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The Existence of Optimal Bang-Bang Controls for GMxB Contracts*

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Abstract. A large collection of financial contracts offering guaranteed minimum benefits are often posed as control problems, in which at any point in the solution domain, a control is able to take any one of an uncountable number of values from the *admissible set*. Often, such contracts specify that the holder exerts control at a finite number of deterministic times. The existence of an *optimal bang-bang control*, an optimal control taking on only a finite subset of values from the admissible set, is a common assumption in the literature. In this case, the numerical complexity of searching for an optimal control is considerably reduced. However, no rigorous treatment as to when an optimal bang-bang control exists is present in the literature. We provide the reader with a bang-bang principle from which the existence of such a control can be established for contracts satisfying some simple conditions. The bang-bang principle relies on the convexity and monotonicity of the solution and is developed using basic results in convex analysis and parabolic partial differential equations. We show that a *Guaranteed Lifelong Withdrawal Benefits* (GLWB) contract admits an optimal bang-bang control. In particular, we find that the holder of a GLWB can maximize a writer's losses by only ever performing nonwithdrawal, withdrawal at exactly the contract rate, or full surrender. We demonstrate that the related *Guaranteed Minimum Withdrawal Benefits* contract is not convexity preserving, and hence does not satisfy the bang-bang principle other than in certain degenerate cases.

Key words. bang-bang controls, GMxB guarantees, convex optimization, optimal stochastic control

AMS subject classifications. 91G, 93C20, 65N06

1. Introduction.

1.1. Main results. A large collection of financial contracts offering guaranteed minimum benefits (GMxBs) are often posed as control problems [3], in which the control is able to take any one of an uncountable number of values from the *admissible set* at each point in its domain. For example, a contract featuring regular withdrawals may allow the holder to withdraw any portion of their account. In the following, we consider a control which maximizes losses for the writer of the contract; hereafter referred to as an *optimal control*.

A typical example is a *Guaranteed Minimum Withdrawal Benefit* (GMWB). If withdrawals are allowed at any time (i.e. "continuously"), then the pricing problem can be formulated as a singular control [23, 13, 18, 19] or an impulse control [9] problem.

In practice, the contract usually specifies that the control can only be exercised at a finite number of deterministic *exercise times* $t_1 < t_2 < \ldots < t_{N-1}$ [3, 10]. The procedure for pricing such a contract using dynamic programming proceeds backwards from the expiry time t_N as follows:

1. Given the solution as $t \to t_{n+1}^-$, the solution as $t \to t_n^+$ is acquired by solving an initial value problem.

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2. The solution as $t \to t_n^-$ is then determined by applying an optimal control, which is found by considering a collection of optimization problems.

If, for example, a finite difference method is used to solve the initial value problem from t_{n+1}^- to t_n^+, the optimal control is determined by solving an optimization problem at each grid node, in order to advance the solution to t_n^-. Continuing in this way, we determine the solution at the initial time.

If there exists an optimal bang-bang control, an optimal control taking on only a finite subset of values from the admissible set, the numerical algorithm simplifies considerably. The existence of such a control is a common assumption in insurance applications [2, 24, 17], although no rigorous treatment is present in the literature.

Our main result in this paper is the specification of sufficient conditions (a bang-bang principle), which can be used to guarantee the existence of an optimal bang-bang control. The bang-bang principle relies on the convexity and monotonicity of the solution and follows from a combination of basic results in convex analysis and parabolic partial differential equations (PDEs). We demonstrate our results on two common contracts in the GMxB family:

- The *Guaranteed Lifelong Withdrawal Benefits* (GLWB) (a.k.a. *Guaranteed Minimum Lifelong Withdrawal Benefits (GMLWB)*) admits an optimal bang-bang control. In particular, we prove that a holder can maximize the writer's losses by only ever performing
  - nonwithdrawal
  - withdrawal at the contract rate (i.e. never subject to a penalty)
  - a full surrender (i.e. maximal withdrawal; may be subject to a penalty)
- On the other hand, the *Guaranteed Minimum Withdrawal Benefit*s (GMWB) is not necessarily convexity preserving, and does not (in general) satisfy the bang-bang principle other than in certain degenerate cases.

In the event that it is not possible to determine the optimal control analytically, numerical methods are required. Standard techniques in optimization are not always applicable, since these methods cannot guarantee convergence to a global extrema. In particular, without a priori knowledge about the objective functions appearing in the family of optimization problems corresponding to optimal holder behaviour at the exercise times, a numerical method needs to resort to a linear search over a discretization of the admissible set. Convergence to a desired tolerance is achieved by refining this partition [28]. Only with this approach can we be assured of a convergent algorithm. However, if an optimal bang-bang control exists, discretizing the control set becomes unnecessary. Theoretically, this simplifies convergence analysis. More importantly, in practice, this reduces the amount of work per local optimization problem, often the bottleneck of any numerical method.

1.2. Insurance applications. The GLWB is a response to a general reduction in the availability of defined benefit pension plans [8], allowing the buyer to replicate the security of such a plan via a substitute. The GLWB is bootstrapped via a lump sum payment x_0 to an insurer, which is invested in risky assets. We term this the *investment account*. Associated with the GLWB contract is the *guaranteed withdrawal benefit account*, referred to as the withdrawal benefit for brevity. This account is also initially set to x_0. At a finite set of deterministic *withdrawal times*, the holder is entitled to withdraw a predetermined fraction of the withdrawal benefit (or any lesser amount), even if the investment account diminishes to zero. This predetermined fraction is referred to as the *contract withdrawal rate*. If the holder wishes to withdraw in excess of the contract withdrawal rate, they can do so upon the payment of a penalty. Typical GLWB contracts include penalty rates that are decreasing functions of time.

These contracts are often bundled with *ratchets* (a.k.a. step-ups), a contract feature that periodically increases the withdrawal benefit to the investment account, provided that the latter has grown larger than the former. Moreover, *bonus* (a.k.a. roll-up) provisions are also often present, in which the withdrawal benefit is increased if the holder does not withdraw at a given withdrawal time. Upon death, the holder's estate receives the entirety of the investment account. We show that a holder can maximize the writer's costs by only ever performing *nonwithdrawal*, *withdrawal at exactly the contract rate*, or *surrendering the entirety of their account*. Such a holder will never withdraw a nonzero amount strictly below the contract rate or perform a partial surrender. However, this result requires a special form for the penalty and lapsation functions, which is not universal in all contracts. Pricing GLWB contracts has been previously considered in [26, 17, 14, 1].

Much like the GLWB contract, a GMWB is composed of an investment account and withdrawal benefit initially set to x_0, in which x_0 is a lump sum payment to an insurer. At a finite set of withdrawal times, the holder is entitled to withdraw up to a predetermined amount. Note that this amount is not a fraction of the withdrawal benefit, as in the GLWB, but rather a constant amount irrespective of the withdrawal benefit's size. Furthermore, unlike the GLWB, the action of withdrawing decreases both the investment account and withdrawal benefit on a dollar-for-dollar basis.

The GMWB promises to return at least the entire original investment, regardless of the performance of the underlying risky investment. The holder may withdraw more than the predetermined amount subject to a penalty. Upon death, the contract is simply transferred to the holder's estate, and hence mortality risk need not be considered. Pricing GMWB contracts has been previously considered in [23, 13, 10, 18, 19].

1.3. Overview.

In §2 we introduce the GLWB and GMWB contracts. In §3, we generalize this to model a contract that can be controlled at finitely many times, a typical case in insurance practice (e.g. yearly or quarterly exercise). In §4, we develop sufficient conditions for the existence of a bang-bang control, and show that the GLWB satisfies these conditions. §5 discusses a numerical method for finding the cost of funding GLWB and GMWB contracts, demonstrating the bang-bang principle for the former and providing an example of where it fails for the latter.

2. Guaranteed Minimum Benefits (GMxBs).

We introduce mathematical models for the GLWB and GMWB contracts in this section. To distinguish the two contracts, we will refer to the cost of funding a GLWB as V^{L}, and that of a GMWB as V^{M}. In general, we use the superscripts L and M to distinguish between quantities that pertain to the GLWB and GMWB, respectively. For both contracts, we assume a finite number of exercise times gathered in the set

$$\mathcal{T} \equiv \{t_1, t_2, \ldots, t_N\}$$

with the order

$$0 \equiv t_0 < t_1 < \ldots < t_N \equiv T.$$

0 and T are referred to as the *initial* and *expiry* times[1], respectively.

[1] Note that $0 \notin \mathcal{T}$ (i.e. the initial time is not an exercise time). This assumption is taken purely to simplify notation.

2.1. Guaranteed Lifelong Withdrawal Benefits (GLWB). Let $\mathcal{M}(t)$ be the mortality rate at time t (i.e. $\int_{t_1}^{t_2}\mathcal{M}(t)\,dt$ is the fraction of the original holders who pass away in the interval $[t_1,t_2]$), so that the survival probability at time t is

$$\mathcal{R}(t)=1-\int_0^t\mathcal{M}(s)\,ds.$$

We assume $\mathcal{R}$ is monotone decreasing with $\mathcal{R}(t)\in[0,1]$ for all t. Furthermore, we assume the existence of a time $t^\star>0$ s.t. $\mathcal{R}(t^\star)=0$. That is, survival beyond $t^\star$ is impossible (i.e. no holder lives forever). We assume that mortality risk is diversifiable. Since most GLWB contracts offer withdrawals on anniversary dates, to simplify notation, we restrict our attention to annual withdrawals, taking $t_0=0$, $t_1=1$, …, $t_N=N$. In particular, N is chosen s.t. $N\geqslant t^\star$ to ensure that all holders have passed away at the expiry of the contract. As is often the case in practice, we assume ratchets are prescribed to occur on a subset of the anniversary dates (e.g. triennially).

In order to ensure that the writer can, at least in theory, hedge a short position in the contract with no risk, we assume that the holder of a GLWB will employ a loss-maximizing strategy. That is, the holder will act so as to maximize the cost of funding a GLWB. This represents the worst-case scenario for the writer. This worst-case cost is a function of the holder's investment account and withdrawal benefit. As such, we write $\mathbf{x}\equiv(x_1,x_2)$, where x_1 is the value of the investment account and x_2 is the withdrawal benefit. Both of these quantities are nonnegative. We denote the worst-case cost of funding a GLWB, adjusted to account for the effects of mortality, by $V^{\mathrm{L}}\colon[0,\infty)^2\times[0,N]\to\mathbb{R}$. Since N was picked sufficiently large so that the insurer has no obligations at the N^{th} anniversary (recall $N\geqslant t^\star$) and hence we have the terminal condition φ^{L} with

$$\varphi^{\mathrm{L}}(\mathbf{x})=0. \tag{2.1}$$

2.1.1. Across exercise times. The GLWB allows a holder to withdraw at most a predetermined fraction of the withdrawal benefit (the contract withdrawal rate) or a higher amount at a penalty on each anniversary. Consider the point $(\mathbf{x},t_n)\equiv(x_1,x_2,t_n)=(x_1,x_2,n)\in[0,\infty)^2\times\mathcal{T}$. We begin by introducing the conditions at the exercise times using a heuristic approach and give a formal statement in §3.1. We relate the cost of the contract at $n-\varepsilon$ to $n+\varepsilon$ as $\varepsilon\to0^+$ as follows:

1. If the holder does not withdraw, the withdrawal account is amplified by $1+\beta$, in which $\beta\geqslant0$ is referred to as the *bonus rate*. In this case,

$$V^{\mathrm{L}}(x_1,x_2,n-\varepsilon)=V^{\mathrm{L}}(x_1,x_2(1+\beta),n). \tag{2.2}$$

2. Let δ denote the contract withdrawal rate s.t. δx_2 is the maximum a holder can withdraw without incurring a penalty. To account for the effects of mortality, we adjust cash flows by $\mathcal{R}(n)$, the survival probability at time n [14]. If the holder withdraws $\lambda\delta x_2$ with $\lambda\in(0,1]$,

$$V^{\mathrm{L}}(x_1,x_2,n-\varepsilon)=V^{\mathrm{L}}((x_1-\lambda\delta x_2)\vee0,x_2,n)+\mathcal{R}(n)\lambda\delta x_2, \tag{2.3}$$

in which $a\vee b\equiv\max(a,b)$.

