fraction of wealth outcomes \( \Rightarrow \text{larger is better.} \)