3. Let $\kappa_n\in[0,1]$ denote the *penalty rate* at the n^{th} anniversary and $x_1'(\mathbf{x})\equiv(x_1-\delta x_2)\vee0$ be the state of the investment account after a withdrawal at (exactly) the contract rate. If the holder

performs a partial or full surrender (i.e. withdraws over the contract rate),

$$V^{\mathrm{L}}(x_1,x_2,n-\varepsilon)=V^{\mathrm{L}}\left((2-\lambda)x_1'(\mathbf{x}),(2-\lambda)x_2,n\right)+\mathcal{R}(n)\left(\delta x_2+(\lambda-1)(1-\kappa_n)x_1'(\mathbf{x})\right), \quad (2.4)$$

where $\lambda \in (1,2]$ corresponds to a partial surrender and $\lambda = 2$ corresponds to a full surrender. A holder employing a loss-maximizing strategy will pick one of nonwithdrawal (2.2), withdrawal at or below the contract rate with $\lambda \in (0,1]$ (2.3), or surrender with $\lambda \in (1,2]$ (2.4) s.t. $V^{\mathrm{L}}(\mathbf{x},n-\varepsilon)$ is maximized.

If a ratchet is prescribed to occur on the n^{th} anniversary, the withdrawal benefit is increased to the value of the investment account, provided that the latter has grown larger than the former. We can capture both ratcheting and nonratcheting anniversaries by letting

$$\mathbb{I}_n=\begin{cases}1 & \text{if a ratchet is prescribed to occur on the } n^{\mathrm{th}} \text{ anniversary}\\ 0 & \text{otherwise}\end{cases} \quad (2.5)$$

and taking

$$V^{\mathrm{L}}(x_1,x_2,n)=V^{\mathrm{L}}(x_1,(\mathbb{I}_n x_1)\vee x_2,n+\varepsilon). \quad (2.6)$$

We can relate V^{L} at $n-\varepsilon$ to V^{L} at $n+\varepsilon$ by first taking compositions of the above functions and only then taking $\varepsilon \to 0^+$, yielding two one-sided limits. The costs at $n-\varepsilon$ and $n+\varepsilon$ as $\varepsilon \to 0^+$ can be thought of as the costs of funding the contract "immediately before" and "immediately after" the exercise time n, respectively. We stress that this is a heuristic definition only, since the final statement relating V^{L} at $n-\varepsilon$ to V^{L} at $n+\varepsilon$ is not necessarily true for any $\varepsilon > 0$.

2.1.2. Between exercise times. Let α denote the *hedging fee*, the rate continuously deducted from the investment account x_1 to provide the premium for the contract. We assume that the investment account of the GLWB follows geometric Brownian motion (GBM) as per

$$\frac{dx_1}{x_1}=(\mu-\alpha)\,dt+\sigma dZ \quad (2.7)$$

where Z is a Wiener process, tracking the index $\hat{x}_1$ satisfying

$$\frac{d\hat{x}_1}{\hat{x}_1}=\mu dt+\sigma dZ. \quad (2.8)$$

We assume that it is not possible to short the investment account x_1 for fiduciary reasons [10], so that the obvious arbitrage opportunity is prohibited. It can be shown [14], by an application of Itô's lemma, that between exercise times n and $n+1$, the cost to fund the GLWB satisfies[2]

$$\partial_t V^{\mathrm{L}}+\mathcal{L}V^{\mathrm{L}}+\mathcal{M}x_1=0 \text{ on } [0,\infty)^2\times(n,n+1) \quad (2.9)$$

where

$$\mathcal{L}\equiv\frac{1}{2}\sigma^2x_1^2\partial_{x_1x_1}+(r-\alpha)x_1\partial_{x_1}-r \quad (2.10)$$

and r is the risk-free rate of return.

[2] We discuss what it means for a function to satisfy this PDE in Appendix A. We also assume the regularity established in Appendix A.

2.2. Guaranteed Minimum Withdrawal Benefits (GMWB). As with the GLWB, to simplify notation, we restrict our attention to annual withdrawals with $t_0 = 0$, $t_1 = 1$, $\ldots$, $t_N = N$. As usual, we assume that the holder of a GMWB will employ a loss-maximizing strategy. We write $\mathbf{x} \equiv (x_1, x_2)$, where x_1 is the value of the investment account and x_2 is the withdrawal benefit. Since the GMWB contract is transferred to the holder's estate upon death, mortality risk is not considered. We denote the worst-case cost of funding a GMWB by $V^{\mathrm{M}}\colon [0,\infty)^2 \times [0,N] \to \mathbb{R}$. We have the terminal condition [13]

$$\varphi^{\mathrm{M}}(\mathbf{x}) = \max\left(x_1, (1-\kappa)x_2\right),$$

corresponding to the greater of a full surrender at the penalty rate or the entirety of the investment account.

2.2.1. Across exercise times. Let $G \geqslant 0$ denote the predetermined contract withdrawal amount associated with the GMWB so that $G \wedge x_2$ (where $a \wedge b \equiv \min(a,b)$) is the maximum the holder can withdraw without incurring a penalty. Consider the point $(\mathbf{x}, t_n) \equiv (x_1, x_2, t_n) = (x_1, x_2, n) \in [0,\infty)^2 \times \mathcal{T}$. Let $\varepsilon > 0$. We relate the cost of the contract at $n - \varepsilon$ to $n + \varepsilon$ as $\varepsilon \to 0^+$ as follows:

1. The maximum a holder can withdraw without incurring a penalty is $G \wedge x_2$. If the holder withdraws the amount λx_2 with $\lambda x_2 \in [0, G \wedge x_2]$,
$$V^{\mathrm{M}}(x_1, x_2, n-\varepsilon) = V^{\mathrm{M}}\left((x_1 - \lambda x_2) \vee 0, x_2 - \lambda x_2, n\right) + \lambda x_2. \tag{2.11}$$
2. Let $\kappa_n \in [0,1]$ denote the penalty rate at the n^{th} anniversary. If the holder withdraws the amount λx_2 with $\lambda x_2 \in (G \wedge x_2, x_2]$,
$$V^{\mathrm{M}}(x_1, x_2, n-\varepsilon) = V^{\mathrm{M}}\left((x_1 - \lambda x_2) \vee 0, x_2 - \lambda x_2, n\right) + \lambda x_2 - \kappa_n(\lambda x_2 - G). \tag{2.12}$$
Here, $\lambda x_2 \in (G \wedge x_2, x_2)$ corresponds to a partial surrender and $\lambda x_2 = x_2$ (i.e. $\lambda = 1$) corresponds to a full surrender.

A holder employing a loss-maximizing strategy will pick $\lambda \in [0,1]$ so as to maximize $V^{\mathrm{M}}(\mathbf{x}, n-\varepsilon)$.

2.2.2. Between exercise times. As with the GLWB, we assume the investment account x_1 follows the dynamics (2.7), tracking the index $\hat{x}_1$ following (2.8). It can be shown, by an application of Itô's lemma, that between exercise times n and $n+1$, the cost to fund the GMWB satisfies

$$\partial_t V^{\mathrm{M}} + \mathcal{L} V^{\mathrm{M}} = 0 \text{ on } [0,\infty)^2 \times (n, n+1) \tag{2.13}$$

where $\mathcal{L}$ is defined in (2.10) [10].

3. General formulation.

In describing the general formulation, we use the symbol V to distinguish it from V^{L} and V^{M}. Any claims made about V apply to V^{L} and V^{M}.

3.1. Exercise times. We assume the contract is alive on the finite time-horizon $[0,T]$. As usual, let $\mathcal{T} \equiv \{t_1, \ldots, t_{N-1}\}$ along with the order $0 \equiv t_0 < t_1 < \ldots < t_N \equiv T$. We use the symbol $V\colon \Omega \times [0,T] \to \mathbb{R}$ to denote the cost of hedging the contract at a particular state ($\mathbf{x} \in \Omega$) and time ($t \in [0,T]$) under a loss-maximizing strategy. We assume that Ω is a convex subset of $\mathbb{R}^m$.

Remark 3.1. *We restrict V to be a* càdlàg *function of time so that the process $V(t) \equiv V(\mathbf{S}(t), t)$ is nonpredictable whenever $\mathbf{S}$ is nonpredictable [11]. Specifically, for all $\mathbf{x} \in \Omega$ and $t \in [0,T]$,*

1. $V(\mathbf{x}, t^-) \equiv \lim_{s \uparrow t_n} V(\mathbf{x}, s)$ exists.

2. $V(\mathbf{x},t^+) \equiv \lim_{s\downarrow t_n} V(\mathbf{x},s)$ exists and is equal to $V(\mathbf{x},t)$.

Intuitively, when $t = t_n \in \mathcal{T}$, $V(\mathbf{x},t_n^-)$ and $V(\mathbf{x},t_n^+)$ can be thought of as the value of the contract "immediately before" and "immediately after" the exercise time t_n.

At the point $(\mathbf{x},t_n) \in \Omega \times \mathcal{T}$, the holder is able to perform one of a number of *actions* based on the state of the contract. The set of all such actions is the *admissible set* $\Lambda_n \subset \mathbb{R}^{m'}$, which we allow to vary w.r.t. the exercise time (as suggested by the subscript n).

Remark 3.2.*Convexity preservation, a property that helps establish the bang-bang principle, depends on the admissible set Λ_n being independent of the state of the contract, $\mathbf{x}$. This is discussed in Remark 4.17.*

Consider the point $(\mathbf{x},t_n) \in \Omega \times \mathcal{T}$. Let

$$v_{\mathbf{x},n}(\lambda) \equiv V\left(\mathbf{f}_{\mathbf{x},n}(\lambda), t_n^+\right) + f_{\mathbf{x},n}(\lambda) \tag{3.1}$$

be the value of the contract assuming the holder performs action $\lambda \in \Lambda_n$. Here, $f_{\mathbf{x},n} \colon \Lambda_n \to \mathbb{R}$ represents cash flow from the writer to the holder, while $\mathbf{f}_{\mathbf{x},n} \colon \Lambda_n \to \Omega$ represents the state of the contract after the exercise time. We write $v_{\mathbf{x},n}(\lambda)$ instead of $v_n(\mathbf{x},\lambda)$ to stress that for each fixed $(\mathbf{x},n)$, we consider an optimization problem involving the variable $\lambda \in \Lambda_n$. In particular, recalling that we are interested in loss-maximizing strategies, define

$$\Lambda_n^\star(\mathbf{x}) \equiv \operatorname*{argsup}_{\lambda \in \Lambda_n} v_{\mathbf{x},n}(\lambda) \tag{3.2}$$

to be the set of all optimal (i.e. loss-maximizing) actions at $(\mathbf{x},t_n)$. Let $\lambda_n^\star(\mathbf{x})$ denote an arbitrary member of $\Lambda_n^\star(\mathbf{x})$, and assume for the time being that $\Lambda_n^\star(\mathbf{x})$ is nonempty. Assuming the holder acts so as to maximize the value of the contract, we have

$$V\left(\mathbf{x},t_n^-\right) = v_{\mathbf{x},n}\left(\lambda_n^\star(\mathbf{x})\right) \text{ on } \Omega \times \mathcal{T}. \tag{3.3}$$

Lastly, we assume a terminal condition of the form

$$V\left(\mathbf{x},t_N^+\right) = \varphi(\mathbf{x}) \text{ on } \Omega.$$

Example 3.3.*Consider the GLWB. We write the withdrawal and ratchet conditions (2.2), (2.3), (2.4) and (2.6) in the form (3.1), (3.2) and (3.3). To do so, we take $\Lambda_n^L \equiv [0,2]$,*

$$\mathbf{f}_{\mathbf{x},n}^L(\lambda) \equiv \begin{cases} \langle x_1,\, x_2(1+\beta) \vee \mathbb{I}_n x_1 \rangle & \text{if } \lambda = 0 \\ \langle x_1 - \lambda\delta x_2 \vee 0,\, x_2 \vee \mathbb{I}_n[x_1 - \lambda\delta x_2] \rangle & \text{if } \lambda \in (0,1] \\ (2-\lambda)\,\mathbf{f}_{\mathbf{x},n}(1) & \text{if } \lambda \in (1,2] \end{cases} \tag{3.4}$$

and

$$f_{\mathbf{x},n}^L(\lambda) \equiv \mathcal{R}(n) \cdot \begin{cases} 0 & \text{if } \lambda = 0 \\ \lambda\delta x_2 & \text{if } \lambda \in (0,1] \\ \delta x_2 + (\lambda-1)(1-\kappa_n)(x_1 - \delta x_2 \vee 0) & \text{if } \lambda \in (1,2] \end{cases}. \tag{3.5}$$

Here, $\vee$ is understood to have lower operator precedence than the arithmetic operations (e.g. $a+b\vee c = (a+b)\vee c$). $\mathbb{I}_n$ is defined by (2.5).

Remark 3.4. *At each point* $(\mathbf{x}, t_n)$*, one needs to find a member of* $\Lambda_n^\star(\mathbf{x})$ *(defined by (3.2)) in order to evaluate (3.3). We remark that for the case of the GLWB, the admissible set* $\Lambda_n^L \equiv [0,2]$ *is undesirably large (i.e. a continuum). We will apply the results established in §4 to show that an optimal strategy taking on values only from* $\{0,1,2\}$ *exists. In other words, an equivalent problem can be constructed by substituting the admissible set* $\{0,1,2\}$ *for the original admissible set* $\Lambda_n^L \equiv [0,2]$ *in the optimization problem (3.2). The resulting problem is (computationally) easier than the original one.*

Example 3.5. *Consider the GMWB. We write the withdrawal and ratchet conditions (2.11) and (2.12) in the form (3.1), (3.2) and (3.3). To do so, we take* $\Lambda_n^M \equiv [0,1]$,

$$\mathbf{f}_{\mathbf{x},n}^M(\lambda) \equiv \langle x_1 - \lambda x_2 \vee 0, (1-\lambda) x_2 \rangle \tag{3.6}$$

and

$$f_{\mathbf{x},n}^M(\lambda) \equiv \begin{cases} \lambda x_2 & \text{if } \lambda x_2 \in [0, G \wedge x_2] \\ G + (1-\kappa_n)(\lambda x_2 - G) & \text{if } \lambda x_2 \in (G \wedge x_2, x_2] \end{cases}. \tag{3.7}$$

Note that for $x_2 > 0$*, the conditions* $\lambda x_2 \in [0, G \wedge x_2]$ *and* $\lambda x_2 \in (G \wedge x_2, x_2]$ *are equivalent to* $\lambda \in [0, G/x_2 \wedge 1]$ *and* $\lambda \in (G/x_2 \wedge 1, 1]$*, respectively. We will use the equivalent form involving* λ *(as opposed to* λx_2*) in §5.3.2.*

4. Bang-bang principle.

In an effort to remain self-contained, we provide the reader with several elementary (but useful) definitions. In practice, we only consider vector spaces over $\mathbb{R}$ and hence restrict our definitions to this case.

Definition 4.1. *Let* W *be a vector space over the field* $\mathbb{R}$. $X \subset W$ *is a* convex set *if for all* $x, x' \in X$ *and* $\theta \in (0,1)$, $\theta x + (1-\theta) x' \in X$.

Definition 4.2. *Let* X *be a convex set and* Y *be a vector space over the field* $\mathbb{R}$ *equipped with a partial order* $\leqslant_Y$. $h\colon X \to Y$ *is a* convex function *if for all* $x, x' \in X$ *and* $\theta \in (0,1)$,

$$h\left(\theta x + (1-\theta) x'\right) \leqslant_Y \theta h(x) + (1-\theta) h\left(x'\right).$$

Definition 4.3. *Let* Y *be a vector space over the field* $\mathbb{R}$. $P \subset Y$ *is a* convex polytope *if there exists* $p_1, \ldots, p_M \in Y$ *s.t.*

$$P = \left\{ p \in Y \mid \exists \theta_1, \ldots, \theta_M \in [0,1] \colon \sum_j \theta_j = 1 \text{ and } p = \sum_j \theta_j p_j \right\}.$$

W.l.o.g., we assume no p_j *is a convex combination of the other* $M-1$ *points of the form* p_k*. In this case,* $p_1, \ldots, p_M$ *are said to be the vertices of* P*.*

Definition 4.4. *An* extreme point *of a convex set* X *is a point* $x \in X$ *which cannot be written* $x = \theta x' + (1-\theta) x''$ *for any* $\theta \in (0,1)$ *and* $x', x'' \in X$ *with* $x' \neq x''$*. Note that the extreme points of a convex polytope are its vertices.*

Definition 4.5. *Let* X *and* Y *be sets equipped with partial orders* $\leqslant_X$ *and* $\leqslant_Y$*, respectively.* $h\colon X \to Y$ *is* monotone *if for all* $x, x' \in X$, $x \leqslant_X x'$ *implies* $h(x) \leqslant_Y h(x')$.

Lemma 4.6. *Let* A *be a convex set, and* B *and* C *be vector spaces over the field* $\mathbb{R}$ *equipped with partial orders* $\leqslant_B$ *and* $\leqslant_C$*, respectively. If* $h_1\colon A \to B$ *and* $h_2\colon B \to C$ *are convex functions with* h_2 *monotone, then* $h_2 \circ h_1$ *is a convex function.*

Remark 4.7. *W.r.t* $\mathbb{R}^m$*, we define* $\leqslant_{\mathbb{R}^m}$ *as follows: if* $\mathbf{x}, \mathbf{y} \in \mathbb{R}^m$, $\mathbf{x} \leqslant_{\mathbb{R}^m} \mathbf{y}$ *whenever* $x_i \leqslant y_i$ *for all* i*. We omit the subscript in subsequent sections and simply write* $\leqslant$.

Throughout this section, we use the shorthand $V_n^+(\mathbf{x}) \equiv V(\mathbf{x}, t_n^+)$ and $V_n^-(\mathbf{x}) \equiv V(\mathbf{x}, t_n^-)$.

4.1. Across exercise times. Throughout this section, we consider the n^{th} exercise time $t_n \in \mathcal{T}$. Assume

(A1) V_n^+ is convex and monotone (as a function of $\mathbf{x}$).
(A2) For each fixed $\mathbf{x} \in \Omega$, $v_{\mathbf{x},n}$ is bounded above (as a function of λ).

4.1.1. Maximum principle. Throughout this section, we consider a particular point $\mathbf{y} \in \Omega$. To arrive at our main result, we require some assumptions.

(B1) There exists a collection $\mathcal{P}_n(\mathbf{y}) \subset 2^{\Lambda_n}$ s.t. $\bigcup_{P\in\mathcal{P}(\mathbf{y})} P = \Lambda_n$ and each $P \in \mathcal{P}_n(\mathbf{y})$ is compact and convex.
(B2) For each $P \in \mathcal{P}_n(\mathbf{y})$, the restrictions $\mathbf{f}_{\mathbf{y},n}|_P$ and $f_{\mathbf{y},n}|_P$ are convex (as functions of λ).

Remark 4.8.*(B1) simply states that we can "cut up" the admissible set Λ_n into (possibly overlapping) compact and convex sets. (B2) states that the restrictions of $\mathbf{f}_{\mathbf{y},n}$ and $f_{\mathbf{y},n}$ on each of these convex sets are convex functions of λ.*

Lemma 4.9.*Suppose (A1), (B1) and (B2). For each $P \in \mathcal{P}_n(\mathbf{y})$, the restriction $v_{\mathbf{y},n}|_P$ is convex (as a function of λ).*

Proof. By (3.3), (A1), (B2) and Lemma 4.6.■

Lemma 4.10 (Maximum principle).*Suppose (A1), (A2), (B1) and (B2). Take $P \in \mathcal{P}_n(\mathbf{y})$ and let $E(P)$ denote the set of extreme points of P. Then,*

$$\sup v_{\mathbf{y},n}(P) = \sup v_{\mathbf{y},n}(E(P)). \tag{4.1}$$

Proof. Let $w \equiv v_{\mathbf{y},n}|_P$. Note that $w(P) = v_{\mathbf{y},n}(P)$, and hence no generality is lost in considering w. Lemma 4.9 establishes the convexity of w. Naturally, $\sup w(P)$ exists (and hence $\sup w(E(P))$ exists too) due to (A2). Lastly, it is well-known from elementary convex analysis that the supremum of a convex function on a compact and convex set P lies on the extreme points of P, $E(P)$. See [27, Ch. 32].■

Corollary 4.11.*Suppose (A1), (A2), (B1) and (B2). If $P \in \mathcal{P}_n(\mathbf{y})$ is a convex polytope, then $\sup v_{\mathbf{y},n}(P) = \sup v_{\mathbf{y},n}(\{p_1,\ldots,p_M\})$ where $p_1, \ldots, p_M$ are the vertices of P.*

Proof. The extreme points of P are its vertices. ■

Corollary 4.12 (Bang-bang principle).*Suppose (A1), (A2), (B1) and (B2). Then,*

$$\sup v_{\mathbf{y},n}(\Lambda_n) = \sup v_{\mathbf{y},n}\left(\bigcup_{P\in\mathcal{P}_n(\mathbf{y})} E(P)\right).$$

Proof. By (B1), we have that $\Lambda_n = \bigcup_{P\in\mathcal{P}_n(\mathbf{y})} P$. We can, w.l.o.g., assume that all members of $\mathcal{P}_n(\mathbf{y})$ are nonempty (otherwise, construct $\mathcal{P}_n'(\mathbf{y})$ from $\mathcal{P}_n(\mathbf{y})$ by removing all empty sets). $\sup v_{\mathbf{y},n}(\Lambda_n)$ exists due to (A2). Since for each $P \in \mathcal{P}_n(\mathbf{y})$, $\sup v_{\mathbf{y},n}(P) = \sup v_{\mathbf{y},n}(E(P))$ (Lemma 4.10), by Lemma B.1,

$$\begin{aligned}\sup v_{\mathbf{y},n}(\Lambda_n) &= \sup v_{\mathbf{y},n}\left(\bigcup_{P\in\mathcal{P}_n(\mathbf{y})} P\right)\\ &= \sup\left\{u \in \mathbb{R} \mid \exists P \in \mathcal{P}_n(\mathbf{y}) : u = \sup v_{\mathbf{y},n}(P)\right\}\\ &= \sup\left\{u \in \mathbb{R} \mid \exists P \in \mathcal{P}_n(\mathbf{y}) : u = \sup v_{\mathbf{y},n}(E(P))\right\}\\ &= \sup\left(\bigcup_{P\in\mathcal{P}_n(\mathbf{y})} E(P)\right).\end{aligned}$$

■

Corollary 4.12 leaves us with a much "smaller" region over which to search for an optimal control. We refer to this as the *bang-bang principle*. When each $P \in \mathcal{P}_n(\mathbf{y})$ is a convex polytope (Corollary 4.11) and $\mathcal{P}_n(\mathbf{y})$ is finite, the situation is even nicer: the set $\bigcup_{P\in\mathcal{P}_n(\mathbf{y})} E(P)$ is finite (a finite union of finite sets), and hence only a finite subset of Λ_n needs to be considered in solving the optimal control problem at $(\mathbf{y}, t_n)$. Thus, when the bang-bang principle is satisfied, and for each $\mathbf{y}$ and each t_n, $\mathcal{P}_n(\mathbf{y})$ is a finite collection of compact and convex sets, an optimal bang-bang control exists, by the construction above.

Example 4.13. *We now find $\mathcal{P}_n^L(\mathbf{y})$ s.t. (B1) and (B2) are satisfied for the GLWB. Let $\mathbf{y} \in \Omega$ be arbitrary. Take $P_1 \equiv [0,1]$, $P_2 \equiv [1,2]$ and $\mathcal{P}_n^L(\mathbf{y}) \equiv \{P_1, P_2\}$. Note that $\bigcup_{P\in\mathcal{P}_n^L(\mathbf{y})} P = [0,2] = \Lambda$ and hence (B1) is satisfied. It is trivial to show that the functions $\mathbf{f}_{\mathbf{y},n}^L|_{P_j}$ and $f_{\mathbf{y},n}^L|_{P_j}$ defined in (3.4) and (3.5) are convex as functions of λ (the maximum of convex functions is a convex function), thereby satisfying (B2). We conclude (whenever (A1) and (A2) hold), by Corollary 4.12, that the supremum of $v_{\mathbf{y},n}^L$ occurs on*

$$\bigcup_{P\in\mathcal{P}_n^L(\mathbf{y})} E(P) = E(P_1) \cup E(P_2) = E([0,1]) \cup E([1,2]) = \{0,1\} \cup \{1,2\} = \{0,1,2\}$$

(corresponding to nonwithdrawal, withdrawal at exactly the contract rate, and full surrender). Since $\mathbf{y}$ was arbitrary and $\mathcal{P}_n^L(\mathbf{y})$ was picked independent of $\mathbf{y}$, we can apply this argument to any point in Ω.

4.1.2. Preservation of convexity and monotonicity.

Since the convexity and monotonicity of V are desirable properties upon which the bang-bang principle depends (i.e. (A1)), we would like to ensure that they are preserved "across" exercise times (i.e. from t_n^+ to t_n^-). To do so, we require some additional assumptions.

Remark 4.14. *For the remainder of this section, we use $\lambda_n^\star(\mathbf{x})$ to denote some member of the optimal control set $\Lambda_n^\star(\mathbf{x})$ as defined by (3.2). Note that for each $\mathbf{x}$, $\Lambda_n^\star(\mathbf{x})$ is nonempty due to (A2).*

(C1) For each fixed $\lambda \in \Lambda_n$, $\mathbf{f}_{\mathbf{x},n}(\lambda)$ and $f_{\mathbf{x},n}(\lambda)$ are convex[3] as functions of $\mathbf{x}$.

[3] Note that this is not the same as (B2). Here, we mean that for each fixed $\lambda \in \Lambda_n$ and for all $\mathbf{x}, \mathbf{x}' \in \Omega$ and $\theta \in (0,1)$,

$$\mathbf{f}_{\theta\mathbf{x}+(1-\theta)\mathbf{x}',n}(\lambda) \leqslant \theta\mathbf{f}_{\mathbf{x},n}(\lambda) + (1-\theta)\mathbf{f}_{\mathbf{x}',n}(\lambda) \tag{4.2}$$

and

$$f_{\theta\mathbf{x}+(1-\theta)\mathbf{x}',n}(\lambda) \leqslant \theta f_{\mathbf{x},n}(\lambda) + (1-\theta) f_{\mathbf{x}',n}(\lambda).$$

The order $\leqslant$ used in (4.2) is that on $\Omega \subset \mathbb{R}^m$, inherited from the order on $\mathbb{R}^m$ established in Remark 4.7.

(C2) For each $\mathbf{x},\mathbf{x}' \in \Omega$ s.t. $\mathbf{x} \leqslant \mathbf{x}'$, there exists $\lambda' \in \Lambda_n$ s.t. $\mathbf{f}_{\mathbf{x},n}(\lambda_n^\star(\mathbf{x})) \leqslant \mathbf{f}_{\mathbf{x}',n}(\lambda')$ and $f_{\mathbf{x},n}(\lambda_n^\star(\mathbf{x})) \leqslant f_{\mathbf{x}',n}(\lambda')$.

Remark 4.15. *(C2) simply states that for each position $\mathbf{x} \leqslant \mathbf{x}'$, there is an action λ' s.t. the position and cash flow after the event at $\mathbf{x}'$ under action λ' are greater than the position and cash flow after the event at $\mathbf{x}$ under an optimal action (e.g. $\lambda_n^\star(\mathbf{x})$). Intuitively, this guarantees us that the position $\mathbf{x}'$ is more desirable than $\mathbf{x}$ (from the holder's perspective). This is not a particularly restrictive assumption, and should hold true for any model of a contract in which a larger position is more desirable than a smaller one.*

Lemma 4.16. *Suppose (A1) and (C1). V_n^- is convex (as a function of $\mathbf{x}$).*

Proof. Fix $\mathbf{x},\mathbf{x}' \in \Omega$ and $\theta \in (0,1)$ and let $\mathbf{z} \equiv \theta\mathbf{x} + (1-\theta)\mathbf{x}'$ for brevity. Then, by (A1) and (C1),

$$\begin{aligned}
V_n^-(\mathbf{z}) &= v_{\mathbf{z},n}(\lambda_n^\star(\mathbf{z})) \\
&= V_n^+(\mathbf{f}_{\mathbf{z},n}(\lambda_n^\star(\mathbf{z}))) + f_{\mathbf{z},n}(\lambda_n^\star(\mathbf{z})) \\
&\leqslant V_n^+(\theta\mathbf{f}_{\mathbf{x},n}(\lambda_n^\star(\mathbf{z})) + (1-\theta)\mathbf{f}_{\mathbf{x}',n}(\lambda_n^\star(\mathbf{z}))) + \theta f_{\mathbf{x},n}(\lambda_n^\star(\mathbf{z})) + (1-\theta) f_{\mathbf{x}',n}(\lambda_n^\star(\mathbf{z})). \\
&\leqslant \theta\left[V_n^+(\mathbf{f}_{\mathbf{x},n}(\lambda_n^\star(\mathbf{z}))) + f_{\mathbf{x},n}(\lambda_n^\star(\mathbf{z}))\right] + (1-\theta)\left[V_n^+(\mathbf{f}_{\mathbf{x}',n}(\lambda_n^\star(\mathbf{z}))) + f_{\mathbf{x}',n}(\lambda_n^\star(\mathbf{z}))\right] \\
&= \theta v_{\mathbf{x},n}(\lambda_n^\star(\mathbf{z})) + (1-\theta) v_{\mathbf{x}',n}(\lambda_n^\star(\mathbf{z}))
\end{aligned}$$

Employing the optimality of $\lambda_n^\star(\mathbf{x})$ and $\lambda_n^\star(\mathbf{x}')$,

$$\begin{aligned}
V_n^-(\mathbf{z}) &\leqslant \theta v_{\mathbf{x},n}(\lambda_n^\star(\mathbf{x})) + (1-\theta) v_{\mathbf{x}',n}\left(\lambda_n^\star\left(\mathbf{x}'\right)\right) \\
&= \theta V_n^-(\mathbf{x}) + (1-\theta) V_n^-\left(\mathbf{x}'\right).
\end{aligned}$$

Remark 4.17. *Note that the proof of Lemma 4.16 evaluates $v_{\mathbf{x},n}$ and $v_{\mathbf{x}',n}$ at $\lambda_n^\star(\mathbf{z})$. If the control set Λ_n is instead a function of the contract state (i.e. $\Lambda_n \equiv \Lambda_n(\mathbf{x})$), then it is not necessarily true that $\lambda_n^\star(\mathbf{z}) \in \Lambda_n(\mathbf{x}), \Lambda_n(\mathbf{x}')$, and hence $v_{\mathbf{x},n}(\lambda_n^\star(\mathbf{z}))$ and $v_{\mathbf{x}',n}(\lambda_n^\star(\mathbf{z}))$ may not be well-defined.*

Lemma 4.18. *Suppose (A1) and (C2). V_n^- is monotone (as a function of $\mathbf{x}$).*

Proof. Let $\mathbf{x},\mathbf{x}' \in \Omega$ s.t. $\mathbf{x} \leqslant \mathbf{x}'$. By (A1) (specifically, since V_n^+ is monotone) and (C2), there exists $\lambda' \in \Lambda(t_n)$ s.t.

$$\begin{aligned}
V_n^-(\mathbf{x}) &= v_{\mathbf{x},n}(\lambda_n^\star(\mathbf{x})) \\
&= V_n^+(\mathbf{f}_{\mathbf{x},n}(\lambda_n^\star(\mathbf{x}))) + f_{\mathbf{x},n}(\lambda_n^\star(\mathbf{x})) \\
&\leqslant V_n^+\left(\mathbf{f}_{\mathbf{x}',n}\left(\lambda'\right)\right) + f_{\mathbf{x}',n}\left(\lambda'\right) \\
&= v_{\mathbf{x}',n}\left(\lambda'\right) \leqslant v_{\mathbf{x}',n}\left(\lambda_n^\star\left(\mathbf{x}'\right)\right) = V_n^-\left(\mathbf{x}'\right).
\end{aligned}$$

Example 4.19. *We now show that the GLWB satisfies (C1) and (C2) given (A1) and (A2). It is trivial to show that the functions $\mathbf{f}_{\mathbf{x},n}^L(\lambda)$ and $f_{\mathbf{x},n}^L(\lambda)$ defined in (3.4) and (3.5) are convex in $\mathbf{x}$ (the maximum of convex functions is a convex function), thereby satisfying (C1). (C2) is slightly more tedious to verify. Let $\mathbf{x},\mathbf{x}' \in \Omega$ s.t. $\mathbf{x} \leqslant \mathbf{x}'$. W.l.o.g., we can assume $\mathbf{x}' > 0$ as the case of $\mathbf{x} = \mathbf{x}' = 0$ is trivial. By (A1), (A2) and the argument in Example 4.13, we can, w.lo.g., assume $\lambda_n^\star(\mathbf{x}) \in \{0,1,2\}$. Hence, we need only consider three cases:*

1. *Suppose* $\lambda_n^\star(\mathbf{x}) = 0$. *Take* $\lambda' = 0$ *to get* $\mathbf{f}_{\mathbf{x},n}^L(0) \leqslant \mathbf{f}_{\mathbf{x}',n}^L(\lambda')$ *and* $f_{\mathbf{x},n}^L(0) = f_{\mathbf{x}',n}^L(\lambda')$.
2. *Suppose* $\lambda_n^\star(\mathbf{x}) = 1$. *Take* $\lambda' = x_2/x_2'$ *to get* $\mathbf{f}_{\mathbf{x},n}^L(1) \leqslant \mathbf{f}_{\mathbf{x}',n}^L(\lambda')$ *and* $f_{\mathbf{x},n}^L(1) = f_{\mathbf{x}',n}^L(\lambda')$.
3. *Suppose* $\lambda_n^\star(\mathbf{x}) = 2$. *If* $x_1 \leqslant \delta x_2$, *then* $\mathbf{f}_{\mathbf{x},n}^L(2) = \langle 0,0\rangle \leqslant \mathbf{f}_{\mathbf{x},n}^L(1)$ *and* $f_{\mathbf{x},n}^L(2) = f_{\mathbf{x},n}^L(1)$, *and we can once again take* $\lambda' = x_2/x_2'$ *to get* $\mathbf{f}_{\mathbf{x},n}^L(2) = \langle 0,0\rangle \leqslant \mathbf{f}_{\mathbf{x}',n}^L(\lambda')$ *and* $f_{\mathbf{x},n}^L(2) = f_{\mathbf{x}',n}^L(\lambda')$. *Therefore, we can safely assume that* $x_1 > \delta x_2$ *so that*

$$f_{\mathbf{x},n}^L(2) = \mathcal{R}(n)\left[(1-\kappa)x_1 + \kappa\delta x_2\right] \leqslant \mathcal{R}(n)\, x_1. \tag{4.3}$$

   (a) *Suppose* $x_1' \leqslant \delta x_2'$. *Take* $\lambda' = 1$ *to get* $\mathbf{f}_{\mathbf{x},n}^L(2) = \langle 0,0\rangle \leqslant \mathbf{f}_{\mathbf{x}',n}^L(1)$ *and*

$$f_{\mathbf{x},n}^L(2) \leqslant \mathcal{R}(n)\, x_1 \leqslant \mathcal{R}(n)\,\delta x_2' = f_{\mathbf{x}',n}^L(1)$$

   by (4.3).

   (b) *Suppose* $x_1' > \delta x_2'$. *Take* $\lambda' = 2$ *to get* $\mathbf{f}_{\mathbf{x},n}^L(2) = \langle 0,0\rangle = \mathbf{f}_{\mathbf{x}',n}^L(2)$ *and*

$$f_{\mathbf{x},n}^L(2) \leqslant \mathcal{R}(n)\left[(1-\kappa_n)\, x_1' + \kappa\delta x_2'\right] = f_{\mathbf{x}',n}^L(2).$$

4.2. Between exercise times. As previously mentioned, to apply Corollary 4.12, we need to check the validity of (A1) (i.e. that the solution is convex and monotone at t_n^+). In light of this, we would like to identify scenarios in which V_n^+ is convex and monotone provided that V_{n+1}^- is convex and monotone (i.e. convexity and monotonicity are preserved between exercise times).

Example 4.20.*If we assume that both GLWB and GMWB are written on an asset that follows GBM, then Appendix A establishes the convexity and monotonicity (under sufficient regularity) of* V_n^+ *given the convexity and monotonicity of* V_{n+1}^-. *The general argument is applicable to contracts written on assets whose returns follow multidimensional drift-diffusions with parameters independent of the level of the asset (a local volatility model, for example, is not included in this class). Convexity and monotonicity preservation are retrieved directly from a property of the corresponding Green's function.*

Convexity and monotonicity preservation are established for a stochastic volatility model in [4]. For the case of general parabolic equations, convexity preservation is established in [20]. This result is further generalized to parabolic integro-differential equations, arising from problems involving assets whose returns follow jump-diffusion processes [5].

4.3. Existence of an optimal bang-bang control. Once we have established that convexity and monotonicity are preserved between and across exercise times (i.e. §4.1 and §4.2, respectively), we need only apply our argument inductively to establish the existence of an optimal bang-bang control. Instead of providing a proof for the general case, we simply focus on the GLWB contract here. For the case of a general contract, assuming the dynamics followed by the assets preserve the convexity and monotonicity of the cost of funding the contract between exercise times (e.g. GBM, as in Appendix A), the reader can apply the same techniques used for the GLWB contract here to establish the existence of a bang-bang control.

Example 4.21.*Consider the GLWB. Suppose for some* n *s.t.* $1 \leqslant n \leqslant N$, $V^L(\mathbf{x}, n^+)$ *is convex and monotone as a function of* $\mathbf{x}$ *(satisfying (A1)), and satisfies the growth condition*

$$\left|V^L\left(\mathbf{x}, n^+\right)\right| \leqslant K \exp\left(k\left|\log \mathbf{x}\right|^2\right) \text{ on } [0,\infty)^2 \tag{4.4}$$

Figure 1. *Dynamic programming for pricing contracts with finitely many exercise times.*

Data: payoff of the contract at the expiry, V_N^+
Result: price of the contract at time zero, $V_0 \equiv V_0^+$
1 **for** $n \leftarrow N$ **to** 1 **do**
2   **for** $\mathbf{x} \in \Omega$ **do**
3         $V_n^- (\mathbf{x}) := \sup_{\lambda \in \Lambda_n} V_n^+ (\mathbf{f}_{\mathbf{x},n} (\lambda)) + f_{\mathbf{x},n} (\lambda)$
4   **end**
5   use V_n^- to determine V_{n-1}^+
6 **end**

for positive constants K and k. It immediately follows that for all $\mathbf{x}$, $v_{\mathbf{x},n}^L$ is bounded (satisfying (A2)). Since (A1) and (A2) are satisfied, we can use Example 4.13 to conclude that the supremum of $v_{\mathbf{x},n}^L$, for each $\mathbf{x} \in \Omega$, occurs on $\{0,1,2\}$. By Example 4.19, $V^L (\mathbf{x}, n^-)$ is convex and monotone. Furthermore, it can be shown by a routine computation that $V^L (\mathbf{x}, n^-)$ also satisfies a growth condition of the form (4.4). Under sufficient regularity, as discussed in Example 4.20, the argument in Appendix A ensures that $V^L \left(\mathbf{x}, (n-1)^+\right)$ is convex and monotone as a function of $\mathbf{x}$ as well. Furthermore, under the presumed regularity in Appendix A, $V^L \left(\mathbf{x}, (n-1)^+\right)$ satisfies a growth condition of the form (4.4).

Recall that since $t_N = N$ was picked large enough as to ensure that all holders have passed away at time N (i.e. $\mathcal{R}(N) = 0$), the insurer has no obligations at the N^{th} anniversary and hence by (2.1)

$$V^L \left(\mathbf{x}, N^+\right) = \varphi^L (\mathbf{x}) = 0. \tag{4.5}$$

Note that φ^L is trivially convex and monotone as a function of $\mathbf{x}$, and satisfies a growth condition of the form (4.4). We can then apply the above argument inductively to establish the existence of an optimal bang-bang control. For the case of the GLWB, since the range of this control is finite (Example 4.19), we conclude that an optimal bang-bang control exists for the GLWB.

5. Demonstrating the bang-bang principle. To demonstrate the bang-bang principle in practice, we implement a numerical method to solve the GLWB and GMWB problems and examine loss-maximizing withdrawal strategies.

5.1. Contract pricing algorithm. Figure 1 highlights the usual dynamic programming approach to pricing contracts with finitely many exercise times. Note that line 5 is purposely non-specific; the algorithm does not presume anything about the underlying dynamics of the stochastic process(es) that V is a function of, and as such does not make mention of a particular numerical method used to solve V_{n-1}^+ given V_n^-. Establishing the bang-bang principle for a particular contract allows us to replace Λ_n appearing on line 3 with $\bigcup_{P \in \mathcal{P}(\mathbf{x})} E(P)$.

5.2. Numerical method. The numerical method discussed here applies to both GLWB and GMWB contracts. Each contract is originally posed on $\Omega = [0, \infty)^2$. We employ the algorithm in Figure 1 but instead approximate the solution using a finite difference method on the truncated domain $[0, x_1^{\max}] \times [0, x_2^{\max}]$. As such, since $\mathbf{f}_{\mathbf{x},n} (\lambda)$ will not necessarily land on a mesh node, linear interpolation is used to approximate $V_n^+ (\mathbf{f}_{\mathbf{x},n} (\lambda))$ on line 3. A local optimization problem is solved for each point on the finite difference grid. Details of the numerical scheme can be found in [1, 14].

Between exercise times, the cost of funding each contract satisfies one of (2.9) or (2.13). Corresponding to line 5 of Algorithm 1, we determine V_{n-1}^{+} from V_n^{-} using an implicit finite difference discretization. No additional boundary condition is needed at $x_1 = 0$ or $x_2 = 0$ ((2.9) and (2.13) hold along $\partial\Omega \times [t_n, t_{n+1})$). The same is true of $x_2 = x_2^{\max} \gg 0$. At $x_1 = x_1^{\max} \gg 0$, we impose

$$V\left(x_1^{\max}, x_2, t\right) = g\left(t\right) x_1^{\max} \tag{5.1}$$

for some *càdlàg* function g differentiable everywhere but possibly at the exercise times $t_n \in \mathcal{T}$. Substituting the above into (2.9) or (2.13) yields an ordinary differential equation which is solved numerically alongside the rest of the domain. Errors introduced by the above approximations are small in the region of interest, as verified by numerical experiments.

Remark 5.1.*Since we advance the numerical solution from n^- to $n-1^+$ using a convergent method, the numerical solution converges pointwise to a solution V that is convexity and monotonicity preserving. Although it is possible to show—for special cases—that convexity and monotonicity are preserved for finite mesh sizes, this is not necessarily true unconditionally.*

Remark 5.2.*Although we have shown that an optimal bang-bang control exists for the GLWB problem, we do not replace Λ_n with $\{0,1,2\}$ on line 3 of the algorithm in Figure 1 when computing the cost to fund a GLWB so as to demonstrate that our numerical method, having preserved convexity and monotonicity, selects an optimal bang-bang control. For both GLWB and GMWB, We assume that nothing is known about $v_{\mathbf{x},n}$ and hence form a partition*

$$\lambda_1 < \lambda_2 < \ldots < \lambda_p$$

of the admissible set and perform a linear search[4] to find $\max_i v_{\mathbf{x},n}\left(\lambda_i\right)$. Convergence is achieved by refining this partition.

5.3. Results.

5.3.1. Guaranteed Lifelong Withdrawal Benefits. Figure 2 shows withdrawal strategies for the holder under the parameters in Table 1 on the first four contract anniversaries. We can clearly see that the optimal control is bang-bang from the Figures. At any point $(\mathbf{x}, n)$, we see that the holder performs one of nonwithdrawal, withdrawal at exactly the contract rate, or full surrender (despite being afforded the opportunity to withdraw any amount between nonwithdrawal and full surrender).

When the withdrawal benefit is much larger than the investment account, the optimal strategy is withdrawal at the contract rate (the guarantee is in the money). Conversely, when the investment account is much larger than the withdrawal benefit, the optimal strategy is surrender (the guarantee is out of the money), save for when the holder is anticipating the triennial ratchet (time $n = 2$ and $n = 3$). Otherwise, the optimal strategy includes nonwithdrawal (to receive a bonus) or withdrawal at the contract rate. Note that the strategy is constant along any straight line through the origin since V^{L} is homogeneous of order one in $\mathbf{x}$, as discussed by [14].

5.3.2. Guaranteed Minimum Withdrawal Benefits. For the GMWB, (C1) is violated. In particular, for $\kappa_n > 0$, the function $f_{\mathbf{x},n}^{\mathrm{M}}\left(\lambda\right)$ is concave as a function of $\mathbf{x}$ (Figure 3). However, when $\kappa_n = 0$

[4]It is worthwhile to note that for the general problem, if nothing is known about the smoothness of $v_{\mathbf{x},n}$ but it is known that $v_{\mathbf{x},n}$ is (piecewise-)unimodal, one can approximate $\sup_{\Lambda_n} v_{\mathbf{x},n}$ using one (or more) golden section search(es) [21, 7] to obtain an extremum. However, if this method is used when nothing is known about the unimodality of $v_{\mathbf{x},n}$, the resulting numerical method will not necessarily be convergent. In these situations, to guarantee convergence to the relevant solution, one must resort to a linear search with successive refinement to guarantee convergence.

Table 1
GLWB parameters.

Parameter		Value
Volatility	σ	0.20
Risk-free rate	r	0.04
Hedging fee	α	0.015
Contract rate	δ	0.05
Bonus rate	β	0.06
Expiry	N	57
Initial investment	x_0	100
Initial age at time zero		65
Mortality data		[25]
Ratchets		Triennial
Withdrawals		Annual

Anniversary n	Penalty κ_n
1	0.03
2	0.02
3	0.01
$\geqslant 4$	0.00

or $G = 0$ ($G = 0$ is considered in [19]), the function $f^{\mathrm{M}}_{\mathbf{x},n}(\lambda)$ (see (3.7)) is linear in $\mathbf{x}$, and hence the convexity of $V^{\mathrm{M}}(\mathbf{x}, n^-)$ as a function of $\mathbf{x}$ can be guaranteed given $V^{\mathrm{M}}(\mathbf{x}, n^+)$ convex and monotone in $\mathbf{x}$. In this case, it is possible to use the same machinery as was used in the GLWB case to arrive at a bang-bang principle. The case of $\kappa_n = 0$ corresponds to zero surrender charges at the n^{th} anniversary, while $G = 0$ corresponds to enforcing that all withdrawals (regardless of size) be charged at the penalty rate.

Now, consider the data in Table 2. Since $\kappa_n = 0$ for all $n \geqslant 7$, the convexity of V in $\mathbf{x}$ is preserved for all $t \in [6, N]$. However, since $\kappa_6 > 0$, the convexity is violated as $t \to 6^-$. Figure 4 demonstrates this preservation and violation of convexity. As a consequence, V will not necessarily be convex in $\mathbf{x}$ as $t \to 5^+$, and the contract fails to satisfy the bang-bang principle at each anniversary date $n \leq 5$.

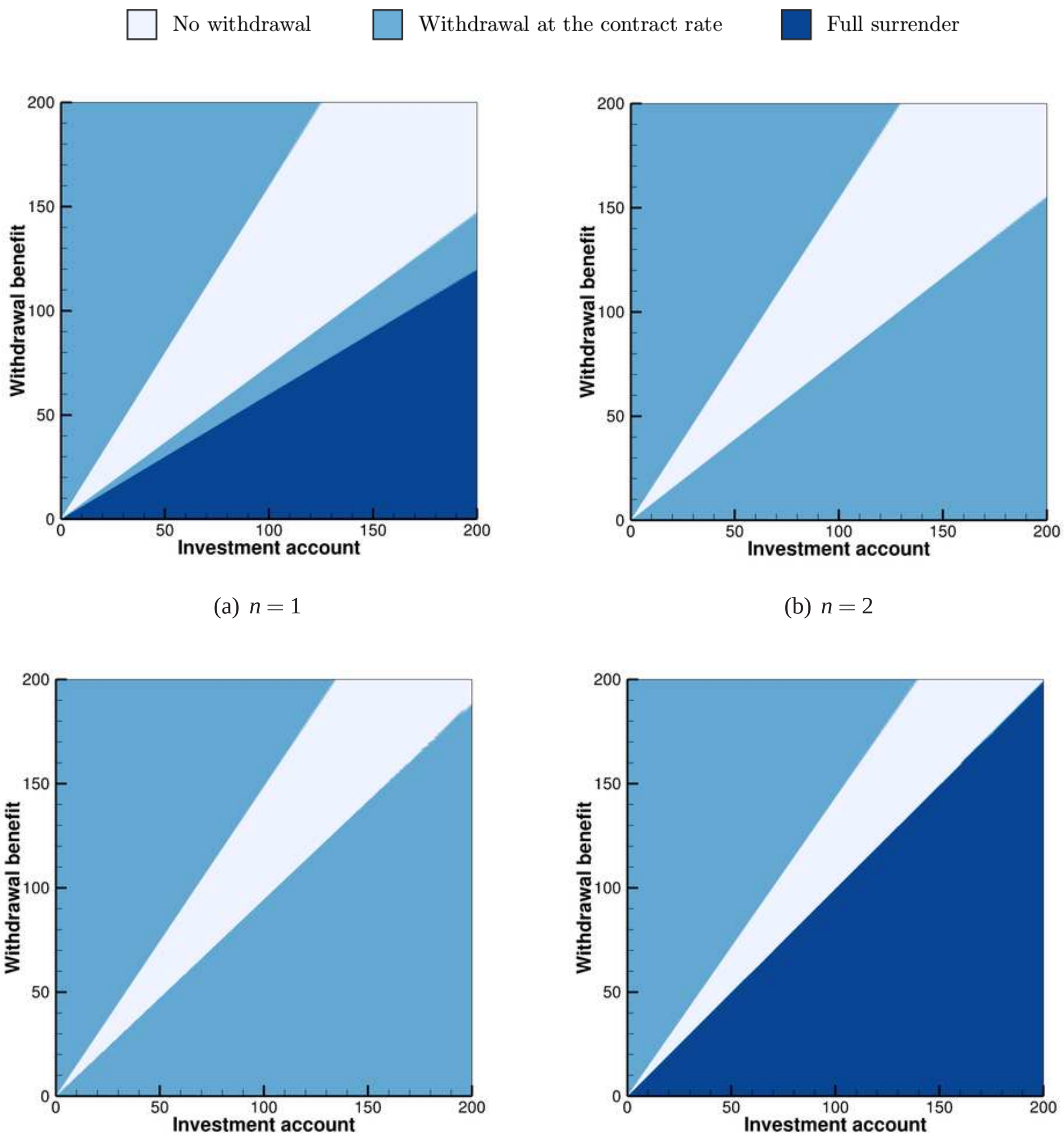
Assuming that $V^{\mathrm{M}}(\mathbf{x}, n^+)$ is convex and monotone (along with the usual assumptions on boundedness) and taking $\mathcal{P}^{\mathrm{M}}_n(\mathbf{x}) \equiv \{P_1, P_2\}$ with $P_1 \equiv [0, G/x_2 \wedge 1]$ and $P_2 \equiv [G/x_2 \wedge 1, 1]$ yields that there exists an optimal control s.t. at any point $(\mathbf{x}, n)$, with $x_2 > 0$, the optimal control takes on one of the values $\{0, G/x_2, 1\}$. These three actions correspond to nonwithdrawal, withdrawing the predetermined amount G, or performing a full surrender. This is verified by Figure 5, which shows withdrawal strategies under the parameters in Table 2 at times $n = 6$ and $n = 7$. As predicted, along any line $x_2 = \text{const.}$, the optimal control takes on one of three values ($\lambda^\star_n(\mathbf{x}) \in \{0, G/x_2, 1\}$, or equivalently, $\lambda^\star_n(\mathbf{x})\, x_2 \in \{0, G, x_2\}$). Since at $n = 6$, $\kappa_n > 0$, we see that the holder is more hesitant to surrender the contract whenever $x_1 \gg x_2$ (compare with the same region at $n = 7$).

6. Conclusion. Although it is commonplace in the insurance literature to assume the existence of optimal bang-bang controls, there does not appear to be a rigorous statement of this result. We have rigorously derived sufficient conditions which guarantee the existence of optimal bang-bang controls for GMxB guarantees.

These conditions require that the contract features are such that the solution to the optimal control can be formulated as maximizing a convex objective function, and that the underlying stochastic process assumed for the risky assets preserve convexity and monotonicity.

These conditions are non-trivial, in that the conditions are satisfied for the GLWB contract but not

Figure 2. *Optimal control for the GLWB for data in Table 1. As predicted, there exists an optimal control consisting only of nonwithdrawal, withdrawal at the contract rate, and full surrender.*



(a) $n = 1$ (b) $n = 2$

(c) $n = 3$ (d) $n = 4$

454 for the GMWB contract with typical contract parameters. Our numerical experiments indicate that the
455 GMWB controls are not bang-bang precisely where our conditions are violated.

456 From a practical point of view, the existence of bang-bang controls allows for the use of very
457 efficient numerical methods.

458 Although we have focused specifically on the application of our results to GMxB guarantees, the
459 reader will have no difficulty in applying the sufficient conditions to other optimal control problems in

Figure 3. $f^M_{\mathbf{x},n}(\lambda)$ *for fixed* x_1, λ *and* n *with* $\kappa > 0$. *Convexity does not hold across the kink at* $x_2 = G/\lambda$ *(see (3.7)).*

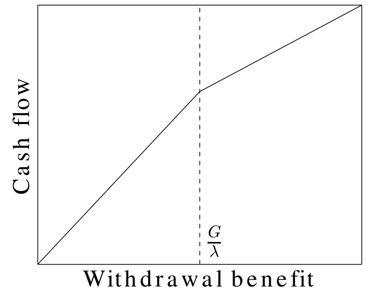
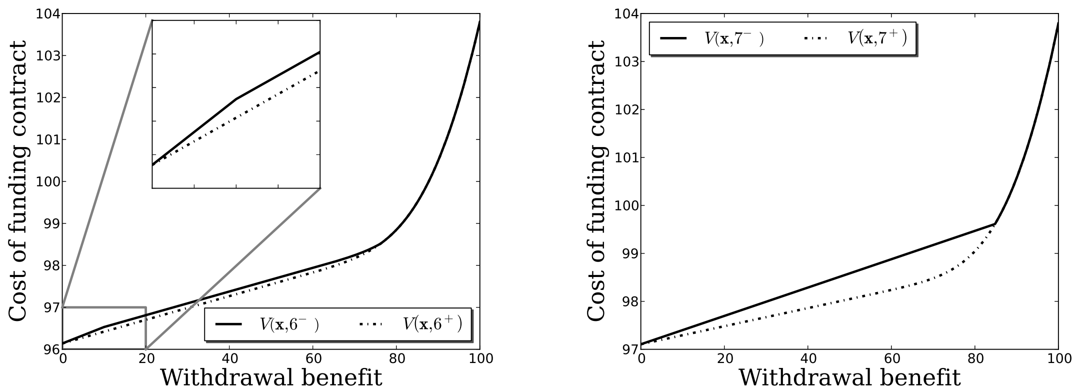


Figure 4. $V^M(\mathbf{x},t)$ *for fixed* $x_1 = 100$ *under the data in Table 2. Points where* $V^M(\mathbf{x},n^-) = V^M(\mathbf{x},n^+)$ *correspond to nonwithdrawal. To the left of these points, the holder withdraws at the contract rate (see Figure 5).*



(a) Convexity is not preserved from $t \to 6^+$ to $t \to 6^-$.

(b) Convexity is preserved from $t \to 7^+$ to $t \to 7^-$.

finance. We believe that we can also use an approach similar to that used here to show the existence of bang-bang controls for general impulse control problems. In the impulse control case, these conditions will require that the intervention operator have a particular form, and that the stochastic process (without intervention) preserve convexity and monotonicity. We leave this generalization for future work.

Appendix A. Preservation of convexity and monotonicity.

In this Appendix, we establish the convexity and monotonicity of a contract whose payoff is convex and monotone and written on assets whose returns follow (multi-dimensional, possibly correlated) GBM. We do so by considering the PDE satisfied by V and considering the fundamental solution corresponding to the operator appearing in the log-transformed version of this PDE. Considering the log-transformed PDE allows us to eliminate the parabolic degeneracy at the boundaries and to argue that the resulting fundamental solution for the log-transformed operator should be of the form $\Gamma(\mathbf{y},\mathbf{y}',\tau,\tau') \equiv \Gamma(\mathbf{y}-\mathbf{y}',\tau,\tau')$.

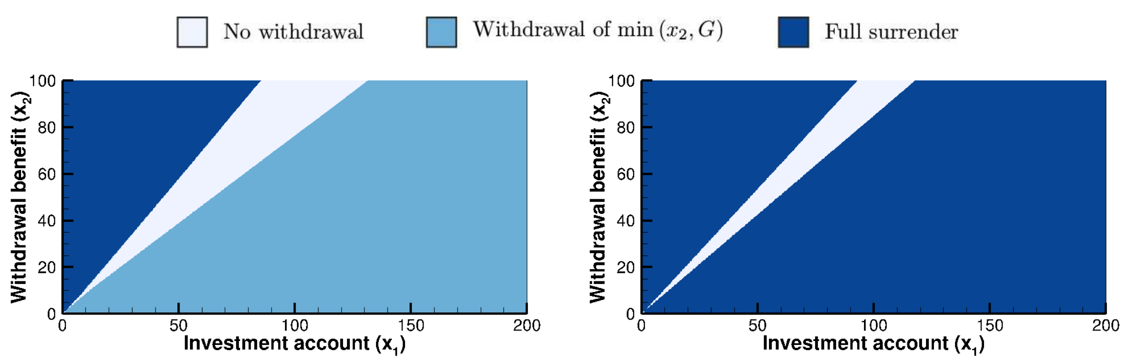
We begin by describing some of the notation used in this appendix as follows:

Table 2
GMWB parameters [10].

Parameter		Value
Volatility	σ	0.15
Risk-free rate	r	0.05
Hedging fee	α	0.01
Contract rate amount	G	10
Expiry	N	10
Initial investment	x_0	100
Withdrawals		Annual

Anniversary n	Penalty κ_n
1	0.08
2	0.07
3	0.06
4	0.05
5	0.04
6	0.03
$\geqslant 7$	0

Figure 5. *Optimal control $\lambda_n^\star(\mathbf{x})$ scaled by x_2 for the data in Table 2.*



(a) $n = 6$ (b) $n = 7$

- Let $\Omega \equiv \Omega_1 \times \Omega_2$ where $\Omega_1 \equiv [0, \infty)^m$ and Ω_2 is a convex subset of a partially ordered vector space A over the field $\mathbb{R}$ with order $\leqslant_A$. Ω can thus be considered as a convex subset of the vector space $\mathcal{A} \equiv \mathbb{R}^m \times A$ over the field $\mathbb{R}$.
- We write an element of Ω in the form $(\mathbf{x}, x_{m+1}) \equiv (x_1, \ldots, x_m, x_{m+1})$ with $\mathbf{x} \in \Omega_1$ and $x_{m+1} \in \Omega_2$ in order to distinguish between elements of Ω_1 and Ω_2.
- The partial order we consider on $\mathcal{A}$ is simply inherited from the orders $\leqslant_{\mathbb{R}^m}$ (Remark 4.7) and $\leqslant_A$. Specifically, $(\mathbf{x}, x_{m+1}) \leqslant_{\mathcal{A}} \left(\mathbf{x}', x'_{m+1}\right)$ if and only if $\mathbf{x} \leqslant_{\mathbb{R}^m} \mathbf{x}'$ and $x_{m+1} \leqslant_A x'_{m+1}$.
- We denote by $|\cdot|$ the Euclidean norm.

Suppose V satisfies

$$\partial_t V + \mathcal{L} V + \omega = 0 \text{ on } \Omega \times (t_n, t_{n+1}) \tag{A.1}$$

and

$$V\left(\mathbf{x}, x_{m+1}, t_{n+1}^-\right) = \varphi\left(\mathbf{x}, x_{m+1}\right) \text{ on } \Omega \tag{A.2}$$

where

$$\mathcal{L} \equiv \frac{1}{2} \sum_{i,j=1}^{m} a_{i,j} x_i x_j \partial_{x_i} \partial_{x_j} + \sum_{i=1}^{m} b_i x_i \partial_{x_i} + c. \tag{A.3}$$

In the above, $\omega \equiv \omega(\mathbf{x}, t)$. We will, for the remainder of this appendix, assume that

(D1) $a_{i,j} \equiv a_{i,j}(t)$, $b_i \equiv b_i(t)$ and $c \equiv c(t)$ (i.e. The functions $a_{i,j}$, b_i and c are independent of $(\mathbf{x}, x_{m+1})$).

(D2) $\mathcal{L}$ is uniformly elliptic on $\Omega_1 \setminus \partial\Omega_1$.

Example A.1. *For the GLWB guarantee, $\mathcal{L}$ is given in (2.10) and $\omega = \mathcal{M}(t)x_1$.*

Remark A.2. *We say V satisfies (A.1) if V is twice differentiable in (the components of) $\mathbf{x}$ and once differentiable in t on $\Omega \times (t_n, t_{n+1})$*[5]*, continuous on $\Omega \times [t_n, t_{n+1})$*[6] *and satisfies (A.1) pointwise.*

We now describe the log-transformed problem. For ease of notation, let

$$\begin{aligned} e^{\mathbf{y}} &\equiv (e^{y_1}, \ldots, e^{y_m}) & a'_{i,j}(\tau) &\equiv a_{i,j}(t_{n+1} - \tau) \\ \varphi'(\mathbf{y}, y_{m+1}) &\equiv \varphi(e^{\mathbf{y}}, y_{m+1}) & b'_i(\tau) &\equiv b_i(t_{n+1} - \tau) \\ \omega'(\mathbf{y}, y_{m+1}, \tau) &\equiv \omega(e^{\mathbf{y}}, y_{m+1}, t_{n+1} - \tau) & c'(\tau) &\equiv c(t_{n+1} - \tau) \end{aligned}$$

Let V be a solution of the Cauchy problem (A.1) and (A.2). Let

$$u(\mathbf{y}, y_{m+1}, \tau) \equiv V(e^{\mathbf{y}}, y_{m+1}, t_{n+1} - \tau)$$

and $\Delta \equiv t_{n+1} - t_n$. Then, u satisfies

$$\mathcal{L}'u - \partial_\tau u + \omega' = 0 \text{ on } \Omega' \times (0, \Delta) \tag{A.4}$$

and

$$u\left(\mathbf{y}, y_{m+1}, 0^+\right) = \varphi'(\mathbf{y}) \tag{A.5}$$

where

$$\mathcal{L}' \equiv \frac{1}{2} \sum_{i,j=1}^{m} a'_{i,j} \partial_{y_i} \partial_{y_j} + \sum_{i=1}^{m} b'_i \partial_{y_i} + c'.$$

Note that (D2) implies that $\mathcal{L}'$ is uniformly elliptic on $\mathbb{R}^m$.

In order to guarantee that a solution u to the log-transformed Cauchy problem (A.4) and (A.5) exists and is unique, sufficient regularity must be imposed on the functions $a'_{i,j}$, b'_i, c', ω' and φ', and u. We summarize below.

(E1) φ' is continuous on $\mathbb{R}^m$.

(E2) $a'_{i,j}$, b'_i, c', ω' are sufficiently regular (for an accurate detailing of the required regularity, see [15, Ch. 1: Thm. 12, Thm. 16]).

(E3) For all $y_{m+1} \in \Omega_2$,

$$\left|\omega'(\mathbf{y}, y_{m+1}, \tau)\right| \leqslant K \exp\left(k|\mathbf{y}|^2\right) \text{ on } \mathbb{R}^m \times (0, \Delta)$$

and

$$\left|\varphi'(\mathbf{y}, y_{m+1})\right| \leqslant K \exp\left(k|\mathbf{y}|^2\right) \text{ on } \mathbb{R}^m$$

for positive constants K and k.

[5] i.e. $V|_{\Omega \times (t_n, t_{n+1})} \in C^{2,1}(\Omega \times (t_n, t_{n+1}))$.

[6] i.e. $V|_{\Omega \times [t_n, t_{n+1})} \in C(\Omega \times [t_n, t_{n+1}))$.

(E4) For all $y_{m+1} \in \Omega_2$, u satisfies the growth condition

$$\left|u\left(\mathbf{y},y_{m+1},\tau\right)\right| \leqslant K' \exp\left(k' \left|\mathbf{y}\right|^2\right) \text{ on } \mathbb{R}^m \times (0,\Delta)$$

for positive constants K' and k'.

When (D2) and (E1)-(E4) are satisfied, u can be written

$$u\left(\mathbf{y},y_{m+1},\tau\right) = \int_{\mathbb{R}^m} \Gamma\left(\mathbf{y},\mathbf{y}',\tau,0\right) \varphi'\left(\mathbf{y}',y_{m+1}\right) d\mathbf{y}' + \int_0^\Delta \int_{\mathbb{R}^m} \Gamma\left(\mathbf{y},\mathbf{y}',\tau,\tau'\right) \omega'\left(\mathbf{y}',y_{m+1},\tau'\right) d\mathbf{y}' d\tau' \text{ on } \mathbb{R}^m \times (0,\Delta) \tag{A.6}$$

where Γ is the fundamental solution for $\mathcal{L}'$. We first note that (E1) follows immediately if φ is convex, as shown below.

Lemma A.3. *If φ is convex w.r.t. the order $\leqslant_{\mathcal{A}}$ (see Definition 4.2), then for all x_{m+1}, $\varphi' \equiv \varphi'\left(\mathbf{y};x_{m+1}\right)$ is continuous in $\mathbf{y}$ on $\mathbb{R}^m$.*

Proof. We have assumed that $\varphi \equiv \varphi\left(\mathbf{x},x_{m+1}\right)$ is convex w.r.t. $\leqslant_{\mathcal{A}}$ on Ω. From this it follows that for all $x_{m+1} \in \Omega_2$, φ is convex in $\mathbf{x}$ on $\Omega_1 \equiv [0,\infty)^m$ w.r.t. to the order $\leqslant_{\mathbb{R}^m}$. This in turn yields that for all $x_{m+1} \in \Omega_2$, φ is continuous in $\mathbf{x}$ on the open set $\Omega_1 \setminus \partial\Omega_1 = (0,\infty)^m$. Therefore, $\varphi' \equiv \varphi'\left(\mathbf{y};x_{m+1}\right)$ is continuous in $\mathbf{y}$ on $\mathbb{R}^m$. ■

Theorem A.4. *Suppose (D1), (D2) and (E2)-(E4). Suppose that φ is convex and monotone w.r.t. the order $\leqslant_{\mathcal{A}}$ (see Definition 4.2 and Lemma 4.6). Suppose further that for all $t \in [t_n,t_{n+1})$, ω is convex and monotone in $\left(\mathbf{x},x_{m+1}\right)$ on Ω w.r.t. the order $\leqslant_{\mathcal{A}}$. Then, for all $t \in [t_n,t_{n+1})$, V is convex and monotone in $\left(\mathbf{x},x_{m+1}\right)$ on Ω w.r.t. the order $\leqslant_{\mathcal{A}}$. In particular, V_n^+ is convex and monotone.*

Proof. Γ appearing in (A.6) depends on $\mathbf{y}'$ and $\mathbf{y}$ through $\mathbf{y}'-\mathbf{y}$ alone since by (D1), $a'_{i,j}$, b'_i and c' are independent of the spatial variables [15, Ch. 9: Section 2]. Therefore

$$u\left(\mathbf{y},y_{m+1},\tau\right) = \int_{\mathbb{R}^m} \Gamma\left(\mathbf{y}'-\mathbf{y},\tau,0\right) \varphi'\left(\mathbf{y}',y_{m+1}\right) d\mathbf{y}' + \int_0^\Delta \int_{\mathbb{R}^m} \Gamma\left(\mathbf{y}'-\mathbf{y},\tau,\tau'\right) \omega'\left(\mathbf{y}',y_{m+1},\tau'\right) d\mathbf{y}' d\tau' \text{ on } \mathbb{R}^m \times (0,\Delta].$$

Let $\log \mathbf{x} \equiv (\log x_1,\ldots,\log x_m)$. From the above, whenever $x_i > 0$ for all i,

$$V\left(\mathbf{x},x_{m+1},t\right) = \int_{\mathbb{R}^m} \Gamma\left(\mathbf{y}'-\log\mathbf{x},t_{n+1}-t,0\right) V\left(e^{\mathbf{y}'},x_{m+1},t_{n+1}^-\right) d\mathbf{y}' + \int_0^\Delta \int_{\mathbb{R}^m} \Gamma\left(\mathbf{y}'-\log\mathbf{x},t_{n+1}-t,\tau'\right) \omega\left(e^{\mathbf{y}'},x_{m+1},t_{n+1}-\tau'\right) d\mathbf{y}' d\tau' \text{ on } \Omega \times [t_n,t_{n+1}).$$

Denote by $\mathbf{x} \circ \mathbf{x}' \equiv (x_1x'_1,\ldots,x_mx'_m)$ the element-wise product of $\mathbf{x}$ and $\mathbf{x}'$. The substitution $\mathbf{y}' = \log\left(\mathbf{x}\circ\mathbf{x}'\right)$ into the above yields

$$V\left(\mathbf{x},x_{m+1},t\right) = \int_0^\infty \cdots \int_0^\infty \Gamma\left(\log\mathbf{x}',t_{n+1}-t,0\right) V\left(\mathbf{x}\circ\mathbf{x}',x_{m+1},t_{n+1}^-\right) \frac{1}{\prod_i x'_i} d\mathbf{x}' + \int_0^\Delta \int_0^\infty \cdots \int_0^\infty \Gamma\left(\log\mathbf{x}',t_{n+1}-t,\tau'\right) \omega\left(\mathbf{x}\circ\mathbf{x}',x_{m+1},t_{n+1}-\tau'\right) \frac{1}{\prod_i x'_i} d\mathbf{x}' d\tau' \text{ on } \Omega \times [t_n,t_{n+1}).$$

Since Γ is >0 [15, 16, Ch. 2, Ch. IV, resp.], from the convexity and monotonicity of V_{n+1}^{-} and ω, it follows immediately that $V_n(\mathbf{x}, x_{m+1}, t)$ is convex and monotone on $(\Omega_1 \setminus \partial\Omega_1) \times \Omega_2$ for any $t \in [t_n, t_{n+1})$. The presumed continuity of V allows us to extend this to Ω. ■

Remark A.5. *The construction of the parametrix approximating the fundamental solution was originally used to establish the existence of fundamental solutions to parabolic PDEs [22]. More recently, it has found its way to finance in both technical [6] and numerical [12] settings.*

Appendix B. Commutativity of union and supremum.

Let T be a poset with order $\leqslant$ satisfying the least-upper-bound property. All supremums are taken w.r.t. T.

Lemma B.1. *Let $\mathcal{S} \equiv \{S_\alpha\}_{\alpha \in \mathcal{A}}$ be an indexed family of nonempty subsets of T. Let $S \equiv \bigcup_{\alpha \in \mathcal{A}} S_\alpha$ and*

$$U \equiv \{v \in T \mid \exists \alpha \in \mathcal{A} \colon v = \sup S_\alpha\}.$$

Then, $\sup S = \sup U$ whenever S is bounded above.

Proof. Suppose $\mathcal{A}$ is empty. Then both S and U are empty, and hence the expressions agree.

Suppose $\mathcal{A}$ is nonempty and that S is bounded above. Since S is bounded above, its supremum u must occur in T. For each α, u is an upper bound of S_α, and since S_α is a nonempty subset of T, $\sup S_\alpha = u_\alpha$ for some $u_\alpha \in T$. Thus, $U = \{u_\alpha\}_{\alpha \in \mathcal{A}} \subset S$. Since $u_\alpha \leqslant u$ for each α, u is an upper bound of U. Since $\mathcal{A}$ is nonempty, U is nonempty and hence U has a least upper bound $u' \in T$ with $u' \leqslant u$. Let $x \in S$. Then $x \in S_\beta$ for some β, and hence $x \leqslant u_\beta \leqslant u'$ and hence u' is an upper bound of S. Since $\sup S = u$, $u \leqslant u'$ and hence $u' = u$. ■

REFERENCES

[1] P. Azimzadeh, P. Forsyth, and K. Vetzal, *Hedging costs for variable annuities under regime-switching*, in Hidden Markov Models in Finance Volume II, R. Mamon and R. Elliot, eds., Springer, New York, 2014. to appear.

[2] A. R. Bacinello, P. Millossovich, A. Olivieri, and E. Pitacco, *Variable annuities: A unifying valuation approach*, Insurance: Mathematics and Economics, 49 (2011), pp. 285–297.

[3] D. K. Bauer, A. Kling, and J. Russ, *A universal pricing framework for guaranteed minimum benefits in variable annuities*, ASTIN Bulletin-Actuarial Studies in Non Life Insurance, 38 (2008), p. 621.

[4] Y. Z. Bergman, B. D. Grundy, and Z. Wiener, *General properties of option prices*, The Journal of Finance, 51 (1996), pp. 1573–1610.

[5] B. Bian and P. Guan, *Convexity preserving for fully nonlinear parabolic integro-differential equations*, Methods Appl. Anal, 15 (2008), pp. 39–51.

[6] I. Bouchouev and V. Isakov, *Uniqueness, stability and numerical methods for the inverse problem that arises in financial markets*, Inverse problems, 15 (1999), pp. R95–R116.

[7] R. P. Brent, *Algorithms for minimization without derivatives*, Courier Dover Publications, 1973.

[8] B. A. Butrica, H. M. Iams, K. E. Smith, and E. J. Toder, *The disappearing defined benefit pension and its potential impact on the retirement incomes of baby boomers*, Social Security Bulletin, 69 (2009).

[9] Z. Chen and P. A. Forsyth, *A numerical scheme for the impulse control formulation for pricing variable annuities with a guaranteed minimum withdrawal benefit (GMWB)*, Numerische Mathematik, 109 (2008), pp. 535–569.

[10] Z. Chen, K. Vetzal, and P. A. Forsyth, *The effect of modelling parameters on the value of GMWB guarantees*, Insurance: Mathematics and Economics, 43 (2008), pp. 165–173.

[11] R. Cont and P. Tankov, *Financial Modelling With Jump Processes*, Chapman & Hall, London, 2004.

[12] F. Corielli, P. Foschi, and A. Pascucci, *Parametrix approximation of diffusion transition densities*, SIAM Journal on Financial Mathematics, 1 (2010), pp. 833–867.

[13] M. Dai, Y. Kuen Kwok, and J. Zong, *Guaranteed minimum withdrawal benefit in variable annuities*, Mathematical Finance, 18 (2008), pp. 595–611.

[14] P. A. Forsyth and K. Vetzal, *An optimal stochastic control framework for determining the cost of hedging of variable annuities*. Working paper, University of Waterloo, 2013.

[15] A. Friedman, *Partial differential equations of parabolic type*, Englewood Cliffs, NJ, (1964).

[16] M. G. Garroni and J. L. Menaldi, *Green functions for second order parabolic integro-differential problems*, vol. 275, Longman Scientific & Technical Harlow, 1992.

[17] D. Holz, A. Kling, and J. Russ, *GMWB for life an analysis of lifelong withdrawal guarantees*, Zeitschrift für die gesamte Versicherungswissenschaft, 101 (2012), pp. 305–325.

[18] Y. Huang and P. Forsyth, *Analysis of a penalty method for pricing a guaranteed minimum withdrawal benefit (GMWB)*, IMA Journal of Numerical Analysis, 32 (2012), pp. 320–351.

[19] Y. T. Huang and Y. K. Kwok, *Analysis of optimal dynamic withdrawal policies in withdrawal guarantees products*. Working paper, Hong Kong University of Science and Technology, 2013.

[20] S. Janson and J. Tysk, *Preservation of convexity of solutions to parabolic equations*, Journal of Differential Equations, 206 (2004), pp. 182–226.

[21] J. Kiefer, *Sequential minimax search for a maximum*, Proceedings of the American Mathematical Society, 4 (1953), pp. 502–506.

[22] E. E. Levi, *Sulle equazioni lineari totalmente ellittiche alle derivate parziali*, Rendiconti del circolo Matematico di Palermo, 24 (1907), pp. 275–317.

[23] M. A. Milevsky and T. S. Salisbury, *Financial valuation of guaranteed minimum withdrawal benefits*, Insurance: Mathematics and Economics, 38 (2006), pp. 21–38.

[24] A. Ngai and M. Sherris, *Longevity risk management for life and variable annuities: The effectiveness of static hedging using longevity bonds and derivatives*, Insurance: Mathematics and Economics, 49 (2011), pp. 100–114.

[25] U. Pasdika, J. Wolff, Gen Re, and MARC Life, *Coping with longevity: The new german annuity valuation table DAV 2004 R*, in The Living to 100 and Beyond Symposium, Orlando Florida, 2005.

[26] G. Piscopo and S. Haberman, *The valuation of guaranteed lifelong withdrawal benefit options in variable annuity contracts and the impact of mortality risk*, North American Actuarial Journal, 15 (2011), p. 59.

[27] R. T. Rockafellar, *Convex analysis*, vol. 28, Princeton university press, 1997.

[28] J. Wang and P. A. Forsyth, *Maximal use of central differencing for Hamilton-Jacobi-Bellman PDEs in finance*, SIAM Journal on Numerical Analysis, 46 (2008), pp. 1580–1601.